Relational Contracts: Public versus Private Savings*

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Abstract

We study relational contracting with an agent who has consumption-smoothing preferences as well as the ability to save to defer consumption (or to borrow). Our focus is the comparison of principal-optimal relational contracts in two settings. The first where the agent's consumption and savings decisions are private, and the second where these decisions are publicly observed. In the first case, the agent smooths his consumption over time, the agent's effort and payments eventually decrease with time, and the balances on his savings account eventually increase. In the second, the agent consumes earlier than he would like, consumption and the balance on savings fall over time, and effort and payments to the agent increase. There is convergence to efficiency in the long run. Our results suggest a possible explanation for low savings rates in certain industries where compensation often comes in the form of discretionary payments.

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1 Introduction

Work on repeated moral hazard with a risk-averse agent — for instance Rogerson (1985) and Fudenberg et al. (1990) — highlighted the value of controlling the agent's consumption through formal contracts. Optimal dynamic contracts, as studied by Rogerson, force the agent to consume more than he would like early in the relationship (the agent would gain by secretly saving and deferring some consumption to a later date).[1] Fudenberg et al. show that a sequence of short-term contracts can often implement the outcome of a long-term contract if the short-term contracts can stipulate agent consumption or savings. Yet, in modern employment relationships, workers' consumption expenditures are largely at their discretion and free from formal agreements. This motivated a literature on "private savings" in dynamic contracting (see Abraham et al., 2011, Edmans et al., 2012, He, 2012, and Di Tella and Sannikov, 2018), where the agent's consumption and savings are unobserved by the principal.

The assumption of private savings, however, fails to recognize that some of a worker's consumption and savings decisions are observable to the employer. For instance, an employer might observe "conspicuous consumption" choices such as spending on clothes, cars, or leisure activities.[2] Some savings decisions (such as pension contributions) might be observed as direct deductions from employee paychecks. It therefore seems important to consider an alternative possibility to the literature above: consumption and saving decisions might be observed, but incentives are only *relational*. By relational incentives, we mean those stemming from informal agreements sustained by an implicit threat of punishments that the players carry out themselves in equilibrium; see, for instance, Bull (1987), MacLeod and Malcomson (1989) and Levin (2003). The aim of this paper is to study the role of consumption and savings and their observability in a relational contracting setting. We do so by introducing savings and consumption-smoothing preferences in a canonical relational contracting model.[3]

Setting. We study a simple relational contracting setting in which the agent exerts costly effort and has concave utility from consumption, with concave utility implying a preference for smooth consumption. We consider two polar opposite cases: one where consumption is perfectly jointly observed, and the other where consumption is unobservable to the principal. Our setting

[1] Other papers where the principal controls the level of agent consumption include Lambert (1983), Spear and Srivastava (1987), Rey and Salanie (1990), Phelan and Townsend (1991), Sannikov (2008) and Garrett and Pavan (2015).

[2] The term "conspicuous consumption" was introduced by Veblen (1899). See Chanen (2008) for a discussion of how US employers are increasingly monitoring employee behavior, including using this information to make retention decisions. There has also been an increase in monitoring recently with work-at-home arrangements due to Covid-19.

[3] The vast majority of relational contracting models specify quasi-linear preferences for money, so any role for saving and the timing of consumption is assumed away. Bull (1987) is the only exception to our knowledge, but his paper is restricted to the sustainability of first-best policies in an overlapping-generations model.

is deterministic, with the output enjoyed by the principal equal to the agent's effort. In each period, first the agent chooses effort and consumption, then the principal makes a discretionary payment to the agent. The initial balance on the agent's savings account is common knowledge, and the evolution of these savings are determined by pay and consumption. Neither party can make any commitments, and a relational contract can be understood simply as an agreed stream of effort, consumption and pay.

As with other papers on relational contracts, we focus on agreements that are "self-enforcing". We thus impose two key sets of constraints that ensure neither party wishes to publicly breach the contract, given that any such breach results in termination of a productive relationship (i.e., no further effort by the agent, and no further payments from the principal).[4] One set of constraints ensures willingness of the agent never to quit the relationship by deviating from the agreed effort and consumption, if observed, and then continuing to optimally consume his savings. The other ensures credibility of the principal's payments to the agent. That is, it ensures that the principal's discounted future profits from continuing the relationship always exceed the payments promised to the agent. We investigate self-enforcing agreements that maximize the principal's discounted profits.

Unobservable consumption. We begin by studying the problem with unobservable consumption (or "private savings"). Dynamic contracting problems with hidden consumption have often been viewed as challenging due to the difficulties associated with "double deviations" (see for example Di Tella and Sannikov, 2018). For instance, an agent in such problems may find it attractive to shirk on effort while saving more to mitigate anticipated punishment through low pay. In our setting, an agent who plans on shirking on effort at a given date, smooths consumption by choosing constant consumption from the beginning. This observation yields a condition for the incentive compatibility of the prescribed effort. We show that, since lifetime earnings accumulate the longer the agent obediently exerts effort, and since agent utility is concave, the pay needed to keep the agent from deviating from a given level of effort increases with time.

As regards the principal's optimal contract, the agent's effort is constant whenever the principal and agent are sufficiently patient. For lower levels of patience, effort is eventually decreasing with time. The reason is that the payment required to compensate a given level of effort gradually increases, as explained above. This means that the contract becomes gradually less profitable, making it more difficult to sustain agreement at later dates. We thus show that the payments that the principal can credibly promise eventually decline, and so effort decreases as well. These effects are novel to contracting settings with private savings, since they are borne out of the principal's inability to commit (while existing work assumes full commitment). Also,

[4] Termination of a productive relationship represents an optimal punishment in the sense that it is as severe on both players as possible.

full commitment models generally permit only weak predictions on the timing of payments, since a principal who fully commits can “save for the agent” which implies indeterminacy on this dimension.[5]

Observable consumption. The above dynamics stand in contrast to the case where consumption is instead jointly observed. Note that the possibility for the informal agreement to condition on consumption enlarges the set of sustainable outcomes for two reasons. One is that non-constant consumption paths become possible, due to relational incentives on the level of consumption. The other is that, if the agent plans to deviate from the specified effort at a given date, he cannot lower his consumption at earlier dates without this being detected, triggering premature termination of his pay. This effectively reduces the profitability of deviations from the agreed effort (compared to what the agent could obtain were he able to reduce consumption at earlier dates and continue being paid).

The dynamics of an optimal relational contract are then as follows. When the players are sufficiently patient, effort is again constant. For lower discount factors, effort strictly increases with time. The reason is that the optimal contract stipulates high consumption for the agent early in the relationship, which drives down the balance on his account. Since the balance on his savings decreases, the agent needs to be paid less at later dates to keep him willing to work. Since the principal’s profits in the relationship grow with time, the pay she can credibly offer increases, and so the contract can call for higher effort.

The key to understanding these dynamics lies in recognizing the value of increasing the profitability of the relationship at later dates, which in turn relaxes the principal’s credibility constraint and permits high pay early on. As noted, the channel for increasing future profits is to induce high consumption by the agent, reducing the balance on his account and making him more willing to work. We show that this process continues indefinitely, with all contractual variables converging over time. The limiting contract is a first-best contract; i.e., there are no distortions in the long run.

Possible interpretations. Our results suggest a possible theory of high conspicuous consumption and low savings that favors the profits of the principal. The idea could be relevant in industries such as banking, where high remuneration (often through discretionary bonuses) is accompanied by a propensity for high spending.[6] High spending would increase the willingness of bankers to continue devoting long hours, which seems advantageous to firms and might even be encouraged by them. For instance, aherence to norms surrounding consumption habits might

[5]The literature focuses, for convenience, on contracts where the agent consumes what he is paid.

[6]There are numerous reports of bankers under pressure to engage in high consumption spending. An article in *Huffpost Business* notes that bankers appear “under constant social pressure to spend and spend some more”, ‘Bankers Explain How They Cannot Possibly Live On $1 Million Pay’ by Mark Gongloff, *Huffpost Business*, 1 May 2013, https://www.huffpost.com/entry/bankers-1-million-pay_n_3188177.

be rewarded by promotion through the firm or perhaps protection from termination.[7] In many respects, this view is similar to Postlewaite (1998), who suggests that pressure to conform with social norms could, in some circumstances, lead individuals to solve the labor-leisure trade-off in favor of high consumption and long work hours. As explained by Postlewaite, this norms-based view can be contrasted with the (perhaps more common) view that high conspicuous consumption is often explained by signaling.[8]

Other literature. The literature on relational contracts has been reviewed in MacLeod (2007) and Malcomson (2015). Notable examples include Levin (2003), Fuchs (2007), Chassang (2010), Halac (2012), Li and Matouschek (2013), Yang (2013), Malcomson (2016) and Fong and Li (2017). Unlike our paper, much of the interest in these works lies in the role of exogenous uncertainty, which is often a source of private information for one of the parties.

As mentioned above, most relational contracting models specify payoffs quasi-linear in payments. Some papers, such as Pearce and Stacchetti (1998) and Thomas and Worrall (1990), consider models with risk-averse agents, but do not permit the agent to save.[9] Bull (1987) specifies overlapping generations of agents with concave utility of consumption who can save. However, this paper does not characterize the optimal contract when the first best is not achievable.

An important driver of dynamics in our model is evolution of the agent's outside option of ceasing productive effort and "living off" the balance on his account. The role of the value of the agent's outside option in repeated relationships has been emphasized in work such as Baker et al. (1994) and McAdams (2011). In Baker et al., the outside option is endogenously determined by the possibility of contracting on objective performance measures, while in McAdams (2011) it is endogenously determined by the opportunities of partners to a relationship to rematch. In general, the higher the outside options of the parties to a relationship, the harder it is for a productive relationship to be sustained. In some papers, as in ours, the outside option evolves dynamically. An example is Garicano and Rayo (2017), where the agent is paid by increments in productive knowledge which increases the value of his outside option.[10]

[7] In particular, incentives in reality may be less stark than in the equilibrium of our model. At the same time, there may of course be reasons outside our model to view incentives for high consumption as undesirable.

[8] More recent theory on consumption as a form of signaling includes Hopkins and Kornienko (2004) and Moav and Neeman (2012).

[9] See also Thomas and Worrall (1994), which considers separately cases where the agent can accumulate capital (arguably akin to saving), and where the agent is risk averse.

[10] Also related is Fudenberg and Rayo (2019); the agent's outside option improves over time also due to the accumulation of knowledge, but the paper focuses on the case where the principal can commit to a long-term contract.

2 Setting

Environment and preferences. A principal and agent meet in discrete time at dates $t = 1, 2, \ldots$. Letting $r > 0$ be the interest rate that will apply to the balance on the agent's savings account, we suppose the players have a common discount factor $\delta = \frac{1}{1+r}$. In every period t, first the agent exerts an effort e_t and consumes an amount c_t. Then, the principal makes a discretionary payment w_t to the agent. These variables are all restricted to be non-negative.

The agent has initial savings balance $b_1 > 0$ as well as access to a savings technology (with the interest rate r as specified above). The initial balance will be common knowledge between the principal and agent, including in our model of private savings in Section 4. The agent's balance at time $t + 1 > 1$ then satisfies

$$b_{t+1} = \frac{b_t + w_t - c_t}{\delta} = b_1 \delta^{-t} + \sum_{s=1}^{t} \delta^{s-t-1} (w_s - c_s). \tag{1}$$

Balances can, in principle, be negative (i.e., the agent can borrow). We say that the agent's intertemporal budget constraint is satisfied in case

$$\sum_{t=1}^{\infty} \delta^{t-1} c_t \leq b_1 + \sum_{t=1}^{\infty} \delta^{t-1} w_t. \tag{2}$$

The agent's felicity from consumption c_t in any period t is denoted $v(c_t)$, where $v : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$. We assume that $v(c)$ is real-valued for $c > 0$, and takes value $-\infty$ at $c = 0$. We further assume that v, when evaluated on positive consumption values, is twice continuously differentiable, strictly increasing and strictly concave. In addition, v is onto all of $\mathbb{R}$, implying $\lim_{c \searrow 0} v(c) = -\infty$.

The agent's disutility of effort e_t is $\psi(e_t)$. We assume that ψ is continuously differentiable, strictly increasing, strictly convex, and such that $\psi(0) = \psi'(0) = 0$, and that $\lim_{e \to \infty} \psi'(e) = \infty$.

The agent's period-t payoff will be $v(c_t) - \psi(e_t)$, while the principal's will be $e_t - w_t$; hence, we interpret effort as equal to the output enjoyed by the principal.

Relational contracts. We focus for tractability on *deterministic* relational contracts.[11] We identify relational contracts with their outcomes; denote them $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t \geq 1}$. We restrict attention to contracts that satisfy the following feasibility constraints.

Definition 2.1. A *feasible relational contract* is a sequence $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t \geq 1}$ satisfying the following conditions:

[11] This assumption is in common with some other work such as Ray (2002). In examining contracts that are optimal for the principal, whether random contracts can improve on deterministic ones might be expected to depend on the nature of risk aversion (e.g., whether v exhibits increasing or decreasing risk aversion). Our results for deterministic contracts do not depend on these considerations.

1. **Positivity:** $\tilde{e}_t, \tilde{w}_t, \tilde{c}_t \geq 0$ for all t.

2. **Balance dynamics and constraint:** Conditions (1) and (2) hold.

3. **Bounded consumption:** The sequences of consumption, pay and effort $((\tilde{c}_t)_{t\geq 1}$, $(\tilde{w}_t)_{t\geq 1}$, and $(\tilde{e}_t)_{t\geq 1})$ are bounded.

While the first and second conditions reflect features of the environment introduced above, the third condition guarantees that the players' payoffs are well-defined in a feasible contract.

3 First best and principal full commitment

Consider first the problem of maximizing the principal's payoff by choice of a feasible relational contract subject only to the constraint that the agent is initially willing to participate. If the agent does not participate, a possibility we describe as "autarky", we stipulate that he consumes $(1-\delta)b_1$ per period. This is the optimal consumption for the agent among consumption streams satisfying the intertemporal budget constraint in Equation (2) given that all payments are set to zero. Therefore, we consider maximizing the principal's payoff over feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ such that the payoff of the agent

$$\sum_{t=1}^{\infty} \delta^{t-1} \left(v(\tilde{c}_t) - \psi\left(\tilde{e}_t\right)\right) \tag{3}$$

is no lower than his autarky value, $\frac{1}{1-\delta}v((1-\delta)b_1)$.

Proposition 3.1. *Consider maximizing the principal's discounted payoff by choice of feasible contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$, subject to ensuring the agent a payoff at least his autarky value $\frac{1}{1-\delta}v((1-\delta)b_1)$. In any optimal feasible contract, effort and consumption are constant at $e^{FB}(b_1) > 0$ and $c^{FB}(b_1) > (1-\delta)\, b_1$, respectively, being the unique solutions to:*

1. ***First order condition:*** *$\psi'(e^{FB}(b_1)) = v'(c^{FB}(b_1))$, and*

2. ***Agent's indifference condition:*** *$v(c^{FB}(b_1)) - \psi(e^{FB}(b_1)) = v((1-\delta)b_1)$.*

Furthermore, the payoff of the principal is $V^{FB}(b_1) \equiv \frac{1}{1-\delta}(e^{FB}(b_1) - (c^{FB}(b_1) - (1-\delta)b_1))$, which is a strictly decreasing function of b_1.

The proof of this and other results are provided in the Appendix. Note that the first-best policies depend on both b_1 and δ, since they depend on the value of autarky consumption $(1-\delta)b_1$ (see Condition 2). However, we reduce the notational burden by making dependence only on b_1 explicit. Note also that the proposition does not specify the timing of payments. The only requirements on payments is that they are feasible and satisfy the agent's budget constraint (2) with equality. Payments may be constant, in which case they equal $c^{FB}(b_1) - (1-\delta)b_1$ in each period.

4 Unobservable consumption

We now suppose the principal can observe the agent's effort, but not the consumption choices nor the agent's balance. Given the absence of commitment, we are interested to determine feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ which coincide with outcomes of a perfect Bayesian equilibrium (PBE) of a dynamic game. These represent the outcomes that are sustainable by a relational contract, and among which we can consider optimizing the principal's payoff.

We begin by defining the histories in our game. For $t \geq 0$, a t-history for the agent is $h_t^A = (e_s, w_s, c_s)_{1\leq s<t}$, which gives the observed effort, payments and consumption up until time $t-1$. The set of such histories at date $t \geq 1$ is $\mathcal{H}_t^A = \mathbb{R}_+^{3(t-1)}$ (with the convention that $\mathbb{R}_+^0 = \emptyset$). Note that, given h_t^A and the agent's initial balance b_1, we can completely determine the evolution of the balance up to date t using Equation (1). We denote the date-t balance by $b(h_t^A)$. A t-history for the principal is $h_t^P = (e_s, w_s)_{1\leq s<t}$. The set of such histories at date $t \geq 1$ is $\mathcal{H}_t^P = \mathbb{R}_+^{2(t-1)}$.

A strategy for the agent is then a collection of functions

$$\alpha_t : \mathcal{H}_t^A \to \mathbb{R}_+^2,\ t \geq 1,$$

and a strategy for the principal is a collection of functions

$$\sigma_t : \mathcal{H}_t^P \times \mathbb{R}_+ \to \mathbb{R}_+,\ t \geq 1.$$

Here, α_t maps the t-history of the agent to a pair (e_t, c_t) of effort and consumption. Also, σ_t maps the t-history of the principal, together with the agent's effort choice e_t, to a payment w_t.

As noted above, we will restrict attention to equilibria whose outcomes coincide with a feasible relational contract. However, we do not restrict the strategies that are available to the players. Certain strategies imply, for instance, the violation of the agent's intertemporal budget constraint in Equation (2). To ensure that the agent finds it optimal to satisfy this constraint, we make the following assumption on payoffs. While the principal's payoff is as specified above (and so given by $\sum_{t=1}^{\infty} \delta^{t-1} (e_t - w_t)$), the agent obtains the payoff $\sum_{t=1}^{\infty} \delta^{t-1} (v(c_t) - \psi(e_t))$ if the constraint in Equation (2) is satisfied, and obtains payoff $-\infty$ otherwise.[12]

To obtain the set of feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that are PBE outcomes, we consider PBE where publicly observed deviations from the agreed outcomes are punished by "autarky". This means that, if the agent deviates from the agreed effort $\tilde{e}_t$, or if the principal deviates from the agreed payment $\tilde{w}_t$, the principal makes no payments and the agent exerts no effort from then on; the agent perfectly smoothing the balance of his account over the infinite

[12] Alternative assumptions can be made which yield the same results as documented below. For instance, another possibility involves permitting negative consumption (assigning it a value $-\infty$ in the agent's payoff), but limiting the extent the agent can draw down the balance on his account (i.e., imposing a hard lower bound on $b_t - c_t$).

future.[13] If the agent's balance is negative when autarky begins, the intertemporal budget constraint in Equation (2) is necessarily violated (as the agent receives no further payments), the agent must earn payoff $-\infty$, and so we can specify for instance that the agent consumes zero in every period. Note that deviations by the agent from the specified consumption, provided they are not accompanied by any deviation in effort, go unpunished (i.e., the principal continues to adhere to the payments specified by the agreement).

If the agent plans to always choose effort in accordance with the contract, he optimally consumes

$$\bar{c}_\infty \equiv (1-\delta)\left(b_1 + \sum_{s=1}^{\infty} \delta^{s-1}\tilde{w}_s\right)$$

in every period. Clearly, any contract to which the agent is willing to adhere must then specify $\tilde{c}_t = \bar{c}_\infty$ for all t. To conclude that the agent does not want to deviate from the contract, it is then enough to show that he does not gain by planning to shirk on effort for the first time at any given date t, while making all other choices optimally. Suppose then that the agent plans to shirk for the first time at some date t, and so puts effort equal to $\tilde{e}_s$ for all $s < t$, and then optimally sets it equal to zero at all later dates. Then the agent optimally sets consumption equal to

$$\bar{c}_{t-1} \equiv (1-\delta)\left(b_1 + \sum_{s=1}^{t-1} \delta^{s-1}\tilde{w}_s\right) \tag{4}$$

at all dates, so as to completely smooth consumption and exhaust lifetime earnings.

Given the above, the maximum payoff the agent achieves when deviating in choice of effort for the first time at date t is

$$\frac{1}{1-\delta}v(\bar{c}_{t-1}) - \sum_{s=1}^{t-1}\delta^{s-1}\psi(\tilde{e}_s).$$

Hence, the agent does not want to deviate from the agreement if and only if, for all $t \geq 1$,

$$\frac{1}{1-\delta}v(\bar{c}_{t-1}) - \sum_{s=1}^{t-1}\delta^{s-1}\psi(\tilde{e}_s) \leq \frac{1}{1-\delta}v(\bar{c}_\infty) - \sum_{s=1}^{\infty}\delta^{s-1}\psi(\tilde{e}_s). \tag{AC$_t^{\text{un}}$}$$

This is the incentive compatibility condition described in the Introduction.

The principal remains willing to continue abiding by the agreement if and only if, at each time t, the payment $\tilde{w}_t$ that is due is less than her continuation payoff in the agreement. The exact requirement is that, for all $t \geq 1$,

$$\tilde{w}_t \leq \sum_{s=t+1}^{\infty}\delta^{s-t}(\tilde{e}_s - \tilde{w}_s). \tag{PC$_t$}$$

[13] The reason we can consider autarky punishments is that they deliver the lowest possible individually rational payoffs for the players.

The following result states that the above constraints determine whether a feasible relational contract is the outcome of a PBE.

Proposition 4.1. *Fix a feasible contract* $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$. *It is the outcome of a PBE if and only if, for all* $t \geq 1$, *Conditions* $(\mathrm{AC}_t^{\mathrm{un}})$ *and* (PC_t) *are satisfied, and* $\tilde{c}_t = \bar{c}_\infty$.

Necessity of the conditions in the proposition follow for the reasons described above. To obtain sufficiency, we completely specify PBE strategies and beliefs.

From now on we refer to a contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that satisfies the conditions of Proposition 4.1 as "self-enforceable".[14] Our task reduces to characterizing feasible contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that maximize the principal's payoff subject to the requirement of being self-enforceable. We term such contracts "optimal".

To determine the properties of optimal contracts, we first show that we can restrict attention to contracts with a particular pattern of payments over time. This pattern involves paying the agent as early as possible, subject to satisfying the agent's incentive constraints. This requires that the agent's obedience constraints in Condition $(\mathrm{AC}_t^{\mathrm{un}})$ hold with equality for all $t \geq 1$. Inspired by the terminology of Board (2011), we refer to this condition as "fastest payments". We show the following.

Lemma 4.1. *For any optimal contract, there is another optimal contract* $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ *with the same sequence of efforts and consumption such that, for all* $t \geq 1$,

$$\frac{v\left(\bar{c}_{t-1}\right)}{1-\delta} - \sum_{s=1}^{t-1} \delta^{s-1}\psi(\tilde{e}_s) = \frac{v\left((1-\delta)\, b_1\right)}{1-\delta}. \tag{$\mathrm{FP}_t^{\mathrm{un}}$}$$

An explanation for the result is as follows. First, note that it is optimal to hold the agent to his outside option, and hence

$$\frac{v\left(\bar{c}_{\infty}\right)}{1-\delta} - \sum_{t=1}^{\infty} \delta^{t-1}\psi(\tilde{e}_t) = \frac{v\left((1-\delta)\, b_1\right)}{1-\delta}. \tag{5}$$

If Condition (5) does not hold, $\tilde{e}_1$ can be slightly increased (keeping the rest of the contract the same) so that the constraints $(\mathrm{AC}_t^{\mathrm{un}})$ and (PC_t) continue to hold for all t. Second, when $(\mathrm{FP}_t^{\mathrm{un}})$ holds for all t, the agent is paid as early as possible while preserving the constraints $(\mathrm{AC}_t^{\mathrm{un}})$. The agent cannot be paid earlier, otherwise he will prefer to work obediently for a certain number of periods, save his income at a higher rate than specified in the agreement, and then quit by exerting no effort. It is easily seen that moving payments earlier in time only relaxes the "principal's constraints" (PC_t).

Concerning "fastest payments", we have the following result.

[14] We favor this term to "self-enforcing" because that latter would refer to a complete specification of strategies.

Lemma 4.2. *Consider a feasible relational contract* $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ *that satisfies Condition* $(\text{FP}_t^{\text{un}})$ *at all dates. For any* t*, if* $\tilde{e}_t > 0$*, then*

$$\tilde{w}_t \in \left(\frac{\psi\left(\tilde{e}_t\right)}{v'\left(\bar{c}_{t-1}\right)}, \frac{\psi\left(\tilde{e}_t\right)}{v'\left(\bar{c}_t\right)} \right). \tag{6}$$

This lemma implies that (assuming effort remains strictly positive) the ratio $\frac{\tilde{w}_t}{\psi(\tilde{e}_t)}$ increases with t. The implication is immediate after noticing that $\bar{c}_t$ increases with t. One explanation is as follows. The longer the agent obediently works, the more he is paid in total. Since he can smooth his consumption of these payments over his entire lifetime, and since he has concave utility of consumption, he values additional payments less. Therefore, the payments needed to keep the agent obediently in the relationship, relative to the disutility of effort incurred, increase with time. This observation will be useful for understanding the shape of optimal relational contracts below.

Apart from the observation in Lemma 4.2, the usefulness of Lemma 4.1 is that it permits the design of the relational contract to be reduced to the choice of an effort sequence $(\tilde{e}_t)_{t\geq 1}$. From $(\tilde{e}_t)_{t\geq 1}$ we can obtain $(\bar{c}_t)_{t\geq 1}$ using $(\text{FP}_t^{\text{un}})$ (so the corresponding consumption $\tilde{c}_t = \bar{c}_\infty$ is also pinned down). Then $(\tilde{w}_t)_{t\geq 1}$ is obtained from Equation (4), and $(\tilde{b}_t)_{t\geq 1}$ from Equation (1). We next discuss the implementation of first-best contracts (Section 4.1), before moving to consider optimal contracts when there is no first-best contract that is self-enforceable (Section 4.2).

4.1 Implementing the first-best outcome

Lemma 4.1 is also useful for understanding the conditions under which the principal obtains the first-best payoff. For instance, we can observe that the first-best effort and consumption, which are constant over time and equal to $e^{FB}(b_1)$ and $c^{FB}(b_1)$, can be implemented when the principal can commit to the agreement, but the agent cannot commit. For this, we simply suppose the principal agrees to payments satisfying the conditions in Equation $(\text{FP}_t^{\text{un}})$, provided the agent chooses effort obediently. Any deviation by the agent from the required effort is met with zero payments from then on.

Now consider whether the principal can attain the first-best payoff when neither the principal nor agent can commit; i.e., whether there is a first-best contract that is self-enforceable. According to Lemma 4.2, payments to the agent increase over time. In the long run, payments approach

$$\frac{\psi\left(e^{FB}(b_1)\right)}{v'\left(c^{FB}(b_1)\right)}.$$

Because the principal's constraints (PC_t) tighten over time, verifying they are always satisfied amounts to verifying that

$$\frac{\psi\left(e^{FB}(b_1)\right)}{v'\left(c^{FB}(b_1)\right)} \leq \frac{\delta}{1-\delta}\left(e^{FB}(b_1) - \frac{\psi\left(e^{FB}(b_1)\right)}{v'\left(c^{FB}(b_1)\right)}\right). \tag{7}$$

The right-hand side is the limiting value of the principal's future discounted profits in the agreement, while the left-hand side is the limiting value of the payment to the agent. Because there is no loss in restricting attention to "fastest payments" (due to Lemma 4.1), this condition is also necessary, and so we have the following result.

Proposition 4.2. *Suppose that neither the principal nor agent can commit to the terms of the agreement and that consumption is unobservable. Then the principal attains the first-best payoff in an optimal contract if and only if Condition* (7) *is satisfied.*

While understanding the parameter range for which Condition (7) holds is clearly important for understanding optimal contracts, this is complicated by the dependence of the first-best policy on both b_1 and δ. Nonetheless, if we vary δ while allowing b_1 to adjust, holding $b_1(1-\delta)$ constant, then the first-best consumption and effort remain constant. There is then a threshold value of δ above which Condition (7) is satisfied, and below which it fails.

4.2 Main characterization for unobservable consumption

We now state our main result for the unobservable consumption case, which is a characterization of optimal effort when the first-best effort cannot be sustained.

Proposition 4.3. *An optimal relational contract exists. Suppose the principal cannot attain the first-best payoff in a self-enforceable contract (i.e., Condition* (7) *is not satisfied). Then, for any optimal contract, there is a date $\bar{t} \geq 1$ such that effort is constant up to this date, and is subsequently strictly decreasing. Effort converges to a value $\tilde{e}_\infty > 0$ in the long run. There exist b_1 and δ such that, for any optimal contract, the value $\bar{t}$ is strictly greater than one; in particular, effort can indeed be constant in the initial periods.*[15]

The dynamics of optimal effort when the principal cannot attain the first-best payoff can be explained as follows. There may be some initial periods when the effort is constant. This occurs if the principal's constraint (PC_t) is initially slack. Given that we consider "fastest payments", the payments rise over these periods for the reasons discussed in relation to Lemma 4.2. Given the principal cannot achieve the first-best payoff, it turns out that the principal's constraint eventually binds, and so payments must be reduced. This is only possible by reducing the level of effort. Note that how much effort can be asked without violating the principal's constraint

[15]The proof shows that effort is initially constant in an optimal contract for values of b_1 and δ close to those for which the principal can attain the first-best payoff.

depends on the future profitability of the relationship. Profitability declines over time, both because higher payments must be made relative to the agent's disutility of effort (see Lemma 4.2), and because the effort that can be requested is less. The fact that profitability declines contributes to the decline in effort, which creates a feedback loop.

Our approach to proving Proposition 4.3 relies on variational arguments. For contracts that fail to exhibit the dynamics described in the proposition, we construct more profitable contracts satisfying all the constraints in Proposition 4.1. We demonstrate some of these arguments below.

One useful result towards establishing Proposition 4.3 links the dynamics of effort to the dates at which the principal's constraint (PC_t) is slack (rather than holding with equality).

Lemma 4.3. *Suppose that* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t \geq 1}$ *is an optimal relational contract. Suppose that the principal's constraint is slack at* t^**, i.e.* $\tilde{w}_{t^*} < \sum_{s=t^*+1}^{\infty} \delta^{s-t^*} (\tilde{e}_s - \tilde{w}_s)$*. Then,* $\tilde{e}_{t^*+1} \leq \tilde{e}_{t^*}$*; also, if* $t^* > 1$*, then* $\tilde{e}_{t^*-1} \leq \tilde{e}_{t^*}$*.*

The proof (in the Appendix) proceeds by showing that, if the conclusion of the lemma fails, then effort can be smoothed raising the principal's profits. Such smoothing is profitable given that the disutility of effort is strictly convex (so that differences in effort across periods are inefficient). An immediate implication of the lemma is that effort is constant over any sequence of periods for which the principal's constraint is slack, explaining why effort may be constant in the initial periods.

A further key part of the proof of Proposition 4.3 is to show that effort strictly decreases from a finite date $\bar{t}$ onwards. The main steps of this argument can be explained as follows. Building on Lemma 4.3, we are able to show (in Lemma A.5 in the Appendix) that effort is weakly decreasing with time. Lemma A.6 then establishes that, if the principal's constraint (PC_t) holds with equality at some date $\hat{t}$, then $\tilde{e}_{\hat{t}+1} < \tilde{e}_{\hat{t}}$ and the constraint holds with equality also at $\hat{t}+1$. Hence effort strictly decreases from $\hat{t}$ onwards.

The argument for Lemma A.6 can be summarized as follows. By assumption, the principal's constraint (PC_t) at date $\hat{t}$ holds as an equality, i.e.

$$\tilde{w}_{\hat{t}} = \sum_{s=\hat{t}+1}^{\infty} \delta^{s-\hat{t}} \left(\tilde{e}_s - \tilde{w}_s\right).$$

We are able to show that $\tilde{e}_{\hat{t}+1} - \tilde{w}_{\hat{t}+1} > \tilde{e}_s - \tilde{w}_s$ for all $s > \hat{t}+1$. This follows because $\psi'(\tilde{e}_t) \leq v'(\bar{c}_\infty)$ for all t (as established in Lemma A.1), because effort is weakly decreasing over time (as noted above), and making use of Lemma 4.2 (which recall implies that the ratio of payments to disutility of effort increases with time). Therefore,

$$\tilde{w}_{\hat{t}} = \sum_{s=\hat{t}+1}^{\infty} \delta^{s-\hat{t}} \left(\tilde{e}_s - \tilde{w}_s\right) > \sum_{s=\hat{t}+2}^{\infty} \delta^{s-\hat{t}-1} \left(\tilde{e}_s - \tilde{w}_s\right) \geq \tilde{w}_{\hat{t}+1}$$

where the second inequality is the principal's constraint (PC_t) at date $\hat{t}+1$. Hence, (again using Lemma 4.2) effort is *strictly* lower in period $\hat{t}+1$ (i.e., $\tilde{e}_{\hat{t}+1} < \tilde{e}_{\hat{t}}$). In turn, using Lemma 4.3, the principal's constraint must hold again with equality at $\hat{t}+1$. So we have shown that, if the principal's constraint holds with equality at a given date, it must hold with equality from then on, and so effort strictly decreases with time.

The above argument assumes that the principal's constraint (PC_t) holds with equality at some date. To show this must in fact be the case when the principal cannot attain the first-best payoff, we assume the contrary. Then Lemma 4.3 implies that optimal effort is constant at all dates, say at a value $\tilde{e}_\infty$ (using the notation of the proposition). Letting the payments and the equilibrium consumption $\bar{c}_\infty$ be determined by Equation (FP_t^{un}), payments increase over time, and converge to $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}$. The principal's constraint (PC_t) is then satisfied at all dates if and only if

$$\frac{\psi\left(\tilde{e}_\infty\right)}{v'\left(\bar{c}_\infty\right)} \leq \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi\left(\tilde{e}_\infty\right)}{v'\left(\bar{c}_\infty\right)}\right),$$

where the left-hand side can be read as the limiting payment to the agent, while the right-hand side is the limiting NPV of future profits to the principal. For the most profitable choice of a constant effort $\tilde{e}^*_\infty$, this inequality holds as equality. The principal's constraints (PC_t) tighten over time, but never hold with equality.

Because effort is below the first-best level, we have $\psi'\left(\tilde{e}^*_\infty\right) < v'\left(\bar{c}^*_\infty\right)$, with $\bar{c}^*_\infty$ the level of agent consumption that corresponds to a contract with constant effort $\tilde{e}^*_\infty$. It follows that any sufficiently small adjustment to the effort policy that raises the NPV of effort, together with a change in payments that leaves the agent's payoff in the contract unchanged, raises profits. We therefore suggest a perturbation to the constant-effort contract (see Lemma A.7 in the Appendix) that increases the NPV of effort, but (assuming that payments continue to satisfy (FP_t^{un})) leaves the principal's constraints (PC_t) intact. To be more precise, we consider increasing effort by a little at date one and lowering it by a constant amount in future periods. If we only raise effort at date one, leaving other effort values unchanged and assuming that payments are adjusted to satisfy (FP_t^{un}) at all dates, the principal's constraint (PC_t) is eventually violated (since v is strictly concave and total pay increases, it becomes more costly to compensate the agent for his effort; in particular, payments must increase in all periods). Therefore the reduction in effort at future dates is a "correction" intended to relax the principal's constraint (PC_t) when it is tightest.

We have established then that, when the first-best payoff is not attainable, the principal's constraint (PC_t) holds with equality from some date onwards. At these dates, the principal is indifferent between paying the agent and reneging. This feature is the same as in the optimal contracts of Ray (2002) who provides, in a quite general (though distinct) relational contracting environment, a sense in which agent payoffs are backloaded.

It remains to translate the findings of Proposition 4.3 into predictions for payments and the agent's balance. Note however, that while Lemma 4.1 tells us it is optimal for Condition ($\mathrm{FP}_t^{\mathrm{un}}$) to hold at all dates, other contracts with a different timing for payments may be optimal. We therefore provide a partial converse for Lemma 4.1.

Proposition 4.4. *Suppose the principal cannot attain the first-best payoff in a self-enforceable contract. Fix any optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ *and let* $\bar{t}$ *be the date from which effort is strictly decreasing (see Proposition 4.3). Then, Condition* ($\mathrm{FP}_t^{\mathrm{un}}$) *holds for all* $t > \bar{t}$. *Payments to the agent strictly decrease from date* $\bar{t}+1$ *onwards, while the agent's balances strictly increase.*

The reason payments satisfying Condition ($\mathrm{FP}_t^{\mathrm{un}}$) are strictly decreasing from date $\bar{t}+1$ is explained above. The fact that the agent's balance increases over time then follows straightforwardly from Equation (1) and from Equation (2) taken to hold with equality. In particular, note that

$$\tilde{b}_t = \frac{\bar{c}_\infty}{1-\delta} - \sum_{\tau \geq t} \delta^{\tau - t} \tilde{w}_\tau$$

which strictly increases with t when payments fall over time.

Note that, when $\bar{t} > 1$, the principal's constraint (PC_t) is initially slack. In this case, Condition ($\mathrm{FP}_t^{\mathrm{un}}$) need not hold at $t < \bar{t}$, and so payments before date $\bar{t}$ are not uniquely determined. When this "fastest payments" condition is nonetheless taken to hold, payments in fact increase over time up to date $\bar{t}$ (as was mentioned above).

5 Observed consumption

We now study the case where, at each time t, before making the payment w_t, the principal can observe the agent's past and present-period effort choices $(e_s)_{s=1}^t$ as well as past and present-period consumption choices $(c_s)_{s=1}^t$. Since payments and consumption are commonly observed, the balance b_t at the beginning of each period t is also commonly known (as deduced from Equation (1)).

The game is now one of complete information, and we consider sub-game perfect Nash equilibrium (SPNE). Both players observe at date t the history $h_t = (e_s, w_s, c_s)_{1\leq s<t}$. The set of such histories at each date t is $\mathcal{H}_t = \mathbb{R}_+^{3(t-1)}$. Re-using notation from Section 4 introduces no confusion, so we describe a strategy for the agent as a collection of functions

$$\alpha_t : \mathcal{H}_t \to \mathbb{R}_+^2,\ t \geq 1,$$

and a strategy for the principal as a collection of functions

$$\sigma_t : \mathcal{H}_t \times \mathbb{R}_+^2 \to \mathbb{R}_+,\ t \geq 1.$$

Here, α_t maps the public t-history to a pair (e_t, c_t) of effort and consumption. Also, σ_t maps the public t-history, together with the observed effort and consumption choices (e_t, c_t) of the agent, to a payment w_t. We assume that payoffs are as specified in Section 4 (i.e., the agent earns a payoff $-\infty$ in case his intertemporal budget constraint (2) is violated).

Again we identify a relational contract with the equilibrium outcomes, and we want to characterize contracts that maximize the principal's payoff. A first step is then to determine equilibrium outcomes $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ that are feasible relational contracts. Analogous to the arguments made in the previous section, we begin supposing deviations from the agreed outcomes are punished by "autarky". That is, when either player deviates from the contract, all future effort and payments cease, and the agent perfectly smooths his balance over time. In autarky, the agent consumes $b_t(1-\delta)$ when his balance is $b_t > 0$, and we specify zero consumption in case the balance is $b_t \leq 0$ (in the latter case, the agent can only obtain a payoff of $-\infty$ since violating the intertemporal budget constraint in Equation (2) implies this payoff; hence we might as well set consumption to zero). Now, autarky follows not only deviations in effort and payments, but also in consumption.

Suppose that the agreed contract is $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, and deviations are punished by autarky. The agent's payoff, if complying until date $t-1$ and optimally failing to comply from t onwards, is now

$$\sum_{s=1}^{t-1} \delta^{s-1}(v(\tilde{c}_s) - \psi(\tilde{e}_s)) + \delta^{t-1}\frac{v(\max\{0,(1-\delta)\tilde{b}_t\})}{1-\delta}.$$

This takes into account that the agent who deviates at date t optimally exerts no effort from then on, and consumes $\max\{0,(1-\delta)\tilde{b}_t\}$ per period as explained above. Thus, the agent is willing to follow the prescription of the contract if and only if, at all dates t,

$$\frac{v(\max\{0,(1-\delta)\tilde{b}_t\})}{1-\delta} \leq \sum_{s=t}^{\infty} \delta^{s-t}(v(\tilde{c}_s) - \psi(\tilde{e}_s)). \qquad (\mathrm{AC}_t^{\mathrm{ob}})$$

The reason for the difference to Condition $(\mathrm{AC}_t^{\mathrm{un}})$ is that publicly honoring the agreement up to date $t-1$ ensures that the agent begins period t with the specified balance $\tilde{b}_t$, which in turn determines the wealth he has available to spend in autarky. Condition $(\mathrm{AC}_t^{\mathrm{un}})$, on the other hand, takes into account that the agent who plans to publicly deviate at date t (by shirking on effort) can save in advance for this event, because consumption is not observed.

We can characterize equilibrium outcomes as follows.

Proposition 5.1. *Fix a feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$. It is the outcome of an SPNE in the environment where consumption is observed if and only if, for all $t \geq 1$, Conditions $(\mathrm{AC}_t^{\mathrm{ob}})$ and (PC_t) are satisfied.*

Notice here that the principal's constraint (PC_t) is the one in Section 4. A feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ satisfying the conditions in the proposition is again termed "self-enforceable" and a self-enforceable contract that maximizes the principal's payoff is "optimal". We can now state a result similar to Lemma 4.1.

Lemma 5.1. *For any optimal contract, there exists another optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ *with the same effort and consumption, with* $\tilde{b}_t > 0$ *for all* t*, and where the timing of payments ensures that agent constraints hold with equality in all periods; that is, for all* $t \geq 1$,

$$\frac{v(\tilde{b}_t(1-\delta))}{1-\delta} = \sum_{s=t}^{\infty} \delta^{s-t} \left(v\left(\tilde{c}_s\right) - \psi\left(\tilde{e}_s\right)\right). \tag{8}$$

Lemma 5.1 implies that we can focus on relational contracts where, for all $t \geq 1$,

$$\tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t) = v(\tilde{c}_t) - \psi(\tilde{e}_t) + \tfrac{\delta}{1-\delta} v((1-\delta)\tilde{b}_{t+1}). \tag{FP$_t^{\text{ob}}$}$$

This says that the agent is indifferent between quitting at date t (i.e., ceasing to exert effort) and smoothing the balance $\tilde{b}_t$ optimally over the infinite future, and instead working one more period, exerting effort $\tilde{e}_t$ and consuming $\tilde{c}_t$, before quitting at date $t+1$ and smoothing the balance $\tilde{b}_{t+1}$ over the infinite future.

5.1 Implementing the first-best outcome

Let us turn now to the question of when the principal can attain the first-best payoff in a self-enforceable relational contract. By Lemma 5.1, we can focus on payments such that Equation (8) is satisfied at all dates. Given effort and consumption constant at the first-best levels $e^{FB}(b_1)$ and $c^{FB}(b_1)$, the agent's balance is constant and equal to b_1. Therefore, the payment is constant over time and equal to $w^{FB}(b_1) \equiv c^{FB}(b_1) - (1-\delta)b_1$. The principal's continuation payoff in the contract is constant and equal to

$$V^{FB}(b_1) = \frac{e^{FB}(b_1) - w^{FB}(b_1)}{1-\delta}.$$

Using these observations, we have the following result.

Proposition 5.2. *Suppose that consumption is observable. Then the principal attains the first-best payoff in an optimal relational contract if and only if*

$$w^{FB}(b_1) \leq \frac{\delta}{1-\delta}\big(e^{FB}(b_1) - w^{FB}(b_1)\big). \tag{9}$$

Condition (9) is more easily satisfied than Condition (7) (the condition for the unobservable consumption case). This follows immediately from showing that

$$w^{FB}(b_1) < \frac{\psi\left(e^{FB}(b_1)\right)}{v'\left(c^{FB}(b_1)\right)}. \tag{10}$$

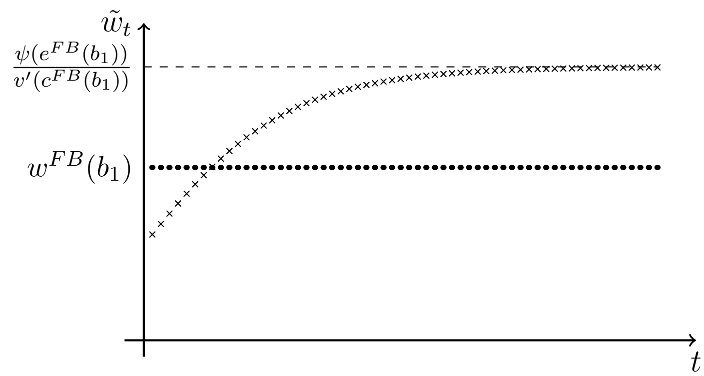

Figure 1: Payments for optimal relational contracts satisfying fastest payments when Equation (7) holds (and so the the principal obtains his first-best payoff), in the unobservable case (crosses) and observable case (circles).

Here, $w^{FB}(b_1)$ is the constant payment to the agent in the observed-consumption case, as specified above. On the other hand, $\frac{\psi\left(e^{FB}(b_1)\right)}{v'(c^{FB}(b_1))}$ is the limiting payment for the unobserved-consumption case (assuming that payments satisfy the "fastest payments" condition in Equation (FP_t^{un})).

The key insight is that, in the observed-consumption case, the principal's constraints (PC_t) are identical in every period, since payments remain constant. In contrast, in the unobserved-consumption case, we saw that they tighten over time. After enough time, the payments in the unobserved-consumption case exceed the constant payments in the observed-consumption case (note that the NPV of payments in both cases is the same), which makes the principal's constraints more difficult to satisfy. Figure 1 illustrates the payments in optimal contracts achieving the first-best payoff for the principal in the unobserved and observed consumption cases.

The reason the fastest payments are later in the unobserved consumption case can be understood as the agent having available more attractive deviations: in particular, the agent can plan to deviate in effort at a given date, reducing consumption from the initial date without this being detected by the principal. In the observed-consumption case, deviations in consumption instead trigger autarky immediately.

To derive the inequality (10) formally, observe that by concavity of v, and because $c^{FB}(b_1) > (1-\delta)\, b_1$, we have

$$v\left(c^{FB}(b_1)\right) - v\left((1-\delta)\, b_1\right) > v'\left(c^{FB}(b_1)\right)\left(c^{FB}(b_1) - (1-\delta)\, b_1\right) = v'\left(c^{FB}(b_1)\right) w^{FB}(b_1).$$

The result then follows because the first-best effort and consumption satisfy $v\left(c^{FB}(b_1)\right) - v\left((1-\delta)\, b_1\right) = \psi\left(e^{FB}(b_1)\right)$ by Condition 2 of Proposition 3.1.

5.2 Optimal contract with observed consumption

Now consider the problem of characterizing an optimal contract when the principal's first-best payoff is not attainable. We can consider the "fastest payments" as given in Lemma 5.1. It is convenient to write the principal's problem recursively, with the balance $\tilde{b}_t$ a state variable for the relationship, applying the principle of optimality. Indeed, suppose for some date t that the continuation contract $(\tilde{e}_s, \tilde{w}_s, \tilde{c}_s, \tilde{b}_s)_{s\geq t}$ does not maximize the continuation value of the relationship to the principal $\sum_{s=t}^{\infty} \delta^{s-t} (e_s - w_s)$, subject to it being self-enforceable; i.e., there is some more profitable self-enforceable continuation contract $(\tilde{e}'_s, \tilde{w}'_s, \tilde{c}'_s, \tilde{b}'_s)_{s\geq t}$ with $\tilde{b}'_t = \tilde{b}_t$, which can be taken to satisfy the agent's indifference conditions (8) at all dates. Then this contract can be substituted, increasing the continuation value $\sum_{s=t}^{\infty} \delta^{s-t} (e_s - w_s)$ (and hence the principal's payoff in the contract overall), maintaining the agent indifference conditions (8) at all dates, and continuing to satisfy the principal's constraints (PC_t).

Since an optimal contract maximizes the principal's continuation profits, an optimal contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ must solve a sequence of sub-problems with value $V(\tilde{b}_t)$ given by

$$V(\tilde{b}_t) = \max_{e_t, b_{t+1}, c_t \in \mathbb{R}_+} \left(e_t - (\delta b_{t+1} - \tilde{b}_t + c_t) + \delta V\left(b_{t+1}\right) \right) \tag{11}$$

subject to the agent's indifference condition

$$v(c_t) - \psi(e_t) + \tfrac{\delta}{1-\delta} v((1-\delta) b_{t+1}) = \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t). \tag{12}$$

and to the principal's constraint

$$\delta b_{t+1} - \tilde{b}_t + c_t \leq \delta V\left(b_{t+1}\right). \tag{13}$$

The left-hand side of (13) can be understood as the date-t payment w_t, which is divided into date-t consumption $c_t \in \mathbb{R}_+$ and savings $\delta b_{t+1} - b_t \in \mathbb{R}$. Non-negativity of the payment $\delta b_{t+1} - \tilde{b}_t + c_t$ is assured by the equality (12) and the concavity of v. The same equality ensures that, given $\tilde{b}_t$ is strictly positive, optimal c_t and b_{t+1} must be strictly positive also.

We show that any optimal policy for the principal can be characterized as follows.

Proposition 5.3. *An optimal contract exists. Suppose that, given the balance b_1, an optimal contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ fails to obtain the first-best payoff $V^{FB}(b_1)$. Then the agent's balance $\tilde{b}_t$ and consumption $\tilde{c}_t$ decline strictly over time, with $\tilde{b}_t \to \tilde{b}_\infty$ for some $\tilde{b}_\infty > 0$. Effort $\tilde{e}_t$ and the payments $\tilde{w}_t$ determined by the Conditions (8) increase strictly over time. We have $V(\tilde{b}_t) \to V^{FB}(\tilde{b}_\infty)$ as $t \to \infty$, and effort and consumption converge to first-best levels for the balance $\tilde{b}_\infty$.*

A heuristic account of the forces behind this result is as follows. When the principal's constraint ((PC_t) or equivalently (13)) binds, effort is suppressed. That is, if the principal could increase effort and credibly increase payments to keep the agent as well off, she would gain by doing so. Also, the principal's value function $V(\cdot)$ is strictly decreasing; intuitively, because a lower balance makes the agent cheaper to compensate to keep him in the agreement. Therefore, for any date t, reducing the balance b_{t+1} increases the principal's continuation payoff $V(b_{t+1})$ and relaxes the principal's date-t constraint (13). Therefore, the principal asks the agent to consume earlier than he would like, driving the balance down over time. This continues to a point where, given the revised balance, the contract is close to first best, and so the value of continuing to distort consumption vanishes.

It is worth pointing out here that the dynamics of $V(\tilde{b}_t)$ are determinative of both the dynamics of effort and payments. When there is no self-enforceable first-best contract, $V(\tilde{b}_t)$ strictly increases with t, and moreover the principal's constraint (13) binds. The latter implies that, for all t, both $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$ and $V(\tilde{b}_t) = \tilde{e}_t - \tilde{w}_t + \delta V(\tilde{b}_{t+1}) = \tilde{e}_t$.[16]

A further part of our analysis worth highlighting is an Euler equation

$$1 - \frac{v'((1-\delta)\tilde{b}_{t+1})}{v'(\tilde{c}_t)} = \frac{v'(\tilde{c}_{t+1})}{\psi'(\tilde{e}_{t+1})}\left(1 - \frac{v'((1-\delta)\,\tilde{b}_{t+1})}{v'(\tilde{c}_{t+1})}\right) \tag{14}$$

which must hold for an optimal contract at all dates t, and which we use to derive several key properties. This condition is derived (in Lemma A.11) by fixing the contract at and before $t-1$, and from date $t+2$ onwards, and then requiring the contractual variables at t and $t+1$ to be chosen optimally. The equation captures the relationship between a static distortion in effort and a dynamic distortion in consumption. In particular, when the principal's first-best payoff cannot be attained, we are able to show that $\psi'(\tilde{e}_{t+1}) < v'(\tilde{c}_{t+1})$ for all t (reflecting a static (downward) distortion in effort), and correspondingly $(1-\delta)\,\tilde{b}_{t+1} < \tilde{c}_{t+1} < \tilde{c}_t$ (i.e., consumption strictly decreases over time, which is a dynamic distortion). A trade-off between the static and dynamic distortions should be anticipated, since asking the agent to consume excessively early in the relationship increases the agent's marginal utility of consumption later on, which makes him easier to motivate and permits higher effort and profits at later dates. In turn, this relaxes the principal's credibility constraint (PC$_t$), permitting higher payments and therefore effort also early in the relationship. As $\tilde{b}_t \to \tilde{b}_\infty$, consumption falls to its lower bound, becoming almost constant, so $\frac{v'(\tilde{c}_{t+1})}{\psi'(\tilde{e}_{t+1})} \to 1$, which accords with convergence of effort and consumption to first-best levels.

Finally, analogous to Proposition 4.4, we provide a result on the uniqueness of the timing of payments.

Proposition 5.4. *Suppose the principal cannot attain the first-best payoff in a self-enforceable*

[16]Note that the conclusion the the principal's constraint (13) binds is obtained under the assumption that Conditions (FP$_t^{\text{ob}}$) hold at all dates; but we establish in Proposition 5.4 below that the satisfaction of Conditions (FP$_t^{\text{ob}}$) is necessary for optimality.

relational contract. Then, in any contract that is optimal for the principal, Condition ($\mathrm{FP}_t^{\mathrm{ob}}$) *holds at all dates. Hence payments to the agent strictly increase over time.*

The logic of this result is that, if the Condition ($\mathrm{FP}_t^{\mathrm{ob}}$) fails, then payments can be made earlier in time, while maintaining the agent constraints ($\mathrm{AC}_t^{\mathrm{ob}}$). This induces slack in the principal's constraint (PC_t), which can then be exploited by increasing payments, consumption and effort, increasing profits. As noted for the case of unobservable consumption, such an observation is related to arguments in Ray (2002).

6 Conclusions

This paper has studied optimal relational contracts in a simple deterministic setting where the agent has consumption-smoothing preferences and can save. We contrasted the case where the agent's consumption is unobservable to the principal and where consumption is observed. In the case where consumption is unobservable, we found that the relationship eventually becomes less profitable with time, implying that the payments the principal can credibly offer must decline. Hence effort eventually declines with time. When consumption is instead observable, the agent consumes inefficiently early (i.e., saves too little), the balance on his savings account gradually declines, the relationship becomes more profitable as the agent grows easier to incentivize, payments to the agent gradually increase, and the agent's effort increases. The contract when the principal observes the agent's consumption is a Pareto improvement on the one when it is not observed. This is in spite the fact there is an additional source of distortion, namely in the timing of consumption. This distortion is more than offset by an improvement in the provision of incentives.

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A Appendix: Omitted proofs

Proof of Proposition 3.1. The proof is standard and so relegated to the Online Appendix.

A.1 Proofs of the results in Section 4.

Proof of Proposition 4.1. Necessity follows the arguments in the main text. Sufficiency is obtained by explicitly constructing equilibria. See the Online Appendix.

Proof of Lemma 4.1

Proof. Fix an optimal relational contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$; that is, a feasible contract that maximizes the principal's discounted payoff subject to the conditions of Proposition 4.1. Then Condition (5) holds, as explained in the main text. Hence, if

$$\frac{v\left(\bar{c}_{t-1}\right)}{1-\delta} - \sum_{s=1}^{t-1} \delta^{s-1}\psi(\tilde{e}_s) \tag{15}$$

exceeds $\frac{v(b_1(1-\delta))}{1-\delta}$ at any date t, then the inequality (AC_t^{un}) must not be satisfied; i.e., the conditions of Proposition 4.1 are not satisfied.

Finally, suppose that the expression (15) is strictly less than $\frac{v(b_1(1-\delta))}{1-\delta}$ at some increasing sequence of dates $(t_n)_{n=1}^N$, where N may be finite or infinite. For each n, there is $\varepsilon_n > 0$ such that

$$\frac{1}{1-\delta}v(\bar{c}_{t_n-1} + \delta^{t_n-2}\varepsilon_n(1-\delta)) - \sum_{s=1}^{t_n-1} \delta^{s-1}\psi(\tilde{e}_s) = \frac{v\left(b_1\left(1-\delta\right)\right)}{1-\delta}.$$

Increase $\tilde{w}_{t_n-1}$ by ε_n, and reduce $\tilde{w}_{t_n}$ by $\frac{\varepsilon_n}{\delta}$; note that this leads to a change in $\bar{c}_{t_n-1}$, but does not affect $\bar{c}_t$ for $t \neq t_n$. After this adjustment has been made for each n, we have a relational contract for which the expression (15) is equal to $\frac{v(b_1(1-\delta))}{1-\delta}$ at all dates t. Also, because ψ is non-negative, $\bar{c}_t$ must be a non-decreasing sequence, and hence all payments $\tilde{w}_t$ in the new relational contract are non-negative. Hence, the new contract is feasible, and we have observed that the agent's constraints (AC_t^{un}) are satisfied. Also, the principal's constraints (PC_t) are satisfied. To see the latter, note that these constraints are affected by the adjustments to the original contract only at dates satisfying $t = t_n$ for some n. At such dates the principal's constraint is *slackened* by the amount $\frac{\varepsilon_n}{\delta}$. □

Proof of Lemma 4.2

Proof. Observe from Condition (FP_t^{un}) evaluated at consecutive dates, we have

$$\frac{v\left(\bar{c}_{t-1}+(1-\delta)\,\delta^{t-1}\tilde{w}_t\right)-v\left(\bar{c}_{t-1}\right)}{1-\delta}=\delta^{t-1}\psi\left(\tilde{e}_t\right).$$

By the Fundamental Theorem of Calculus, we have

$$\int_0^{\tilde{w}_t} v'\left(\bar{c}_{t-1}+(1-\delta)\,\delta^{t-1}x\right)dx=\psi\left(\tilde{e}_t\right)$$

and hence

$$k\tilde{w}_t=\psi\left(\tilde{e}_t\right)$$

for $k\in\left(v'\left(\bar{c}_t\right),v'\left(\bar{c}_{t-1}\right)\right)$, which proves the result. □

Proof of Lemma 4.3

Proof. **Proof that $\tilde{e}_{t^*+1}\leq\tilde{e}_{t^*}$.** Suppose, for the sake of contradiction, that $\tilde{e}_{t^*+1}>\tilde{e}_{t^*}$. We can choose a new contract with efforts $(\tilde{e}'_t)_{t\geq1}$, and payments $(\tilde{w}'_t)_{t\geq1}$ chosen to satisfy Equation (FP_t^{un}), such that they coincide with the original policy except in periods t^* and t^*+1. In these periods, $\tilde{e}'_{t^*}$ and $\tilde{e}'_{t^*+1}$ are such that $\tilde{e}_{t^*}<\tilde{e}'_{t^*}\leq\tilde{e}'_{t^*+1}<\tilde{e}_{t^*+1}$ and

$$\psi\left(\tilde{e}'_{t^*}\right)+\delta\psi\left(\tilde{e}'_{t^*+1}\right)=\psi\left(\tilde{e}_{t^*}\right)+\delta\psi\left(\tilde{e}_{t^*+1}\right),$$

which implies (by convexity of ψ) that $\tilde{e}'_{t^*}+\delta\tilde{e}'_{t^*+1}>\tilde{e}_{t^*}+\delta\tilde{e}_{t^*+1}$. We then have also that $\tilde{w}_{t^*}<\tilde{w}'_{t^*}$ and $\tilde{w}'_{t^*}+\delta\tilde{w}'_{t^*+1}=\tilde{w}_{t^*}+\delta\tilde{w}_{t^*+1}$ (since the NPV of payments does not change, equilibrium consumption does not change in any period t; so the balance at date t^*+1 is larger under the new contract). Provided the changes are small, the principal's constraint (PC_t) at t^* remains satisfied. The above observations imply $\tilde{w}'_{t^*+1}<\tilde{w}_{t^*+1}$, so the principal's constraint is relaxed at date t^*+1. Since the NPV of output goes up, the principal's constraint is relaxed at all periods before t^*.[17] The contract after date t^*+1 is unaffected. The modified contract is thus self-enforceable, and it is strictly more profitable than the original, establishing a contradiction.

Proof that $\tilde{e}_{t^*-1}\leq\tilde{e}_{t^*}$. Analogous and omitted. □

Proof of Propositions 4.2 and 4.3

Proof. The remaining steps in the proof of Proposition 4.3 are divided into nine lemmas. The proof of Proposition 4.2 is provided in the process, in Lemma A.7. Throughout, we restrict

[17] Note that, for the new contract, the principal's constraint at any date $\hat{t}$ may be written as $\sum_{t=\hat{t}}^{\infty}\delta^{t-\hat{t}}\tilde{w}'_t\leq\sum_{t=\hat{t}+1}^{\infty}\delta^{t-\hat{t}}\tilde{e}'_t$. For $\hat{t}<t^*$ this inequality is satisfied strictly since $\sum_{t=\hat{t}}^{\infty}\delta^{t-\hat{t}}\tilde{w}'_t=\sum_{t=\hat{t}}^{\infty}\delta^{t-\hat{t}}\tilde{w}_t$, while $\sum_{t=\hat{t}+1}^{\infty}\delta^{t-\hat{t}}\tilde{e}'_t>\sum_{t=\hat{t}+1}^{\infty}\delta^{t-\hat{t}}\tilde{e}_t$.

attention to payments determined under the restriction to "fastest payments", i.e. satisfying Condition ($\mathrm{FP}_t^{\mathrm{un}}$). Proofs of Lemmas A.4-A.7 are provided in the main text, while the others are in the Online Appendix.

The following result provides a bound on effort in an optimal contract.

Lemma A.1. *In an optimal contract, $\psi'(\tilde{e}_t) \leq v'(\bar{c}_\infty)$ for all t.*

This observation is used to prove existence of an optimal contract.

Lemma A.2. *An optimal relational contract exists.*

We then establish the following regarding the non-degeneracy of optimal contracts.

Lemma A.3. *The principal obtains a strictly positive payoff in any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. Moreover, $\tilde{e}_t$ and $\tilde{w}_t$ are strictly positive at all dates t.*

We now establish an important property of relational contracts: they become (approximately) stationary in the long run.

Lemma A.4. *Suppose that $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is an optimal relational contract satisfying* ($\mathrm{FP}_t^{\mathrm{un}}$). *Then, there exists an effort/payment pair $(\tilde{e}_\infty, \tilde{w}_\infty)$ such that $\lim_{t\to\infty}(\tilde{e}_t, \tilde{w}_t) = (\tilde{e}_\infty, \tilde{w}_\infty)$.*

Proof. **Step 0.** In this step we observe that, for an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ satisfying ($\mathrm{FP}_t^{\mathrm{un}}$),

$$\lim_{t\to\infty}\left(\tilde{w}_t - \frac{\psi(\tilde{e}_t)}{v'(\bar{c}_\infty)}\right) = 0.$$

This follows from Lemma 4.2, after noticing that $(\tilde{e}_t)_{t\geq 1}$ is bounded in an optimal contract.

Step 1. Define $\bar{e} \equiv \limsup_{t\to\infty} \tilde{e}_t$, which we know from Lemma A.1 is no greater than $z(v'(\bar{c}_\infty))$, where z is the inverse of ψ'. We now show that, for any $e \in [0, \bar{e}]$,

$$\frac{\psi(e)}{v'(\bar{c}_\infty)} \leq \frac{\delta}{1-\delta}\left(e - \frac{\psi(e)}{v'(\bar{c}_\infty)}\right). \tag{16}$$

Note, by convexity of ψ, if the inequality (16) is satisfied at $\bar{e}$, then it is satisfied for all $e \in [0, \bar{e}]$. Assume now for the sake of contradiction that the inequality (16) is not satisfied for some $e \in [0, \bar{e}]$. Then we must have

$$\frac{\psi(\bar{e})}{v'(\bar{c}_\infty)} > \frac{\delta}{1-\delta}\left(\bar{e} - \frac{\psi(\bar{e})}{v'(\bar{c}_\infty)}\right). \tag{17}$$

Observe then that there is a sequence $(\varepsilon_t)_{t=1}^{\infty}$ convergent to zero such that, for all $t \geq 1$,

$$\tilde{e}_t - \tilde{w}_t \leq \bar{e} - \frac{\psi(\bar{e})}{v'(\bar{c}_\infty)} + \varepsilon_t.$$

This follows because $\tilde{w}_t - \frac{\psi(\tilde{e}_t)}{v'(\bar{c}_\infty)} \to 0$ as $t \to \infty$ (by Step 0), because $e - \frac{\psi(e)}{v'(\bar{c}_\infty)}$ increases over effort levels e in $[0, \bar{e}]$ (since $\psi'(\bar{e}) \leq v'(\bar{c}_\infty)$ by Lemma A.1), and by definition of $\bar{e}$ as $\limsup_{t\to\infty} \tilde{e}_t$. We therefore have that

$$\limsup_{t\to\infty} \sum_{s=t+1}^{\infty} \delta^{s-t}(\tilde{e}_s - \tilde{w}_s) \leq \frac{\delta}{1-\delta}\left(\bar{e} - \frac{\psi(\bar{e})}{v'(\bar{c}_\infty)}\right) < \frac{\psi(\bar{e})}{v'(\bar{c}_\infty)},$$

where the last inequality holds by Equation (17). However, Step 0 implies that the superior limit of payments to the agent must be $\frac{\psi(\bar{e})}{v'(\bar{c}_\infty)}$, which implies that the principal's constraint (PC_t) is not satisfied at some time t. This contradicts the definition of $\bar{e}$ as $\limsup_{t\to\infty} \tilde{e}_t$.

Step 2. We complete the proof by showing that $\liminf_{t\to\infty} \tilde{e}_t = \bar{e}$. This is immediate if $\bar{e} = 0$, so assume $\bar{e} > 0$. Assume, for the sake of contradiction, that $\liminf_{t\to\infty} \tilde{e}_t < \bar{e}$. In this case, there exists some $t' > 1$ such that $\tilde{e}_{t'} < \min\{\bar{e}, \tilde{e}_{t'+1}\}$.

Step 2a. We have

$$\tilde{w}_{t'} \leq \frac{\delta}{1-\delta}\left(\tilde{e}_{t'+1} - \frac{\psi(\tilde{e}_{t'+1})}{v'(\bar{c}_\infty)}\right). \tag{18}$$

This follows because (i) $\tilde{w}_{t'} \leq \frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)}$ by Lemma 4.2 and the assumption that payments satisfy condition (FP_t^{un}); (ii) $\frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)} \leq \frac{\delta}{1-\delta}\left(\tilde{e}_{t'} - \frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)}\right)$, by assumption that $\tilde{e}_{t'} < \bar{e}$ and by Step 1; and (iii) $\tilde{e}_{t'} - \frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)} \leq e_{t'+1} - \frac{\psi(\tilde{e}_{t'+1})}{v'(\bar{c}_\infty)}$ because $z(v'(\bar{c}_\infty)) \geq \tilde{e}_{t'+1} > \tilde{e}_{t'}$ (recall that the inequality $z(v'(\bar{c}_\infty)) \geq \tilde{e}_{t'+1}$ is established in Lemma A.1).

Step 2b. We now show that the principal's constraint (PC_t) is slack at t'. Note first that, for any $t \geq 1$, we have

$$\begin{aligned}
\tilde{w}_{t+1} - \tilde{w}_t &= \frac{\bar{c}_{t+1} - \bar{c}_t}{\delta^t(1-\delta)} - \frac{\bar{c}_t - \bar{c}_{t-1}}{\delta^{t-1}(1-\delta)} \\
&\geq \frac{v(\bar{c}_{t+1}) - v(\bar{c}_t)}{\delta^t(1-\delta)v'(\bar{c}_t)} - \frac{v(\bar{c}_t) - v(\bar{c}_{t-1})}{\delta^{t-1}(1-\delta)v'(\bar{c}_t)} \\
&= \frac{\psi(\tilde{e}_{t+1}) - \psi(\tilde{e}_t)}{v'(\bar{c}_t)},
\end{aligned}$$

where we used that v is concave and Lemma 4.2. Hence, we have that $\tilde{e}_{t+1} > \tilde{e}_t$ implies $\tilde{w}_{t+1} > \tilde{w}_t$.

Since t' was chosen so that $\tilde{e}_{t'+1} > \tilde{e}_{t'}$, we have $\tilde{w}_{t'+1} > \tilde{w}_{t'}$. Hence,

$$\begin{aligned}\tilde{w}_{t'} &< (1-\delta)\,\tilde{w}_{t'} + \delta\tilde{w}_{t'+1} \\ &\leq \delta\left(\tilde{e}_{t'+1} - \frac{\psi\left(\tilde{e}_{t'+1}\right)}{v'\left(\bar{c}_{\infty}\right)}\right) + \delta\sum_{s=t'+2}^{\infty}\delta^{s-t'-1}\left(\tilde{e}_s - \tilde{w}_s\right) \\ &\leq \sum_{s=t'+1}^{\infty}\delta^{s-t'}\left(\tilde{e}_s - \tilde{w}_s\right),\end{aligned}$$

where the second inequality uses (i) Equation (18) from Step 2a, and (ii) the principal's constraint (PC_t) in period $t'+1$. The third inequality uses that $\tilde{w}_{t'+1} \leq \frac{\psi(\tilde{e}_{t'+1})}{v'(\bar{c}_{\infty})}$, which follows from Lemma 4.2.

Step 2c. We finish the proof with the following observation. The fact the principal's constraint (PC_t) is slack at time t' (proven in Step 2b) contradicts Lemma 4.3, since effort is strictly higher at $t'+1$ than at t'. □

The following lemma determines that, in an optimal contract, effort is weakly decreasing (as mentioned in the main text).

Lemma A.5. *In an optimal contract, the effort policy $(\tilde{e}_t)_{t\geq 1}$ is a weakly decreasing sequence. Therefore, for all t, $\tilde{e}_t \geq \tilde{e}_{\infty} \equiv \lim_{s\to\infty}\tilde{e}_s$.*

Proof. By Lemma A.4, $(\tilde{e}_t)_{t=1}^{\infty}$ is a convergent sequence, so using the notation in its proof, we have $\tilde{e}_{\infty} = \bar{e}$. Step 2 in the proof of Lemma A.4 proves that there is no time t' such that $\tilde{e}_{t'} < \min\{\bar{e}, \tilde{e}_{t'+1}\}$. Hence, there is no t' such that $\tilde{e}_{t'} < \tilde{e}_{\infty}$.

Now suppose, for the sake of contradiction, that $(\tilde{e}_t)_{t=1}^{\infty}$ is not a weakly decreasing sequence. Thus, there exists a date t' where $\max_{t>t'}\tilde{e}_t > \tilde{e}_{t'}$ (the maximum exists by the first part of this proof, and because $\lim_{t\to\infty}\tilde{e}_t = \tilde{e}_{\infty}$ by Lemma A.4). Let $t^*(t')$ be the smallest value $t > t'$ where the maximum is attained, that is, $\tilde{e}_{t^*(t')} = \max_{t>t'}\tilde{e}_t$.

For any $s > t^*(t')$,

$$\tilde{e}_{t^*(t')} - \tilde{w}_{t^*(t')} > \tilde{e}_{t^*(t')} - \frac{\psi(\tilde{e}_{t^*(t')})}{v'(\bar{c}_{t^*(t')})} \geq \tilde{e}_{t^*(t')} - \frac{\psi(\tilde{e}_{t^*(t')})}{v'(\bar{c}_{s-1})} \geq \tilde{e}_s - \frac{\psi(\tilde{e}_s)}{v'(\bar{c}_{s-1})} > \tilde{e}_s - \tilde{w}_s. \tag{19}$$

The first inequality follows from Lemma 4.2; the second inequality follows because $\bar{c}_{s-1} \geq \bar{c}_{t^*(t')}$. The third inequality follows because $e - \frac{\psi(e)}{v'(\bar{c}_{s-1})}$ is increasing in e over $[0, z(v'(\bar{c}_{\infty}))]$, and because $\tilde{e}_s \leq \tilde{e}_{t^*(t')}$ for $s > t^*(t')$ by definition of $t^*(t')$. The fourth inequality follows because $\tilde{w}_s > \frac{\psi(\tilde{e}_s)}{v'(\bar{c}_{s-1})}$ by Lemma 4.2.

Equation (19) implies that

$$\tilde{e}_{t^*(t')} - \tilde{w}_{t^*(t')} > (1-\delta)\sum_{s=t^*(t')+1}^{\infty}\delta^{s-t^*(t')-1}\left(\tilde{e}_s - \tilde{w}_s\right),$$

so that

$$\begin{aligned}\sum_{s=t^*(t')}^{\infty} \delta^{s-t^*(t')} \left(\tilde{e}_s - \tilde{w}_s\right) &= \tilde{e}_{t^*(t')} - \tilde{w}_{t^*(t')} + \delta \sum_{s=t^*(t')+1}^{\infty} \delta^{s-t^*(t')-1} \left(\tilde{e}_s - \tilde{w}_s\right) \\ &> (1-\delta) \sum_{s=t^*(t')+1}^{\infty} \delta^{s-t^*(t')-1} \left(\tilde{e}_s - \tilde{w}_s\right) + \delta \sum_{s=t^*(t')+1}^{\infty} \delta^{s-t^*(t')-1} \left(\tilde{e}_s - \tilde{w}_s\right) \\ &= \sum_{s=t^*(t')+1}^{\infty} \delta^{s-t^*(t')-1} \left(\tilde{e}_s - \tilde{w}_s\right). \qquad (20)\end{aligned}$$

Recall from Lemma 4.3 that the principal's constraint must hold with equality at $t^*(t') - 1$ (since $\tilde{e}_{t^*(t')} > \tilde{e}_{t^*(t')-1}$ by the definition of $t^*(t')$). The inequality (20), then implies (given satisfaction of the principal's constraint (PC_t)) that $\tilde{w}_{t^*(t')-1} > \tilde{w}_{t^*(t')}$. But then, recalling Lemma 4.2, we have

$$\frac{\psi\left(\tilde{e}_{t^*(t')-1}\right)}{v'\left(\bar{c}_{t^*(t')-1}\right)} > \tilde{w}_{t^*(t')-1} > \tilde{w}_{t^*(t')} > \frac{\psi(\tilde{e}_{t^*(t')})}{v'\left(\bar{c}_{t^*(t')-1}\right)}.$$

Hence, $\tilde{e}_{t^*(t')-1} > \tilde{e}_{t^*(t')}$, contradicting the definition of $t^*(t')$. □

Having shown that the effort is weakly decreasing in an optimal relational contract (Lemma A.5) we now show that it is strictly decreasing when the principal's constraint holds with equality.

Lemma A.6. *If the principal's constraint* (PC_t) *holds with equality at some date* t^**, then* $\tilde{e}_{t^*} > \tilde{e}_{t^*+1}$*. Hence, by Lemma 4.3, the principal's constraint also holds with equality at* t^*+1*.*

Proof. The same arguments we used in Lemma A.5 to establish the inequalities in (19) imply that $\tilde{e}_{t^*+1} - \tilde{w}_{t^*+1} > \tilde{e}_s - \tilde{w}_s$ for all $s > t^*+1$. In turn, this means that, if the principal's constraint (PC_t) holds with equality at t^*, then $\tilde{w}_{t^*} > \tilde{w}_{t^*+1}$. Indeed, because the principal's constraint holds with equality at t^*,

$$\begin{aligned}\tilde{w}_{t^*} &= \delta\left(\tilde{e}_{t^*+1} - \tilde{w}_{t^*+1} + \delta \sum_{s=t^*+2}^{\infty} \delta^{s-t^*-2}(\tilde{e}_s - \tilde{w}_s)\right) \\ &> \delta\left((1-\delta) \sum_{s=t^*+2}^{\infty} \delta^{s-t^*-2}(\tilde{e}_s - \tilde{w}_s) + \delta \sum_{s=t^*+2}^{\infty} \delta^{s-t^*-2}(\tilde{e}_s - \tilde{w}_s)\right) \\ &= \sum_{s=t^*+2}^{\infty} \delta^{s-t^*-1}(\tilde{e}_s - \tilde{w}_s) \\ &\geq \tilde{w}_{t^*+1}.\end{aligned}$$

The final inequality follows from the principal's constraint (PC_t) at date t^*+1. Using Lemma 4.2, we have $\tilde{e}_{t^*+1} < \tilde{e}_{t^*}$. □

Lemma A.6 implies that, given payments satisfy condition ($\mathrm{FP}_t^{\mathrm{un}}$), if the principal's constraint (PC_t) holds with equality at some date, then effort is strictly decreasing forever after (and the principal's constraints (PC_t) hold with equality forever after). Our next goal is therefore to establish the condition under which the principal attains the first-best payoff, and, when this condition fails, establish that there is necessarily a date at which the principal's constraint is satisfied with equality.

Lemma A.7. *An optimal contract achieves the first-best payoff of the principal if and only if Condition (7) holds. If this condition is not satisfied, then there is a time $t^* \in \mathbb{N}$ such that the principal's constraint is slack if and only if $t < t^*$. Hence, effort is constant up to date $t^* - 1$ and strictly decreases from date t^*.*

Proof. Consider payments satisfying ($\mathrm{FP}_t^{\mathrm{un}}$), for all t, and determined given the first-best effort (this is $e^{FB}(b_1)$ in Proposition 3.1). Lemma 4.2 shows that the payments increase over time and tend to $\frac{\psi(e^{FB}(b_1))}{v'(c^{FB}(b_1))}$. On the other hand, per-period profits fall over time towards $e^{FB}(b_1) - \frac{\psi(e^{FB}(b_1))}{v'(c^{FB}(b_1))}$. This establishes Condition (7) is both necessary and sufficent for implementation of the first best.

Assume now that Condition (7) fails, and fix an optimal contract that is not first best. Lemma A.6 established that there are two possibilities. First, we might have a finite date $t^* \in \mathbb{N}$, with the principal's constraint (PC_t) holding with equality at t^*, and every subsequent date, but slack at dates $t^* - 1$ and earlier. In this case, effort is constant from the initial date up to $t^* - 1$ (by Lemma 4.3) and strictly decreases from date t^*. Second, we might have that the principal's constraint (PC_t) is slack at all dates. Effort is then constant over all periods (by Lemma 4.3), but not first-best. The result in the lemma is established if we can show this second case does not occur; so assume for a contradiction that it does. Letting $\tilde{e}_\infty$ be the constant effort level and $\bar{c}_\infty$ equilibrium consumption, Proposition 3.1 then implies that $v'(\bar{c}_\infty) \neq \psi'(\tilde{e}_\infty)$. By Lemma A.1, we have $v'(\bar{c}_\infty) > \psi'(\tilde{e}_\infty)$. By Lemma A.3, we have $\tilde{e}_\infty > 0$.

Note that $\tilde{w}_t$ increases over time to $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}$ (from Lemma 4.2). We claim then that

$$\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} = \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}\right). \tag{21}$$

If instead $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} > \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}\right)$, then, for large enough t we must have

$$\tilde{w}_t > \sum_{s=t+1}^{\infty} \delta^{s-t}\left(\tilde{e}_\infty - \tilde{w}_t\right),$$

so the principal's constraint is violated at t. If instead $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} < \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}\right)$, we have $\tilde{w}_t$ remains bounded below $\sum_{s=t+1}^{\infty} \delta^{s-t}\left(\tilde{e}_\infty - \tilde{w}_t\right)$. Without violating ($\mathrm{PC}_t$), effort can be increased by a small constant amount across all periods (with payments adjusted to satisfy (FP_t^{un})). This increases profits.

Note then that Condition (21) can be written as

$$\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} = \delta\tilde{e}_\infty.$$

Because ψ is strictly convex, we have

$$\frac{\psi'\left(\tilde{e}_\infty\right)}{v'\left(\bar{c}_\infty\right)} > \delta.$$

We now consider an adjusted contract in which effort increases at date 1 by $\varepsilon > 0$, raising the disutility of effort at date 1 by $\psi(\tilde{e}_\infty + \varepsilon) - \psi(\tilde{e}_\infty)$. Because payments to the agent increase at all dates under condition (FP_t^{un}), the new policy will not satisfy the principal's constraint (PC_t) if this is the only adjustment (as explained in the main text). We therefore simultaneously reduce effort from some fixed date 2 onwards by $\kappa\left(\varepsilon\right) > 0$ to be determined (i.e., effort is given by $e_t = \tilde{e}_\infty - \kappa\left(\varepsilon\right)$ for $t \geq 2$).

We let $\bar{c}_\infty(\varepsilon, \kappa(\varepsilon))$ denote equilibrium consumption under the new plan (naturally, $\bar{c}_\infty\left(0,0\right)$ is consumption under the original plan). The new consumption satisfies

$$\begin{aligned}\frac{v\left(\bar{c}_\infty\left(\varepsilon,\kappa\left(\varepsilon\right)\right)\right)}{1-\delta} - \frac{v\left(\bar{c}_\infty\left(0,0\right)\right)}{1-\delta} =& \psi\left(\tilde{e}_\infty+\varepsilon\right) - \psi\left(\tilde{e}_\infty\right) \\ & - \frac{\delta}{1-\delta}\left(\psi\left(\tilde{e}_\infty\right) - \psi\left(\tilde{e}_\infty - \kappa\left(\varepsilon\right)\right)\right)\end{aligned}$$

or

$$\bar{c}_\infty(\varepsilon,\kappa(\varepsilon)) = v^{-1}\left(\begin{array}{c}(1-\delta)(\psi(\tilde{e}_\infty+\varepsilon)-\psi(\tilde{e}_\infty)) \\ \quad -\delta(\psi(\tilde{e}_\infty)-\psi(\tilde{e}_\infty-\kappa(\varepsilon))) + v(\bar{c}_\infty(0,0))\end{array}\right)$$

To determine the value for $\kappa(\varepsilon)$, define the following function

$$f(\varepsilon,k) \equiv \frac{\psi\left(\tilde{e}_\infty - k\right)}{v'(\bar{c}_\infty(\varepsilon,k))} - \delta\left(\tilde{e}_\infty - k\right). \tag{22}$$

We then define $\kappa(\varepsilon)$ by $f(\varepsilon,\kappa(\varepsilon)) = 0$ for positive ε in a neighborhood of 0. We will use the implicit function theorem to show that such a local solution $\kappa(\varepsilon)$ exists.

To apply the implicit function theorem, note that $f\left(\varepsilon,k\right)$ is continuously differentiable in a neighborhood of $(\varepsilon,k) = (0,0)$. The derivative of $f(\varepsilon,k)$ with respect to k, evaluated at $(\varepsilon,k) = (0,0)$, is

$$f_2\left(0,0\right) =\delta - \frac{\psi'\left(\tilde{e}_\infty\right)}{v'\left(\bar{c}_\infty\left(0,0\right)\right)} + v''\left(\bar{c}_\infty\left(0,0\right)\right)\left(\frac{\delta\psi'\left(\tilde{e}_\infty\right)}{v'\left(\bar{c}_\infty\left(0,0\right)\right)^3}\right)\psi\left(\tilde{e}_\infty\right).$$

This is strictly negative, using that $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))} > \delta$. The derivative $f(\varepsilon,k)$ instead with respect to ε, evaluated at $(\varepsilon,k)=(0,0)$, is

$$f_1(0,0) = -v''(\bar{c}_\infty(0,0))\left(\frac{(1-\delta)\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))^3}\right)\psi(\tilde{e}_\infty).$$

The implicit function theorem then gives us that κ is locally well-defined by $f(\varepsilon,\kappa(\varepsilon))=0$ on some interval around 0, unique, and continuously differentiable, with derivative approaching

$$\begin{aligned}\kappa'(0) &= -\frac{f_1(0,0)}{f_2(0,0)} \\ &= \frac{v''(\bar{c}_\infty(0,0))\left(\frac{(1-\delta)\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))^3}\right)\psi(\tilde{e}_\infty)}{\delta-\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))}+v''(\bar{c}_\infty(0,0))\left(\frac{\delta\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))^3}\right)\psi(\tilde{e}_\infty)} \\ &< \frac{1-\delta}{\delta} \qquad (23)\end{aligned}$$

as $\varepsilon \to 0$ (the strict inequality follows because $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))} > \delta$).

For small enough ε, the new effort policy and payments defined by condition (FP_t^{un}) satisfy the principal's constraints (PC_t). This follows by observing that, when ε is small, the constraint (PC_t) remains slack at date $t=1$. For all other dates, the satisfaction of the constraint (PC_t) follows from $f(\varepsilon,\kappa(\varepsilon))=0$, and by Lemma 4.2.

It remains to show that, for small enough positive ε, the principal's profits strictly increase. The NPV of effort increases by

$$\varepsilon-\frac{\delta}{1-\delta}\kappa(\varepsilon) = \left(1-\frac{\delta}{1-\delta}\kappa'(0)\right)\varepsilon+o(\varepsilon)$$

(where $o(\varepsilon)$ represents terms that vanish faster than ε as $\varepsilon\to 0$). From the inequality (23) we have $1-\frac{\delta}{1-\delta}\kappa'(0)>0$, and so the increase in effort is strictly positive for ε small enough. Using that payments continue to satisfy Condition (FP_t^{un}), a marginal increase in the NPV of effort is compensated by an increase in the NPV of payments to the agent by $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))}$. Therefore, the principal's payoff under the new policy increases by

$$\left(1-\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))}\right)\left(1-\frac{\delta}{1-\delta}\kappa'(0)\right)\varepsilon+o(\varepsilon)$$

which is strictly positive for small enough ε, recalling that $v'(\bar{c}_\infty(0,0))>\psi'(\tilde{e}_\infty)$. □

We have established that, for any optimal contract that does not attain the first-best payoff of the principal, there is a date $\bar{t}\geq 1$ such that effort is constant up to this date, and subsequently strictly decreasing to a value $\tilde{e}_\infty$, as stated in the proposition. Our next result establishes that $\tilde{e}_\infty>0$, which requires only ruling out $\tilde{e}_\infty=0$.

Lemma A.8. *Suppose the principal cannot attain the first-best payoff. In any optimal contract, the limiting value of effort* $\tilde{e}_\infty \equiv \lim_{t\to\infty} \tilde{e}_t$ *is strictly positive.*

Our final lemma states that, for some configurations of the problem, $\bar{t} > 1$. In this case effort is constant in the initial periods, before strictly decreasing.

Lemma A.9. *For any* v *and* ψ *admitted in the model set-up, there exists a discount factor* δ *and initial balance* b_1 *such that (i) the principal's payoff in an optimal contract is less than the first-best payoff, and (ii) for any optimal contract, the principal's constraint* (PC_t) *is slack for at least* $t = 1, 2$.

(End of the proof of Proposition 4.3.) □

Proof of Proposition 4.4. See the Online Appendix.

A.2 Proofs of the results in Section 5

Proof of Proposition 5.1. The proof follows standard arguments. See the Online Appendix.

Proof of Lemma 5.1. The proof is similar in spirit to that for Lemma 4.1. See the Online Appendix.

Proof of Proposition 5.2

Proof. Follows from the arguments in the main text. □

Proof of Proposition 5.3

Proof. It will be useful to write the recursive problem in the main text by substituting out agent effort. To this end, define a function $\hat{e}$ by

$$\hat{e}(c_t, b_t, b_{t+1}) \equiv \psi^{-1}\Big(v(c_t) + \tfrac{\delta}{1-\delta} v\left((1-\delta)\, b_{t+1}\right) - \tfrac{1}{1-\delta} v((1-\delta) b_t)\Big) \tag{24}$$

for $c_t, b_t, b_{t+1} > 0$ and $v(c_t) + \frac{\delta}{1-\delta} v\left((1-\delta)\, b_{t+1}\right) - \frac{1}{1-\delta} v((1-\delta) b_t) \geq 0$. We will focus throughout on relational contracts that satisfy the "fastest payments" condition (FP_t^{ob}). Hence, given contractual variables $\tilde{c}_t$, $\tilde{b}_t$ and $\tilde{b}_{t+1}$, the date-t effort must be given by $\tilde{e}_t = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1})$.

We can then write the principal's optimal payoff given balance $\tilde{b}_t > 0$ (which we establish below can be attained by a self-enforceable contract) as follows:

$$V(\tilde{b}_t) = \max_{c_t, b_{t+1} > 0} \left(\hat{e}(c_t, \tilde{b}_t, b_{t+1}) - (\delta b_{t+1} - \tilde{b}_t + c_t) + \delta V(b_{t+1})\right) \tag{25}$$

subject to the principal's constraint

$$\delta b_{t+1} - \tilde{b}_t + c_t \leq \delta V\left(b_{t+1}\right) \tag{26}$$

and to the requirement that the implied effort is non-negative, i.e.

$$v(c_t) + \tfrac{\delta}{1-\delta} v\left(\left(1-\delta\right) b_{t+1}\right) - \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t) \geq 0. \tag{27}$$

The proof of Proposition 5.3 will now consist of eight lemmas. The proofs of Lemmas A.10 and A.16 are provided in the Online Appendix. Proofs of the other lemmas are given below.

We begin by observing that no trivial contract can be optimal. Also, in an optimal contract, effort is (weakly) distorted downwards within each period: a marginal increase in effort, compensated by pay that keeps the agent equally well off, would (weakly) raise profits. A strict distortion occurs only if the principal's constraint is binding (since otherwise raising effort and compensating the agent is possible).

Lemma A.10. *In any optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t \geq 1}$, $V(\tilde{b}_t) \in (0, V^{FB}(\tilde{b}_t)]$ *for all* t*. Also,* $\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t > 0$ *for all* t*. In addition,* $\psi'(\tilde{e}_t) \leq v'(\tilde{c}_t)$ *for all* t*, and* $\psi'(\tilde{e}_t) < v'(\tilde{c}_t)$ *only if* $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$*.*

We now establish the Euler equation given in the main text as well as the monotonicity of the consumption plan.

Lemma A.11. *Any optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t \geq 1}$ *satisfies the Euler equation (14) in all periods. Furthermore,* $\tilde{c}_t \geq \tilde{c}_{t+1} > (1-\delta)\tilde{b}_{t+1}$ *for all* t*.*

Proof. We divide the proof in three steps:

Step 1: Fix an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t \geq 1}$. Consider a contract $(\check{e}_t, \check{c}_t, \check{w}_t, \check{b}_t)_{t \geq 1}$, coinciding with the original contract in all periods except for periods t and $t+1$ (so, also, $\check{b}_t = \tilde{b}_t$). We specify that the new contract keeps the agent indifferent between being obedient and optimally deviating in all periods. This requires

$$v(\check{c}_t) - \psi(\check{e}_t) + \tfrac{\delta}{1-\delta} v\left(\tfrac{1-\delta}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t)\right) = \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t), \tag{28}$$

$$v\left(\tfrac{1}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t) + \check{w}_{t+1} - \delta\tilde{b}_{t+2}\right) - \psi(\check{e}_{t+1}) + \tfrac{\delta}{1-\delta} v((1-\delta)\tilde{b}_{t+2}) = \tfrac{1}{1-\delta} v\left(\tfrac{1-\delta}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t)\right), \tag{29}$$

which uses that consumption in period $t+1$ under the new contract is $\check{c}_{t+1} = \frac{1}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t) + \check{w}_{t+1} - \delta\tilde{b}_{t+2}$ (guaranteeing the agent has savings $\tilde{b}_{t+2}$ at date $t+2$).

Fix $\check{e}_t = \tilde{e}_t$ and $\check{w}_{t+1} = \tilde{w}_{t+1}$. Equations (28) and (29) implicitly define $\check{e}_{t+1}$ and $\check{w}_t$ as functions of $\check{c}_t$. Let these functions be denoted $\hat{e}_{t+1}(\cdot)$ and $\hat{w}_t(\cdot)$, respectively. We can use the implicit function theorem to compute the derivatives at $\check{c}_t = \tilde{c}_t$:

$$\hat{e}_{t+1}'(\tilde{c}_t) = \frac{v'(\tilde{c}_t)\big(v'((1-\delta)\tilde{b}_{t+1}) - v'(\tilde{c}_{t+1})\big)}{\delta\psi'\left(\hat{e}_{t+1}(\tilde{c}_t)\right) v'((1-\delta)\tilde{b}_{t+1})} \quad \text{and} \quad \hat{w}_t'(\tilde{c}_t) = 1 - \frac{v'(\tilde{c}_t)}{v'((1-\delta)\tilde{b}_{t+1})}.$$

Note that the original contract is obtained by setting $\check{c}_t = \tilde{c}_t$. If $\check{c}_t$ is changed from $\tilde{c}_t$ to $\tilde{c}_t+\varepsilon$, for some (positive or negative) ε small, the total effect on the continuation payoff of the principal at time t is $(-\hat{w}_t'(\tilde{c}_t) + \delta\hat{e}_{t+1}'(\tilde{c}_t))\varepsilon + o(\varepsilon)$. Hence, a necessary condition for optimality is that $-\hat{w}_t'(\tilde{c}_t) + \delta\hat{e}_{t+1}'(\tilde{c}_t) = 0$, which is equivalent the Euler equation (14).

The Euler equation implies that if $v'(\tilde{c}_{t+1}) = \psi'(\tilde{e}_{t+1})$ we have $\tilde{c}_t = \tilde{c}_{t+1}$. From Lemma A.10 we have that, if instead $v'(\tilde{c}_{t+1}) \neq \psi'(\tilde{e}_{t+1})$, then $v'(\tilde{c}_{t+1}) > \psi'(\tilde{e}_{t+1})$. In this second case, there are three possibilities:

1. If both sides of the Euler equation are strictly positive, then $\tilde{c}_t < \tilde{c}_{t+1} < (1-\delta)\tilde{b}_{t+1}$.
2. If both sides of the Euler equation are zero, then $\tilde{c}_t = \tilde{c}_{t+1} = (1-\delta)\tilde{b}_{t+1}$.
3. If both sides of the Euler equation are strictly negative, then $\tilde{c}_t > \tilde{c}_{t+1} > (1-\delta)\tilde{b}_{t+1}$.

Step 2: We now prove that if $\tilde{c}_t \leq (1-\delta)\tilde{b}_t$ then $\tilde{c}_s \leq \tilde{c}_{s+1} < (1-\delta)\tilde{b}_{s+1}$ for all $s \geq t$. Assume first that there is a period t such that $\tilde{c}_t \leq (1-\delta)\tilde{b}_t$. Hence, since $\tilde{e}_t = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}) > 0$ (recall Lemma A.10) we have $\tilde{b}_{t+1} > \tilde{b}_t$. This shows that each side of the Euler equation is strictly positive, i.e.

$$1 - \frac{v'((1-\delta)\,\tilde{b}_{t+1})}{v'\left(\tilde{c}_t\right)} = \frac{v'\left(\tilde{c}_{t+1}\right)}{\psi'\left(\tilde{e}_{t+1}\right)}\left(1 - \frac{v'((1-\delta)\,\tilde{b}_{t+1})}{v'\left(\tilde{c}_{t+1}\right)}\right) > 0.$$

Since $v'\left(\tilde{c}_{t+1}\right)/\psi'\left(\tilde{e}_{t+1}\right) \geq 1$ (from Lemma A.10), $(1-\delta)\tilde{b}_{t+1} > \tilde{c}_{t+1} \geq \tilde{c}_t$. The result then follows by induction.

Step 3: We prove that $\tilde{c}_t > (1-\delta)\tilde{b}_t$ for all $t > 1$; it then follows immediately from Step 1 that consumption is (weakly) decreasing in t. Assume then, for the sake of contradiction, that there is a $t' > 1$ such that $\tilde{c}_{t'} \leq (1-\delta)\tilde{b}_{t'}$. We will construct a self-enforceable contract that is strictly more profitable than the original, contradicting the optimality of the original.

We first make some preliminary observations. From Step 2, we have that $\tilde{c}_s \leq \tilde{c}_{s+1} < (1-\delta)\tilde{b}_{s+1}$ for all $s \geq t'$. Also, since effort is strictly positive at all times (from Lemma A.10), we have

$$\sum_{s=t'}^{\infty} \delta^{s-t'} v(\tilde{c}_s) > \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_{t'}).$$

Hence, there must be a period $s \geq t'$ where $\tilde{c}_{s+1} > \tilde{c}_{t'}$. Let t'' be the earliest such period, and note that it satisfies $\tilde{c}_{t''+1} > \tilde{c}_{t''}$. Additionally, we can observe that, for all t,

$$\tilde{b}_t + \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{w}_\tau = \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{c}_\tau. \tag{30}$$

If this is not the case (the right-hand side is strictly smaller), then, applying Equation (1) repeatedly, we have $\tilde{b}_t \to \infty$ and so the agent's constraint (AC_t^{ob}) must be violated for large t (given that $(\tilde{c}_t)_{t=1}^{\infty}$ is bounded, as the contract is feasible).

Now let us construct the more profitable contract for the principal, given our assumption that $\tilde{c}_{t'} \leq (1-\delta)\tilde{b}_{t'}$. We first construct a self-enforceable contract $\left(\tilde{e}_t^{new}, \tilde{c}_t^{new}, \tilde{w}_t^{new}, \tilde{b}_t^{new}\right)_{t\geq 1}$ in which the agent obtains a strictly higher payoff than in the original, while the principal obtains the same payoff. We then show how that contract can be further adjusted to obtain one which is strictly better for the principal. In the new contract that is better for the agent, we maintain $\tilde{w}_t^{new} = \tilde{w}_t$ and $\tilde{e}_t^{new} = \tilde{e}_t$ for all t, but specify a different agreed consumption sequence $\tilde{c}_t^{new}$ (and hence different balances $\tilde{b}_t^{new}$).

The change in agent consumption is to specify *constant* consumption $\bar{c}$ in each period from t'' onwards, where

$$\bar{c} = (1-\delta) \sum_{\tau=t''}^{\infty} \delta^{\tau-t''} \tilde{c}_\tau. \tag{31}$$

That is, $\tilde{c}_t^{new} = \bar{c}$ for all $t \geq t''$, while $\tilde{c}_t^{new} = \tilde{c}_t$ for $t < t''$. Notice that, $\bar{c} < (1-\delta)\sum_{\tau=t}^{\infty} \delta^{\tau-t}\tilde{c}_\tau$ for all $t > t''$.

Balances are determined recursively by Equation (1). That is, they are given by $\tilde{b}_t^{new} = \tilde{b}_t$ for $t \leq t''$, and by

$$\tilde{b}_t^{new} = \delta^{t''-t}\tilde{b}_{t''} + \sum_{\tau=t''}^{t-1} \delta^{\tau-t} \left(\tilde{w}_\tau - \bar{c}\right)$$

for all $t > t''$. Observe then that

$$\begin{aligned}
\tilde{b}_t^{new} + \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{w}_\tau &= \delta^{t''-t}\tilde{b}_{t''} + \sum_{\tau=t''}^{\infty} \delta^{\tau-t} \tilde{w}_\tau - \sum_{\tau=t''}^{t-1} \delta^{\tau-t} \bar{c} \\
&= \sum_{\tau=t''}^{\infty} \delta^{\tau-t} \tilde{c}_\tau - \sum_{\tau=t''}^{t-1} \delta^{\tau-t} \bar{c} \\
&= \frac{\bar{c}}{1-\delta},
\end{aligned}$$

where the second equality uses Equation (30) and the third equality uses Equation (31). Therefore, for all $t > t''$,

$$\tilde{b}_t^{new} + \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{w}_\tau = \frac{\bar{c}}{1-\delta} < \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{c}_\tau = \tilde{b}_t + \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{w}_\tau,$$

where the second equality follows from Equation (30). This implies that $\tilde{b}_t^{new} < \tilde{b}_t$ for all $t > t''$.

Now, we want to show that the contract $\left(\tilde{e}_t^{new}, \tilde{c}_t^{new}, \tilde{w}_t^{new}, \tilde{b}_t^{new}\right)_{t\geq 1}$ is self-enforceable. Because effort and payments are unchanged relative to the original contract, the principal's constraints (PC_t) remain intact. Consider then the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) for each period $t \geq 1$. For all $t \leq t''$, the agent anticipates a strictly higher continuation payoff under the new contract, i.e.

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t} \left(v\left(\tilde{c}_\tau^{new}\right) - \psi\left(\tilde{e}_\tau^{new}\right)\right) > \sum_{\tau=t}^{\infty} \delta^{s-t} \left(v\left(\tilde{c}_\tau\right) - \psi\left(\tilde{e}_\tau\right)\right).$$

The strict inequality is immediate from the strict concavity of v, and because consumption from date t'' onwards is constant in the new contract, but the NPV of this consumption is the same as in the original. Since, in addition, $v(\tilde{b}_t^{new}(1-\delta)) = v(\tilde{b}_t(1-\delta))$, the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied at dates $t \leq t''$ as strict inequalities.

To understand how the agent's constraints change at each $t > t''$, define

$$\bar{c}^{(t)} \equiv (1-\delta) \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{c}_\tau.$$

Consider the original contract, and suppose that the agent's consumption is changed from date t onwards, being set equal to $\bar{c}^{(t)}$ in all such periods. The agent's payoff increases from the smoothing of consumption, and so

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t}(v(\bar{c}^{(t)}) - \psi(\tilde{e}_\tau)) \geq \sum_{\tau=t}^{\infty} \delta^{\tau-t} \left(v\left(\tilde{c}_\tau\right) - \psi\left(\tilde{e}_\tau\right)\right) \geq \frac{v(\tilde{b}_t(1-\delta))}{1-\delta}, \tag{32}$$

where the second inequality follows because the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied in the original contract.

Because ψ is non-negative, the inequalities in Equation (32) imply $\bar{c}^{(t)} \geq \tilde{b}_t(1-\delta)$. Therefore, since v is concave, we have

$$v(\bar{c}^{(t)}) - v(\bar{c}^{(t)} - (1-\delta)(\tilde{b}_t - \tilde{b}_t^{new})) \leq v(\tilde{b}_t(1-\delta)) - v(\tilde{b}_t(1-\delta) - (1-\delta)(\tilde{b}_t - \tilde{b}_t^{new})). \tag{33}$$

Note that $\bar{c} = \bar{c}^{(t)} - (1-\delta)\left(\tilde{b}_t - \tilde{b}_t^{new}\right)$. Combining Equations (32) and (33), we therefore have that, for all $t > t''$,

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t} \left(v\left(\bar{c}\right) - \psi\left(\tilde{e}_\tau\right)\right) \geq \frac{v(\tilde{b}_t^{new}(1-\delta))}{1-\delta}.$$

This shows that, for the contract $\left(\tilde{e}_t^{new}, \tilde{c}_t^{new}, \tilde{w}_t^{new}, \tilde{b}_t^{new}\right)_{t\geq 1}$, the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied also at dates $t > t''$.

We have thus shown that $\left(\tilde{e}_t^{new}, \tilde{c}_t^{new}, \tilde{w}_t^{new}, \tilde{b}_t^{new}\right)_{t\geq 1}$ is a self-enforceable contract (in particular, it satisfies all the constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) and ($\mathrm{PC}_t$). Moreover, we saw that the constraints

(AC_t^{ob}) are satisfied strictly at all $t \leq t''$. We can therefore further adjust the contract by raising effort at date t'' by a small amount $\varepsilon > 0$ such that, without any other changes to the contract, all the agent's constraints (AC_t^{ob}) remain intact. The adjusted contract then satisfies all the constraints (AC_t^{ob}) and (PC_t), and the principal obtains a strictly higher payoff than in the original contract, contradicting the optimality of the original. □

We can then provide the key result that balances decrease over time towards a limit point $\tilde{b}_\infty$.

Lemma A.12. *In any optimal contract, $(\tilde{b}_t)_{t\geq 1}$ is a weakly decreasing sequence. It is constant if it attains the first-best payoff, and strictly decreasing towards some $\tilde{b}_\infty > 0$ otherwise. Also, $V(\tilde{b}_\infty) = V^{FB}(\tilde{b}_\infty)$.*

Proof. **Step 0.** If the first-best payoff is achievable at b_1, then equilibrium consumption and effort is uniquely determined by the conditions in Proposition 3.1. Because we restrict attention to payments timed to satisfy Equation (8), the balance is constant as claimed in the statement of this lemma (and explained in the main text). Suppose from now on that $V(b_1) < V^{FB}(b_1)$.

Step 1. Proof that $(\tilde{b}_t)_{t\geq 1}$ is weakly decreasing. Consider an optimal contract $\big(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t\big)_{t\geq 1}$. To show that the balance $\tilde{b}_t$ is weakly decreasing, we suppose for a contradiction that $\tilde{b}_{\hat{t}+1} > \tilde{b}_{\hat{t}}$ for some date $\hat{t}$. We construct a self-enforceable contract that achieves strictly higher profits for the principal.

Step 1a. First, denote a new contract by $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$, which we will choose to coincide with the original contract until $\hat{t}-1$, and with $\tilde{e}'_{\hat{t}} = \tilde{e}_{\hat{t}}$. For dates $t \geq \hat{t}$, let

$$\begin{aligned}\tilde{c}'_t = \bar{c} &\equiv (1-\delta)\sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}\tilde{w}_\tau + (1-\delta)\tilde{b}_{\hat{t}} \\ &= (1-\delta)\sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}\tilde{c}_\tau\end{aligned}$$

where the last equality is for the same reason as Equation (30). For dates $t \geq \hat{t}+1$, let $\tilde{e}'_t = \bar{e}$, where $\bar{e}$ is defined by

$$\psi(\bar{e}) = (1-\delta)\sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}-1}\psi(\tilde{e}_\tau).$$

Let also, for all $t \geq \hat{t}$, $\tilde{w}'_t = \bar{w}$, where $\bar{w} = (1-\delta)\sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}\tilde{w}_\tau$. Thus, we must have $\tilde{b}'_t = \bar{b} \equiv \tilde{b}_{\hat{t}}$ for all $t \geq \hat{t}$.

Step 1b. We now want to show that, for the new contract, the agent's constraint (AC_t^{ob}) is satisfied at all dates. Note that the new contract is stationary from date $\hat{t}+1$ onwards. Let's then consider the agent's constraint for these dates. Note first that, by the previous lemma, we must have $\tilde{c}_{\hat{t}} \geq \bar{c}$. Therefore,

$$\sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}-1} \bar{c} \geq \sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}-1} \tilde{c}_\tau.$$

Also, the NPV of disutility of effort from date $\hat{t}+1$ onwards is the same for both the original contract and the new contract. The fact that the original contract satisfies the agent's constraint (AC_t^{ob}) at date $\hat{t}+1$, plus the observation that $\bar{b} < b_{\hat{t}+1}$, then implies

$$\sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}-1} v(\bar{c}) - \sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}-1} \psi(\bar{e}) > \frac{v\left((1-\delta)\,\bar{b}\right)}{1-\delta}, \tag{34}$$

which means that the agent's constraint is satisfied as a *strict inequality* from $\hat{t}+1$ onwards.

Note then that

$$\sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} v(\bar{c}) \geq \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} v(\tilde{c}_\tau)$$

(with a strict inequality if the consumption levels $\tilde{c}_\tau$ for $\tau \geq \hat{t}$ are non-constant). Also, the NPV of the disutility of effort is the same from $\hat{t}$ onwards under both policies. Therefore, the agent's constraint continues to be satisfied at $\hat{t}$, and by the same logic all earlier periods.

Step 1c. Now we show that the principal's constraint (PC_t) is satisfied in all periods. Because the NPV of disutility of effort from date $\hat{t}+1$ onwards is the same under both contracts; and because ψ is convex, we have $\bar{e} \geq (1-\delta) \sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}-1} \tilde{e}_\tau$. Therefore,

$$\begin{aligned} \sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}} \tilde{e}'_\tau - \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} \tilde{w}'_\tau &= \frac{\delta \bar{e}}{1-\delta} - \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} \tilde{w}_\tau \\ &\geq \sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}} \tilde{e}_\tau - \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} \tilde{w}_\tau \\ &\geq 0 \end{aligned} \tag{35}$$

where the second inequality holds because the principal's constraint is satisfied at date $\hat{t}$ under the original contract. Hence the principal's constraint is satisfied under the new contract at date $\hat{t}$. Because $\tilde{e}'_t$ is constant for $t \geq \hat{t}+1$, and because $\tilde{w}'_t$ is constant for $t \geq \hat{t}$, the same inequality implies the satisfaction of the principal's constraint also from $\hat{t}+1$ onwards. Checking that the principal's constraint is satisfied also at dates before $\hat{t}$ follows the same logic. For $t < \hat{t}$, the principal's constraint is

$$\sum_{\tau=t+1}^{\hat{t}} \delta^{\tau-t} \tilde{e}'_\tau - \sum_{\tau=t}^{\hat{t}-1} \delta^{\tau-t} \tilde{w}'_\tau + \delta^{\hat{t}-t} \left(\sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}} \tilde{e}'_\tau - \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} \tilde{w}'_\tau \right) \geq 0,$$

which is satisfied because (i) $\tilde{e}'_\tau = \tilde{e}_\tau$ for $\tau \leq \hat{t}$, and $\tilde{w}'_\tau = \tilde{w}_\tau$ for $\tau < \hat{t}$, (ii) the first inequality in Equation (35) holds, and (iii) the principal's constraint is satisfied at date t under the original policy.

Step 1d. Finally, we show that the contract can be further (slightly) adjusted to a self-enforceable contract with a strictly higher payoff for the principal. The original contract was taken to satisfy

$$v\left(\tilde{c}_{\hat{t}}\right) - \psi\left(\tilde{e}_{\hat{t}}\right) = \frac{v((1-\delta)\,\bar{b}) - \delta v((1-\delta)\,\tilde{b}_{\hat{t}+1})}{1-\delta} < v\left((1-\delta)\,\bar{b}\right).$$

Hence,

$$\psi\left(\tilde{e}_{\hat{t}}\right) > v\left(\tilde{c}_{\hat{t}}\right) - v\left((1-\delta)\,\bar{b}\right) \geq v\left(\bar{c}\right) - v\left((1-\delta)\,\bar{b}\right) > \psi\left(\bar{e}\right)$$

where the final inequality follows from (34). Hence $\tilde{e}_{\hat{t}} > \bar{e}$. Recall that $\tilde{e}'_{\hat{t}} = \tilde{e}_{\hat{t}}$, and $\tilde{e}'_\tau = \bar{e}$ for $\tau > \hat{t}$; so we have $\tilde{e}'_{\hat{t}} > \tilde{e}'_\tau$ for all $\tau > \hat{t}$.

Now, pick $\tilde{e}''_{\hat{t}}$ and $\tilde{e}''_{\hat{t}+1}$, with

$$\tilde{e}'_{\hat{t}+1} < \tilde{e}''_{\hat{t}+1} < \tilde{e}''_{\hat{t}} < \tilde{e}'_{\hat{t}}$$

and such that

$$\psi\left(\tilde{e}''_{\hat{t}}\right) + \frac{\delta}{1-\delta}\psi\left(\tilde{e}''_{\hat{t}+1}\right) = \psi\left(\tilde{e}'_{\hat{t}}\right) + \frac{\delta}{1-\delta}\psi\left(\tilde{e}'_{\hat{t}+1}\right).$$

Substitute $\tilde{e}''_{\hat{t}}$ for $\tilde{e}'_{\hat{t}}$ and $\tilde{e}''_{\hat{t}+1}$ for $\tilde{e}'_\tau$, for all $\tau \geq \hat{t}+1$, in the contract defined in Step 1a. The agent's value from remaining in the contract from $\hat{t}$ onwards remains unchanged, so the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) remains satisfied at $\hat{t}$, and at all earlier dates. Note that, due to (34), the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) at dates $\hat{t}+1$ onwards are slack under the contract defined in Step 1a, and hence continue to be satisfied under the contract with the further modification, provided the adjustment in effort is small. Moreover, because ψ is strictly convex, the NPV of effort from date $\hat{t}$ onwards increases; so the principal's payoff strictly increases. Also, the principal's constraints (PC_t) clearly continue to be satisfied. Thus, we have constructed a self-enforceable contract that is strictly more profitable for the principal than the original, which completes Step 1.

Step 2. Proof that if $V(\tilde{b}_1) < V^{FB}(\tilde{b}_1)$ then $(\tilde{b}_t)_{t\geq 1}$ is a strictly decreasing sequence.

Step 2a. Consider an optimal contract. We first prove that if $\tilde{b}_{\hat{t}} = \tilde{b}_{\hat{t}+1}$ for some $\hat{t} \geq 1$, then $V(\tilde{b}_{\hat{t}}) = V^{FB}(\tilde{b}_{\hat{t}})$. To do this, note that if $\tilde{b}_{\hat{t}} = \tilde{b}_{\hat{t}+1}$ for some $\hat{t}$, then it is optimal to specify $\tilde{c}_\tau = \tilde{c}_{\hat{t}}$, $\tilde{w}_\tau = \tilde{w}_{\hat{t}}$, and $\tilde{e}_\tau = \tilde{e}_{\hat{t}}$ for all $\tau > \hat{t}$; that is, it must be optimal for the contract to be stationary from period $\hat{t}$ onwards. The Euler equation (14) then requires that $\psi'(\tilde{e}_\tau) = v'(\tilde{c}_\tau)$ for all $\tau \geq \hat{t}+1$,[18] and by stationarity also $\psi'(\tilde{e}_{\hat{t}}) = v'(\tilde{c}_{\hat{t}})$. Then, $\tilde{e}_\tau$ and $\tilde{c}_\tau$ satisfy, for all $\tau \geq \hat{t}$,

[18]To see this, recall from Lemma A.11 that $\tilde{c}_\tau > \tilde{b}_\tau$.

the first-order and agent's indifference conditions in Proposition 3.1, given initial balance $\tilde{b}_{\hat{t}}$. Therefore they are the first-best effort and consumption given balance $\tilde{b}_{\hat{t}}$. This shows that $V(\tilde{b}_{\hat{t}}) = V^{FB}(\tilde{b}_{\hat{t}})$, as desired.

Step 2b. Now we consider any optimal contract, and show the following. If $V(\tilde{b}_1) < V^{FB}(\tilde{b}_1)$, then $V(\tilde{b}_t) < V^{FB}(\tilde{b}_t)$ for all $t \geq 1$, and in addition, $(\tilde{b}_t)_{t\geq 1}$ is strictly decreasing.

Suppose that, for some $\hat{t}$, $V(\tilde{b}_{\hat{t}}) < V^{FB}(\tilde{b}_{\hat{t}})$, which by Step 1 and Step 2a implies $\tilde{b}_{\hat{t}+1} < \tilde{b}_{\hat{t}}$. The result will follow by induction if we can show that $V(\tilde{b}_{\hat{t}+1}) < V^{FB}(\tilde{b}_{\hat{t}+1})$. Hence, suppose for a contradiction that the contract achieves the first-best continuation payoff for the principal at date $\hat{t}+1$, given the balance is $\tilde{b}_{\hat{t}+1}$ (that is, suppose $V(\tilde{b}_{\hat{t}+1}) = V^{FB}(\tilde{b}_{\hat{t}+1})$). This implies that $\tilde{e}_\tau = e^{FB}(\tilde{b}_{\hat{t}+1})$ and $\tilde{c}_\tau = c^{FB}(\tilde{b}_{\hat{t}+1})$ for all $\tau > \hat{t}$. By assumption that Equation (8) holds in all periods, we then have $\tilde{b}_\tau = \tilde{b}_{\hat{t}+1}$ for all $\tau > \hat{t}+1$. Hence, the contract is stationary from $\hat{t}+1$ onwards; in particular, the payment is constant at $\tilde{w}_\tau = \bar{w}$ for $\tau \geq \hat{t}+1$, for some value $\bar{w}$.

From the Euler equation (14) and the fact that $v'(\tilde{c}_{\hat{t}+1}) = \psi'(\tilde{e}_{\hat{t}+1})$, we have $\tilde{c}_{\hat{t}} = \tilde{c}_{\hat{t}+1}$. Hence, using $\tilde{b}_{\hat{t}+2} = \tilde{b}_{\hat{t}+1} < \tilde{b}_{\hat{t}}$, we have (using $(\mathrm{FP}_t^{\mathrm{ob}})$)

$$\begin{aligned}\psi(\tilde{e}_{\hat{t}}) &= v(\tilde{c}_{\hat{t}}) + \tfrac{\delta}{1-\delta}v((1-\delta)\tilde{b}_{\hat{t}+1}) - \tfrac{1}{1-\delta}v((1-\delta)\tilde{b}_{\hat{t}}) \\ &< v(\tilde{c}_{\hat{t}+1}) + \tfrac{\delta}{1-\delta}v((1-\delta)\tilde{b}_{\hat{t}+2}) - \tfrac{1}{1-\delta}v((1-\delta)\tilde{b}_{\hat{t}+1}) = \psi(\tilde{e}_{\hat{t}+1}).\end{aligned}$$

Consequently, $\tilde{e}_{\hat{t}} < \tilde{e}_{\hat{t}+1}$, and so $\frac{\psi'(\tilde{e}_{\hat{t}})}{v'(\tilde{c}_{\hat{t}})} < \frac{\psi'(\tilde{e}_{\hat{t}+1})}{v'(\tilde{c}_{\hat{t}+1})} = 1$. We then know (from Lemma A.10) that the principal's constraint (PC_t) binds at $\hat{t}$, and so

$$\sum_{s=\hat{t}+1}^{\infty} \delta^{s-\hat{t}}\tilde{e}_s = \sum_{s=\hat{t}}^{\infty} \delta^{s-\hat{t}}\tilde{w}_s = \sum_{s=\hat{t}}^{\infty} \delta^{s-\hat{t}}\tilde{c}_s - \tilde{b}_{\hat{t}},$$

where the second equality follows for the same reason as for Equation (30). Using that $\tilde{e}_\tau = e^{FB}(\tilde{b}_{\hat{t}+1})$ for all $\tau \geq \hat{t}+1$, and $\tilde{c}_\tau = c^{FB}(\tilde{b}_{\hat{t}+1})$ for all $\tau \geq \hat{t}$, we have

$$\delta e^{FB}(\tilde{b}_{\hat{t}+1}) = c^{FB}(\tilde{b}_{\hat{t}+1}) - (1-\delta)\tilde{b}_{\hat{t}} < c^{FB}(\tilde{b}_{\hat{t}+1}) - (1-\delta)\tilde{b}_{\hat{t}+1} = \bar{w} = \tilde{w}_{\hat{t}+1}.$$

That $\delta e^{FB}(\tilde{b}_{\hat{t}+1}) < \tilde{w}_{\hat{t}+1}$ means the principal's constraint (PC_t) in period $\hat{t}+1$ (as well as at future dates) is violated, so we reach our contradiction. This completes Step 2.

Step 3. Proof that $\tilde{b}_\infty > 0$ and $V(\tilde{b}_\infty) = V^{FB}(\tilde{b}_\infty)$.

Consider an optimal contract, and suppose that $V(\tilde{b}_1) < V^{FB}(\tilde{b}_1)$. Then $(\tilde{b}_t)_{t\geq 1}$ is a strictly decreasing sequence, as we saw in the previous step. By Lemma 5.1, $\tilde{b}_t > 0$ for all t, so the limit $\lim_{t\to\infty}\tilde{b}_t$ exists and is non-negative. We want to show this limit, call it $\tilde{b}_\infty$, is strictly positive and $V(\tilde{b}_\infty) = V^{FB}(\tilde{b}_\infty)$.

Step 3a. We first show that the function V is continuous at any $b > 0$. Suppose, for the sake of contradiction, that there is a point of discontinuity $\check{b} > 0$. Then there is $\varepsilon > 0$ and a sequence $(b_n)_{n=1}^{\infty}$ convergent to $\check{b}$ with $|V(b_n) - V(\check{b})| \geq \varepsilon$ for all n. Denote $\check{c}$ and $\check{b}'$ the optimal consumption and next-period balance when the balance is $\check{b}$. Then (given our restriction to "fastest payments"), present-period effort is given by $\hat{e}(\check{c}, \check{b}, \check{b}')$. The present-period payment is $\check{w} = \check{c} + \delta\check{b}' - \check{b}$.

Suppose now there is a subsequence (b_{n_k}) along which $V(b_{n_k}) \leq V(\check{b}) - \varepsilon$ for all k. If there is no such subsequence, then there is a subsequence (b_{n_k}) for which $V(\check{b}) \leq V(b_{n_k}) - \varepsilon$; the argument will then be symmetric, and hence is omitted. Consider then a balance b_{n_k}, and choose present-period consumption equal to $c_{n_k} \equiv \check{c} + b_{n_k} - \check{b}$, and next-period balance equal to $b'_{n_k} \equiv \check{b}'$. Note this implies present-period effort is $e_{n_k} \equiv \hat{e}(c_{n_k}, b_{n_k}, b'_{n_k})$, and the payment is $w_{n_k} \equiv c_{n_k} + \delta\check{b}' - b_{n_k} = \check{w}$. Because the principal's next-period continuation payoff is $V(\check{b}')$, the same as in an optimal contract following balance $\check{b}$, and because the payment is the same (i.e., $\check{w}$), the principal's constraint in Equation (26) is satisfied. By continuity of $\hat{e}(\cdot,\cdot,\cdot)$, for large enough k, we have $V(b_{n_k}) > V(\check{b}) - \varepsilon$. This contradicts the assumption that $V(b_{n_k}) \leq V(\check{b}) - \varepsilon$.

Step 3b. We now prove that $\tilde{b}_\infty > 0$. For this, we first show $\lim_{b\searrow 0} \frac{c^{FB}(b) - (1-\delta)b}{e^{FB}(b)} = 0$ and so, by Proposition 5.2, there exists some $\bar{b} > 0$ such that an optimal contract achieves the first-best payoff of the principal for all $b \leq \bar{b}$. This follows after noting that $v(c^{FB}(b)) - v((1-\delta)b) = \psi(e^{FB}(b)) > 0$, so we have that either $\lim_{b\searrow 0} c^{FB}(b) = 0$ or $\lim_{b\searrow 0} e^{FB}(b) = +\infty$. Since $\psi'(e^{FB}(b)) = v'(c^{FB}(b))$ we have, in fact, that both $\lim_{b\searrow 0} c^{FB}(b) = 0$ and $\lim_{b\searrow 0} e^{FB}(b) = +\infty$, which establishes the result. Next, recall from Step 2 that, given $V(b_1) < V^{FB}(b_1)$, the sequence $(\tilde{b}_t)_{t\geq 1}$ of balances in the optimal contract is strictly decreasing and such that $V(\tilde{b}_t) < V^{FB}(\tilde{b}_t)$ for all t. That is, $\tilde{b}_t$ remains above $\bar{b}$, and so converges to some value $\tilde{b}_\infty \geq \bar{b}$.

Step 3c. Finally, we prove that $V(\tilde{b}_\infty) = V^{FB}(\tilde{b}_\infty)$. Recall we assumed that $V(b_1) < V^{FB}(b_1)$. By the continuity of V established in Step 3a, we have that $\lim_{t\to\infty} V(\tilde{b}_t) = V(\tilde{b}_\infty)$. Because the principal's constraint (PC_t) binds for all t, we have $V(\tilde{b}_t) = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1})$ for all t. By continuity of $\hat{e}(\cdot,\cdot,\cdot)$, we have $\lim_{t\to\infty} \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}) = \hat{e}(\tilde{c}_\infty, \tilde{b}_\infty, \tilde{b}_\infty)$, where $\tilde{c}_\infty \equiv \lim_{t\to\infty} \tilde{c}_t$, which exists because $\tilde{c}_t$ is decreasing and remains above $\tilde{b}_\infty$ by Lemma A.11. Therefore,

$$V(\tilde{b}_\infty) = \hat{e}(\tilde{c}_\infty, \tilde{b}_\infty, \tilde{b}_\infty) = \psi^{-1}(v(\tilde{c}_\infty) - v((1-\delta)\tilde{b}_\infty)).$$

Since $V(\tilde{b}_\infty) > 0$ (recall Lemma A.10), $\tilde{c}_\infty > (1-\delta)\tilde{b}_\infty$. Therefore, the Euler equation (14) implies that, necessarily, $\lim_{t\to\infty} \frac{v'(\tilde{c}_{t+1})}{\psi'(\tilde{e}_{t+1})} = 1$, and therefore $\tilde{e}_\infty \equiv \lim_{t\to\infty} \tilde{e}_t$ exists. It is then clear that both Conditions 1 and 2 of Proposition 3.1 hold for $\tilde{e}_\infty$, $\tilde{c}_\infty$, and $\tilde{b}_\infty$ (instead of $e^{FB}(b_1)$, $c^{FB}(b_1)$, and b_1). This establishes the result. □

We now translate the above result into implications for the dynamics of the principal's continuation payoff.

Lemma A.13. *Assume* $V(b_1) < V^{FB}(b_1)$*. Then* $(V(\tilde{b}_t))_{t\geq 1}$ *is a strictly increasing sequence.*

Proof. Recall from Lemma A.12 we have that, if $V(b_1) < V^{FB}(b_1)$, then $(\tilde{b}_t)_{t\geq 1}$ is strictly decreasing. Therefore, the result will follow if we can show $V(\cdot)$ is strictly decreasing.

Step 1. We show that if $V(\cdot)$ fails to be strictly decreasing, then there exists a value $b^* > 0$ such that, for every $\varepsilon > 0$, there is a $\check{b} \in (b^* - \varepsilon, b^*)$ satisfying $V(\check{b}) \leq V(b^*)$.

First, by Step 3a of the proof of the previous lemma, $V(\cdot)$ is continuous on strictly positive values. Suppose $V(\cdot)$ fails to be strictly decreasing, which means that there are values b', b'' with $0 < b' < b''$, and with $V(b') \leq V(b'')$. Consider maximizing V on $[b', b'']$, and note that a maximum exists by continuity of V. Because $V(b') \leq V(b'')$, there is at least one maximizer in $(b', b'']$. We can take any such value to be b^*.

Step 2. Consider an optimal continuation contract when $\tilde{b}_t = b^*$, and consider a change to $\tilde{b}_t = b^* - \nu$ for $\nu > 0$ small enough such that $V(b^* - \nu) \leq V(b^*)$. Then we can reduce $\tilde{c}_t$ by the same amount ν, holding $\tilde{b}_{t+1}$ and $\tilde{w}_t$, as well as all other variables, constant. Note then that, provided ν is small enough,

$$v(\tilde{c}_t - \nu) - \tfrac{1}{1-\delta} v((1-\delta)(\tilde{b}_t - \nu)) > v(\tilde{c}_t) - \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t),$$

which follows again because $\tilde{c}_t > (1-\delta)\tilde{b}_t$ (by Lemma A.11) and by concavity of v. Hence, we have

$$\hat{e}(\tilde{c}_t - \nu, \tilde{b}_t - \nu, \tilde{b}_{t+1}) > \tilde{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}).$$

By construction, the agent remains indifferent to continuing in the contract at all dates (we have that (AC_t^{ob}) holds as an equality at all dates). The continuation of the relationship from $t+1$ onwards is precisely as before, and therefore the principal's constraint at date t is satisfied (since $\tilde{w}_t$ is unchanged). Hence,

$$V(\tilde{b}_t) = \tilde{e}_t - \tilde{w}_t + \delta V(\tilde{b}_{t+1}) < \hat{e}(\tilde{c}_t - \nu, \tilde{b}_t - \nu, \tilde{b}_{t+1}) - \tilde{w}_t + \delta V(\tilde{b}_{t+1}) \leq V(\tilde{b}_t - \nu).$$

However, this contradicts $V(b^* - \nu) \leq V(b^*)$. □

We now show that, if the first-best outcome is not attainable in a self-enforceable relational contract, effort is always downward distorted.

Lemma A.14. *Assume* $V(b_1) < V^{FB}(b_1)$*. Then, in any optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, $v'(\tilde{c}_t) > \psi'(\tilde{e}_t)$ *for all* t*.*

Proof. We first show that, if there is a date $\check{t}$ with $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$, then $v'(\tilde{c}_{\check{t}+1}) = \psi'(\tilde{e}_{\check{t}+1})$. To do so, assume for a contradiction that $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$ and $v'(\tilde{c}_{\check{t}+1}) \neq \psi'(\tilde{e}_{\check{t}+1})$ for some date $\check{t}$.

Then, by Lemma A.10, we have $v'(\tilde{c}_{\check{t}+1}) > \psi'(\tilde{e}_{\check{t}+1})$, and therefore $\tilde{w}_{\check{t}+1} = \delta V(\tilde{b}_{\check{t}+2})$. In turn, this implies

$$\tilde{e}_{\check{t}+1} = \tilde{e}_{\check{t}+1} - \tilde{w}_{\check{t}+1} + \delta V(\tilde{b}_{\check{t}+2}) = V(\tilde{b}_{\check{t}+1}) > V(\tilde{b}_{\check{t}}) = \tilde{e}_{\check{t}} - \tilde{w}_{\check{t}} + \delta V(\tilde{b}_{\check{t}+1}) \geq \tilde{e}_{\check{t}}$$

where the strict inequality follows by the previous lemma, and the weak inequality follows because the principal's constraint is satisfied in an optimal contract at $\check{t}$.

There are two cases: either $\tilde{w}_{\check{t}} < \sum_{s=\check{t}+1}^{\infty} \delta^{s-\check{t}} (\tilde{e}_s - \tilde{w}_s)$ or $\tilde{w}_{\check{t}} = \sum_{s=\check{t}+1}^{\infty} \delta^{s-\check{t}} (\tilde{e}_s - \tilde{w}_s)$. Consider the first. Define a new contract $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$, which is identical to the original, except that $\tilde{e}'_{\check{t}} = \tilde{e}_{\check{t}} + \varepsilon$ and $\tilde{e}'_{\check{t}+1} = \tilde{e}_{\check{t}+1} - \nu(\varepsilon)$, with $\nu(\varepsilon)$ defined by

$$\psi(\tilde{e}_{\check{t}} + \varepsilon) + \delta\psi(\tilde{e}_{\check{t}+1} - \nu(\varepsilon)) = \psi(\tilde{e}_{\check{t}}) + \delta\psi(\tilde{e}_{\check{t}+1}).$$

Thus

$$\nu'(0) = \frac{\psi'(\tilde{e}_{\check{t}})}{\delta\psi'(\tilde{e}_{\check{t}+1})}$$

and so the change in the NPV of effort is

$$\varepsilon - \delta\nu(\varepsilon) = \left(1 - \frac{\psi'(\tilde{e}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}+1})}\right)\varepsilon + o(\varepsilon)$$

which is strictly positive for ε sufficiently small. It is easy to see that the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) is unchanged at all dates except $\check{t}+1$, when the constraint is relaxed. The principal's constraint (PC_t) is unchanged from date $\check{t}+1$ onwards, relaxed at date $\check{t}-1$ and earlier (because the NPV of effort increases), but is tightened at date $\check{t}$. Provided ε is small enough, the date-$\check{t}$ constraint remains intact. Profits increase, contradicting the optimality of the original contract.

Now suppose that $\tilde{w}_{\check{t}} = \sum_{s=\check{t}+1}^{\infty} \delta^{s-\check{t}} (\tilde{e}_s - \tilde{w}_s)$, and note the above adjustment now leads to a violation of the principal's constraint (PC_t) at date $\check{t}$. In this case, we reduce slightly the payment, effort and consumption at date $\check{t}$, keeping the agent's payoff unchanged, but ensuring the principal's constraint (PC_t) is satisfied. This has a negligible effect on profits since $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$. Hence, we again contradict the optimality of the original contract.

Now let us demonstrate precisely an adjustment that yields a self-enforceable contract. We further adjust the modified contract $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$ by reducing the date-$\check{t}$ payment and consumption by an amount $\gamma(\varepsilon)$, and reducing date-$\check{t}$ effort by an amount $\eta(\varepsilon)$ to leave agent payoffs unchanged. The date-$\check{t}$ principal constraint (PC_t) will then hold as equality by setting $\gamma(\varepsilon) = \delta\nu(\varepsilon)$. The requirement on $\eta(\varepsilon)$, that the agent's payoff is unaffected by the adjustment, is

$$v(\tilde{c}_{\check{t}} - \gamma(\varepsilon)) - \psi(\tilde{e}_{\check{t}} + \varepsilon - \eta(\varepsilon)) = v(\tilde{c}_{\check{t}}) - \psi(\tilde{e}_{\check{t}} + \varepsilon).$$

We then have

$$v'(\tilde{c}_{\check{t}})\gamma'(0) = \psi'(\tilde{e}_{\check{t}})\eta'(0).$$

Therefore, the overall increase in date-$\check{t}$ profits from all changes to the contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is

$$\varepsilon - \delta\nu(\varepsilon) - (\eta(\varepsilon) - \gamma(\varepsilon)) = \left(1 - \frac{\psi'(\tilde{e}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}+1})}\right)\varepsilon - \delta\nu'(0)\left(\frac{v'(\tilde{c}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}})} - 1\right)\varepsilon + o(\varepsilon)$$
$$= \left(1 - \frac{\psi'(\tilde{e}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}+1})}\right)\varepsilon + o(\varepsilon)$$

which is strictly positive for ε sufficiently small.

Hence, for small enough ε, the overall effect on profits of all changes to the original contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is positive, with the continuation profits from date $\check{t}$ increasing. The principal's constraint (PC_t) is relaxed at dates $\check{t}-1$ and earlier, it is satisfied by construction at $\check{t}$, and it is unchanged from date $\check{t}+1$ onwards. Again, the fact profits strictly increase contradicts the optimality of $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$.

From the above, and by induction, we have that $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$ at some $\check{t}$ implies $v'(\tilde{c}_t) = \psi'(\tilde{e}_t)$ for all $t \geq \check{t}$. By the Euler equation (14), consumption and effort remain constant from $\check{t}$ onwards. Moreover, the agent's indifference condition in Equation (8) is presumed to hold at all dates. This shows that the agent's balances $\tilde{b}_t$ remains constant from date $\check{t}$ onwards, which given the assumption $V(b_1) < V^{FB}(b_1)$, contradicts the finding of Lemma A.12 that balances strictly decrease. Hence, we cannot have $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$ at any $\check{t}$. □

Lemma A.14 implies by Lemma A.10 that, if $V(b_1) < V^{FB}(b_1)$, the principal's constraint (PC_t) holds with equality in every period. We use this to show the following.

Lemma A.15. *If $V(b_1) < V^{FB}(b_1)$, then, in any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, effort $\tilde{e}_t$ and payments $\tilde{w}_t$ strictly increase over time, while consumption $\tilde{c}_t$ strictly declines over time.*

Proof. The fact that the principal's constraint binds at every date, as argued above, can be stated as $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$ for all t; hence payments are strictly increasing in t by Lemma A.13. We also have $V(\tilde{b}_t) = \tilde{e}_t$ for all t, so effort is strictly increasing as well.

Now consider consumption. By Lemma A.11, we know that $\tilde{c}_{t-1} \geq \tilde{c}_t$ for all $t \geq 2$. Hence, if consumption fails to be strictly decreasing, we must have $\tilde{c}_{t-1} = \tilde{c}_t$ for some t. We then have, by Equation (14) (and noting that $\tilde{c}_t > (1-\delta)\tilde{b}_t$, also by Lemma A.11), that $\psi'(\tilde{e}_t) = v'(\tilde{c}_t)$. However, this contradicts Lemma A.14. □

Lemma A.16. *An optimal contract exists.*

(End of the proof of Proposition 5.3.) □

Proof of Proposition 5.4. See Online Appendix.