Perceived Competition in Networks*

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Abstract

We consider an aggregative game in which agents have an imperfect knowledge about the set of agents they are in competition with. We model this lack of knowledge through a directed graph that we call *the perception network*. In this framework, a natural equilibrium concept emerges, the *Perception-Consistent Equilibrium* (PCE). At a PCE, each agent chooses an action level that maximizes her subjective perceived utility while the action levels of all individuals must be consistent. We prove the existence of PCEs in a large class of aggregative games. We also show that, at any PCE, the efforts are always ordered accordingly to some centrality measure in the perception network. For a specific subclass of aggregative games, we decompose the network into communities and completely characterize the PCEs by identifying which sets of agents are active, as well as their effort level. We prove that, at the unique stable PCE, the agents' action levels are proportional to their eigenvector centrality in the perception network. We illustrate our results with two well-known models: Tullock contest and Cournot competition.

Keywords: Aggregative games, competition, incomplete network knowledge, perception-consistent equilibrium, ordinal centrality, eigenvector centrality.

JEL Classification: C72, D85, Z13.

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1 Introduction

Competition is a central notion in economics. Firms compete for consumers and market shares, researchers compete for funds, etc. More generally, agents compete for scarce resources. The standard way economists have viewed competition is that it is *objective*: agents know exactly the set of agents they are in competition with. In this paper, we are interested in situations in which competition may be subjective or *perceived* rather than objective. Indeed, in the real world, agents may perceive other agents as competitors while the reverse is not necessarily true. Therefore, even though competition is *global*, in many cases, agents may only care about their *local* competitors.[1]

To model such situations, we study a standard aggregative game in which an agent's objective payoff depends on her own effort and on an aggregator of the effort of all agents in the economy. However, each agent only perceives a subset of agents as competitors and is not aware of the others. We model this through a *perception network*, which is naturally directed and, generally, only weakly connected. We assume that agents behave with respect to a *perceived utility function* in which the aggregate effort in the whole economy —which is unknown—is replaced by a shifter of the total effort *of the agents they are aware of.* The shifter therefore captures the *misperception intensity* of the aggregate effort. For example, if we consider the standard Tullock contest game, each agent believes that she competes for some (perceived) resource with the agents she is aware of, while, in reality, she competes for a larger resource with the whole set of agents in the economy. Similarly, in the standard Cournot oligopoly competition with homogenous products, each firm estimates that the product price emerges from the quantity produced by her perceived competitors while, in reality, the price is determined by the quantities produced by all firms in the network.

In this framework, a natural equilibrium concept emerges, the *Perception-Consistent Equilibrium* (PCE). A PCE captures both the agents' local sightedness—each agent chooses an action level that maximizes her *perceived* utility—and the fact that the action levels of all individuals in the network are consistent in equilibrium. Indeed, at a PCE, individual i's perceived subjective utility is equal to her objective payoff. First, we show that a PCE exists for a large class of aggregative games, which we call *competitive aggregative games*. We also show that, at any PCE, the agents' efforts represent an *ordinal centrality measure*, a novel notion introduced by Sadler (2021). Many well-known network centralities, such as degree, eigenvector and Katz-Bonacich centrality, represent ordinal centralities. Our result is quite general, since it holds whether or not the network is symmetric and whether or not it is a game with strategic complements.

[1] In the Online Appendix A, we provide some simple descriptive evidence of 700 street-food vendors in Kolkata (India) for whom competition is not always reciprocal and is perceived as local.

For the remainder of the paper, we focus on a subclass of competitive aggregative games by imposing additional structure on the payoff function. Two key applications that are part of this subclass are Cournot competition and the Tullock contest. Our second contribution is to show that, in this subclass of competitive aggregative games, perception-consistent equilibria provide a behavioral foundation of the *eigenvector centrality* measure. More precisely, we prove that in any weakly connected network and at any PCE, the effort level of each active agent is proportional to her *eigenvector centrality*. This result is very general and holds beyond strongly connected networks, for which eigenvector centrality is usually defined. Other papers have provided a microfoundation of eigenvector centrality. For example, Golub and Jackson (2010, 2012) develop models on DeGroot updating in which eigenvector centrality is the right way to characterize an agent's influence. However, this arises from a heuristic learning process rather than behavior in a game. Banerjee et al. (2013) provide a microfoundation of eigenvector centrality by showing that it is the limit of diffusion centrality.[2] Our model is different in the sense that it provides a behavioral foundation for eigenvector centrality measure based on an aggregative game and PCE. Moreover, in all these models, the network is assumed to be strongly connected and all agents exert strictly positive action in equilibrium. Our model solves for a more general framework in which the network is weakly connected. At our unique stable equilibrium, some agents may exert zero effort and the eigenvector centrality remains well defined.

Focusing on the subclass of competitive aggregative games allows us to provide some additional insights on the structure of the set of PCEs. We further explore the role of the network's architecture in determining who is active at a PCE. A typical PCE contains some agents whose action level is zero. The additional restrictions introduced on the payoff function allow us to decompose the network into communities and to tie these communities to the extensive margin at a PCE. A community is a strongly connected component of the network. Our third contribution, therefore, is to break down the network into communities where, in each community, all agents are either active or inactive. We show that there are typically multiple perception-consistent equilibria.

The multiplicity of perception-consistent equilibria is interesting as it underlines the behavioral richness of our equilibrium concept. Yet, multiplicity of equilibria may be seen as a hurdle, since it clouds predictions on which PCE may be played. To address this issue, we study the stability of the perception-consistent equilibria with respect to a very natural dynamic. At each period of time, agents best replies to their perceived utility function observed in the previous period and the efforts of their direct neighbors until an equilibrium is

[2]Some papers have also provided an axiomatic foundation of eigenvector centrality; see e.g., Palacios-Huerta and Volij (2004); Dequiedt and Zenou (2017); Bloch et al. (2019).

reached in which the perceived utility is equal to the objective utility. Our fourth contribution is to provide a very simple and intuitive characterization of the stable PCE. We show that a community is active at a *stable* PCE if it is "aware" of the largest and densest community in the whole network. As we will see, this implies that there is always a *unique* stable PCE. In this PCE, some agents are active, but typically not all of them.

Next, we conclude our study with some policy implications of our model. We first examine the impact of adding a directed link between two agents. We show that the agent, who is at the source of the link, is the one who obtains the highest benefit from the link addition. Further, we show that adding a link may decrease the number of active agents in the network. We then study the key-player policy (Ballester et al., 2010) and highlight another counter-intuitive result. By removing an agent in the network, we may make several inactive agents (for example, in terms of criminal effort) active. Finally, we show that, by merging two different connected networks (i.e., social mixing), the total activity is higher than the sum of total activity in each disconnected network.

Contribution to the literature

Our paper contributes to the games-on-network literature.[3] In many situations in which networks matter, agents make both binary decisions (*extensive margin*) and quantity decisions (*intensive margin*). Consider, for example, crime. First, an individual has to decide whether to become a criminal (active or not active); this is a binary decision (extensive margin). Then, if she becomes a criminal, she must decide how many crimes to commit (intensive margin). The literature on network games has mostly focused on the intensive margin by assuming that actions are continuous (Jackson and Zenou, 2015). There are, however, some papers that have considered network games with discrete actions (extensive margin); see, for example, Morris (2000), Brock and Durlauf (2001), and Leister et al. (2021). We believe that this is the one of the first papers[4] to consider both extensive and intensive margins. We show that the extensive margin (i.e., who is active in the network) is *community based*, that is, agents belonging to the same community are either all active or all inactive, and depends on the density of the community, whereas the intensive margin is *individual based* and solely determined by the position in the network: the higher is the individual eigenvector centrality, the higher is the effort of each agent.

As with our model, there are also network games that focus on imperfect information about the network and introduce new equilibrium concepts related to our PCE. In particu-

[3]For overviews, see Jackson (2008), Jackson and Zenou (2015), Bramoullé et al. (2016), and Jackson et al. (2017).

[4]Other papers (e.g., Calvó-Armengol and Zenou (2004); Bramoullé and Kranton (2007)) have considered both but without being able to provide a general characterization of the equilibria. Moreover, these models are usually plagued by multiple equilibria.

lar, McBride (2006), Lipnowski and Sadler (2019) and Battigalli et al. (2020) consider *self-confirming* and *peer-confirming equilibria*. Lipnowski and Sadler (2019) apply self-confirming equilibria (SCE) and rationalizable SCE to games where feedback about the actions of others is described by a network topology: agents observe only the actions of their peers (i.e., neighbors), but their payoffs may depend on everybody's actions and are not observed ex-post. The main difference with our PCE is that Lipnowski and Sadler (2019) allow agents to make conjecture about agents who are not their neighbors;[5] in our model, we assume that agents do not even know these agents exist. The peer-confirming equilbrium concept of Lipnowski and Sadler (2019) is such that adding links in the network restricts the set of permissible profiles/conjectures and thus the set of equilibria.[6] This is not true in our model.[7] Moreover, our concept of perception-consistent equilibrium (PCE) is equivalent to a specific kind of the self-confirming equilibrium (SCE) developed by Battigalli et al. (2020) when the game is written in such a way that the feedback agents receive is made explicit. In fact, our equilibrium concept (PCE) is a refinement of SCE whereby agents wrongly believe that they compete locally rather than globally.

Our equilibrium characterization in terms of communities also relates to other network models that partition agents into endogenous community structures, including risk sharing (Ambrus et al., 2014), interaction between market and community (Gagnon and Goyal, 2017), behavioral communities (Jackson and Storms, 2019), information resale and intermediation (Manea, 2021), and technology adoption (Leister et al., 2021). However, the driving forces and policy implications are very different. In particular, all these papers assume a perfect knowledge of the network and use standard equilibrium concepts.

Our paper also contributes to the literature on aggregative games (Jensen, 2018). In this literature, usually the network is not explicitly modeled and agents are assumed to know with certainty their competitors. Tullock contest game is one of our applications; thus we also contribute to the literature on conflicts,[8] especially the more recent literature on conflicts in networks.[9] In this literature, the structure of local conflicts is modeled as a network in which rivals invest in conflict-specific technology to attack their respective neighbors. This literature assumes that the network is undirected (which is a particular case of our network)

[5]Indeed, a strong assumption that is implicit in the definition of peer-confirming equilibrium in Lipnowski and Sadler (2019) is that players know the network structure.

[6]When the network is complete, the set of peer-confirming equilibria coincides with the set of Nash equilibria. For the empty network, peer-confirming equilibria coincide with rationalizable equilibria. Increasing the number of links reduces the number of equilibria. In contrast, the set of PCE may very well increase when links are added.

[7]McBride (2006) applies self-confirming equilibrium (SCE) to games of network formation with asymmetric information in which agents only observe the private information of other linked agents. We instead assume that the network is exogenous and that actions are continuous.

[8]See Jensen (2016) for a recent overview.

[9]For overviews of this literature, see Kovenock and Roberson (2012) and Dziubiński et al. (2016).

and that agents know the network, and solves the model using standard Nash equilibrium concept. Further, these studies usually do not provide a general characterization of all possible equilibria.

Finally, our model contributes to the general literature on competition in Industrial Organization (IO). We believe that this is the first model that introduces the concept of perceived competition in this literature and models it through a network.

2 The model

2.1 Aggregative games

Consider a finite set of agents, denoted by $N = \{1, 2, \cdots, n\}$. Agent $i \in N$ exerts effort $x_i \in \mathbb{R}_+$ and $\mathbf{x} := (x_1, x_2, \cdots, x_n)$ denotes the action profile. We consider an aggregative game (Jensen, 2018) in which every player i's payoff is a function of the player's own action x_i and the *aggregate* of all players' actions $X := \sum_{j \in N} x_j$. Let $\pi_i : \mathbb{R}^n_+ \to \mathbb{R}$ be agent i's payoff function. It is given by:[10]

$$\pi_i(\mathbf{x}) = h(x_i, X), \tag{2}$$

where $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a continuously differentiable map on $\mathbb{R}_+ \times \mathbb{R}_{++}$.

2.2 Perception Networks

We assume that agents are only partially aware of N, the *actual* set of agents they are in competition with. For instance, if $N = \{1, 2, 3\}$, agent 1 may be aware of agents 2 and 3, while agents 2 and 3 may only be aware of each other, but do not know the existence of agent 1. This assumption is captured by a *directed network* $(N, \mathbf{G})$, where $\mathbf{G}$ is an $n \times n$ adjacency matrix with entry $g_{ij} \in \{0, 1\}$ with $g_{ij} = 1$ if and only if agent i is aware of agent j. Since the network is directed $g_{ij} = 1$ does not necessary implies $g_{ji} = 1$. This network will be referred to as the *perception network* and the neigborhood of agent i, $\mathcal{N}_i := \{j \in N : g_{ij} = 1\}$ as the *perception set* of agent i or, equivalently, as i's *perceived neighborhood.*

Agent i is said to be connected to j through a path if there is a sequence $\{j_1, j_2, \cdots, j_m\} \subseteq N$ with $j_1 = i$, $j_m = j$ and such that $g_{j_\ell j_{\ell+1}} = 1$ for each $\ell \in \{1, ..., m-1\}$. We use the

[10] We could generalize our utility function to

$$\pi_i(\mathbf{x}) = h\left(x_i, g(\mathbf{x})\right), \text{ with } g(\mathbf{x}) = \sum_i \phi(x_i) \tag{1}$$

and our main result (Theorem 1) will still be obtained. Observe that in (1), the aggregator g needs to be additively separable and anonymous. Additive separability is the standard way of defining general aggregative games (Jensen, 2010; Acemoglu and Jensen, 2013). We must add "anonymity", which is very important to define the notion of perceived utility and thus PCE.

notation $i \rightrightarrows j$ to indicate that such a path exists between i and j. A directed network $(N, \mathbf{G})$ is *weakly connected* if the underlying undirected graph (i.e., ignoring the directions of edges) is connected. It is *strongly connected* if, for any pair of agents $i, j \in N$, there is a path from i to j (i.e., $i \rightrightarrows j$). Throughout the paper, we consider weakly connected networks satisfying the *no-isolation property*, that is, for each $i \in N$, $\mathcal{N}_i \neq \emptyset$ and there exists j such that $i \in \mathcal{N}_j$.

Remark 1. *All agents are ex-ante identical, i.e., $h_i(.) \equiv h(.)$ for all $i \in N$. Thus, the perception sets are the only source of agents' heterogeneity. This is because we want to examine how the individual's network position affects the effort and outcome of each agent. Note that the perception network is necessarily unweighted, that is, $g_{ij} \in \{0, 1\}$, where $g_{ij} = 1$ means that agent i perceives j a competitor while $g_{ij} = 0$ means that she does not perceive j as a competitor.*[11]

Agents correctly assume that the game is aggregative but only observe the aggregate effort in their perceived neighborhood. This discrepency between the actual aggregate effort and the perceived aggregate effort from agent i's viewpoint is captured by a *shifter* W_i. In other words, W_i represents the *misperception intensity of the aggregate effort* by agent i. Given an effort profile $\mathbf{x} \in \mathbb{R}^n_+$, we can write the *perceived utility* of agent i as[12]

$$u_i(x_i, \mathbf{x}_{-i}; W_i) = h\left(x_i, \frac{x_i + \sum_{j \in N} g_{ij} x_j}{W_i}\right). \tag{3}$$

From a dynamical viewpoint, the interpretation of this shifter is easier to understand because agents repeatedly adjust their effort with the local information they have. In Section 3.5, we describe in detail this dynamic process.

2.3 Two important applications

Our aggregative game with utility function (2) is relatively general and can be applied to many aggregative games such as Tullock contest games, Cournot competition, tournaments, teamwork games, etc. (Alos-Ferrer and Ania, 2005; Jensen, 2010). Here we provide two main applications: Tullock contest games and Cournot competition.

[11]This is an important remark because weakly connected networks are non-generic in the space of weighted directed graphs. Thus, the case for studying them so carefully rests on links being binary rather than weighted. In our case, it makes sense since the network only captures the "perception" that each agent has about agents. It is not a "physical" network such as roads, freeways, bridges or a financial network where links are bank loans.

[12]The perceived payoff is given by $h(x_i, +\infty)$ if $W_i = 0$ and $x_i + \sum_j g_{ij} x_j \neq 0$.

2.3.1 Linear (Tullock) contest games

There is a given resource, available in a fixed amount $V \in \mathbb{R}_+ = [0, \infty)$ to be shared between n agents. Each agent $i \in N$ exerts some effort $x_i \in \mathbb{R}_+$ at marginal cost $c > 0$ and her share of the resource V is given by the following *proportional rule*:

$$\pi_i(\mathbf{x}) = h(x_i, X) := \begin{cases} \frac{x_i}{X} V - c x_i & \text{if } X > 0, \\ \frac{V}{n} & \text{if } X = 0. \end{cases} \tag{4}$$

Equation (4) corresponds to the well-known "Tullock contest function" from the contest literature (Skaperdas, 1996; Kovenock and Roberson, 2012). One important difference is that we do not interpret $\frac{x_i}{X}$ as the *probability* of agent i getting V, but as the *fraction* of resource V that agent i can obtain, given her and the other agents' effort choices.

The actual resources V available in the economy as well as the total efforts of the n agents in the network are *not* observed by any agent in the network; agent i only observes $\sum_{j \in N} g_{ij} x_j$, and acts as if she were competing in a Tullock contest with resources VW_i to be shared only with the agents in her perceived neighborhood. Thus, given W_i, the *perceived utility* of agent i is equal to:

$$u_i(x_i, \mathbf{x}_{-i}; W_i) = \begin{cases} \frac{x_i}{x_i + \sum_{j \in N} g_{ij} x_j} W_i V - c x_i & \text{if } x_i + \sum_{j \in N} g_{ij} x_j > 0, \\ \frac{1}{1 + |\mathcal{N}_i|} W_i V & \text{if } x_i + \sum_{j \in N} g_{ij} x_j = 0. \end{cases} \tag{5}$$

2.3.2 Cournot competition

Consider a standard homogeneous good Cournot oligopoly game on a network with n firms competing in quantities. The profit function for each firm i is given by

$$\pi_i(\mathbf{x}) = (p - c)\, x_i, \text{ where } \quad p = \max\{\bar{\alpha} - X, 0\} \tag{6}$$

We assume that $\overline{\alpha} > c$. In (6), x_i denotes the quantity produced by firm i while p is the price of the product. This means that the profit function of each firm i is still given by (2) but with

$$h(x_i, X) = x_i \left((\bar{\alpha} - X)_+ - c \right).$$

Firms only observe the quantities produced by their neighbors. In other terms, firm i only observes $\sum_j g_{ij} x_j$. Given W_i, the *perceived utility* of firm i is then equal to:

$$u_i(x_i, \mathbf{x}_{-i}; W_i) = \left(\bar{\alpha} - \frac{x_i + \sum_j g_{ij} x_j}{W_i} \right)_+ x_i - c x_i \tag{7}$$

Indeed, firm i believes that the price is an affine transformation in the total demand, which is correct. However, she does not observe the actual demand X, but only the demand in her neighborhood, $x_i + \sum_j g_{ij} x_j$. When observing that the realized price she faces is not her perceived price (the one she anticipated from observing the demand in her perceived neighborhood), she does not know that it is because there are other firms producing the same product in the network. Instead, she believes it is because the slope affecting the price is incorrect.

Observe that, as in the standard Cournot model, there is precisely one price p for the homogeneous good defined in (6), which is not subject to interpretation. This means that firms do not "misperceive" the actual price, because they observe it. However, they "misperceive" how this price emerges because they have a wrong perception of the slope affecting this price.

Remark 2. *Our result can be extended to a Cournot model with a general non-linear demand function, that is, $p = (\bar{\alpha} - k(X))_+$. See the Online Appendix C.*

2.4 Perception-Consistent Equilibrium

The critical assumption of our model is that, in order to choose an effort level, each individual i considers the competition in her perceived neighborhood $\{i\} \cup \mathcal{N}_i$ while, in reality, she is in competition with all the agents in the network. The following equilibrium concept captures this idea.

Definition 1. *Given a perception network $(N, \mathbf{G})$, a **Perception-Consistent Equilibrium** (PCE) is a vector $\mathbf{x}^* \in \mathbb{R}^n_+ \neq \mathbf{0}$ such that,*

(i) for each $i \in N$ and each $x_i \in \mathbb{R}_+$,

$$u_i(x_i^*, \mathbf{x}_{-i}^*; W_i) \geq u_i(x_i, \mathbf{x}_{-i}^*; W_i),$$

(ii) for each $i \in N$,

$$W_i = \frac{x_i^* + \sum_{j \in N} g_{ij} x_j^*}{X^*}.$$

Condition (i) states that, given her perceived competitionand a vector of actions $\mathbf{x}$, each individual i chooses an effort that maximizes her *perceived utility.* Each agent i takes W_i as given, and chooses the action x_i that maximizes $u_i(x_i, \mathbf{x}_{-i}, W_i)$. Note, however, the subtle part of condition (i): taking W_i as given, each agent i is only best responding to the choice of actions of agents in her perception set.

Condition (ii) is a *consistency* requirement imposed in equilibrium. Indeed, at a PCE, individual i's perceived utility has to be equal to her *objective* payoff function in the underlying aggregative game. This is why we call it a *perception-consistent equilibrium.*

Remark 3. *Perception-consistent equilibria and Nash equilibria coincide if and only if the network is complete, in which case the PCE is the Nash equilibrium of the aggregative game.*

Indeed, a perception-consistent equilibrium of our aggregative game on a *complete network* is simply a Nash equilibrium on the same game, since all agents observe the whole set of agents. As soon as one link is missing, at least one agent is not aware of the existence of some other agents and PCEs no longer coincide with Nash equilibria. Observe that the PCE is neither a refinement of the concept of Nash equilibrium nor a superset (such as correlated equilibria or the concept of peer-confirming equilibria defined in Lipnowski and Sadler (2019)). Rather, it is the outcome of a decentralized optimization problem where each agent must choose the action that maximizes their perceived utility and where each perceived utility must be ex-post consistent with the realized outcome.

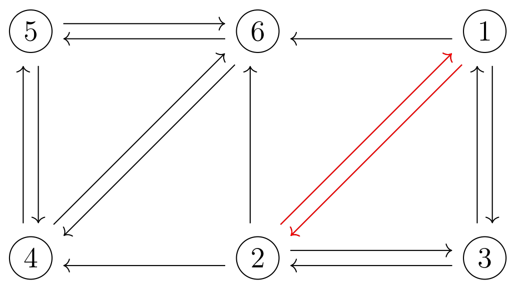
2.5 Perception-consistent equilibrium: An illustration

To understand our perception-consistent equilibrium concept (Definition 1), let us consider the following two-part example in which we compare the set of perception-consistent equilibria in two closely related networks for the linear (Tullock) contest games (Section 2.3.1) where the utility of each agent i given by (5). This will be part of our leading examples in this paper.

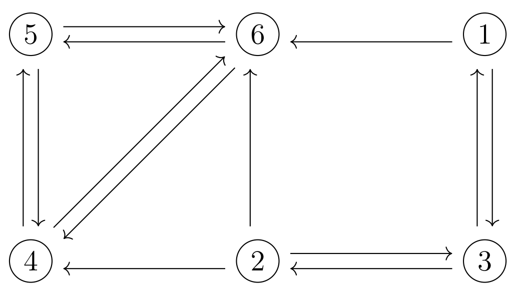
Example 1. Consider the two networks displayed in Figure 1.

Consider an economy with six agents that are in competition for a fixed amount of resources V. Each firm i chooses effort x_i, and the resource V is distributed according to the sharing rule (4). Suppose that the interactions are repeated every year and the perception sets are as follows: $\mathcal{N}_1 = \{2, 3, 6\}$, $\mathcal{N}_2 = \{1, 3, 4, 6\}$, $\mathcal{N}_3 = \{1, 2\}$, $\mathcal{N}_4 = \{5, 6\}$, $\mathcal{N}_5 = \{4, 6\}$, $\mathcal{N}_6 = \{4, 5\}$. The corresponding directed network is depicted in Figure 1(a). At the end of a year, V is shared according to this rule and all firms observe the efforts of those in their perception set. To keep things simple, suppose that all firms initially choose the same effort and hence earns each $V/6$. From the viewpoint of firms 4, 5, and 6, which believe that they are only in competition with each other, $V/6$ corresponds to the total resource $V/2$, shared between them. On the other hand, firm 2 obtains the same share, $V/6$, but has a different perception, since she is aware that she is competing with firms $1, 3, 4$, and 6. Hence, she believes that the total resource is $5V/6$ and that it is shared between 5 firms. As a result, the following year, firm 2 will make more effort than firms $4, 5, 6$, since she thinks that there is more total resources ($5V/6 > V/2$). This pattern continues and reinforces itself over time, so that firms 1 and 2 make more and more effort, which induces firm 3 to also increase its effort. On the

Figure 1: Two similar networks with different densities



(a) A dense network



(b) A less dense network

contrary, firms $4, 5, 6$ —which are not aware of the effort increase by the other firms in the economy—believe that the total resource V decreases over time (they believe that there are fewer and fewer resources in the economy year after year) without knowing why. It turns out that, after some time, firms 4, 5, and 6 will end up making no effort and all resources will be shared between firms $1, 2,$ and 3. This steady state is a PCE.[13]

Suppose now that agents 1 and 2 are no longer aware of each other. This corresponds to the network displayed in Figure 1(b). After the first year, firms $1, 3, 4, 5,$ and 6 believe that the amount of resource to be shared is equal to $V/2$, because they are all aware of exactly two other firms. Firm 2, in turn, is aware of three other competitors and hence believes that the amount of resource to be shared is $2V/3$. Thus, in the following year, firm 2 will make more effort than the other firms. The following year, firm 2's effort increase will trigger firm 3 increase its effort. And so forth. At this point it is not possible to provide more intuition, but we will show that, at the PCE, all firms will be active. Thus, adding one link dramatically

[13] At this point of the paper, it is not possible to explain why this outcome will be reached. We explicitely compute this PCE in Section 3.4.

changes the set of active firms in equilibrium.

2.6 Existence

Let us show that there exists a Perception Consistent Equilibrium (PCE) in a rather large class of aggregative games, which includes both Tullock and Cournot models.

Definition 2. *We say that there is a* ***competitive aggregative game*** *on a directed network* $(N, \mathbf{G})$ *if,*

(i) *h is (weakly) concave in the first argument, i.e.,* $\frac{\partial^2 h}{\partial x^2}(x, X) \leq 0$;

(ii) *h is strictly decreasing in the second argument, i.e.,* $\frac{\partial h}{\partial X} < 0$ *for any* $x > 0$;

(iii) *the cross derivative of h is strictly negative, i.e.,* $\frac{\partial^2 h}{\partial x \partial X} < 0$;

(iv) *the map* $x \mapsto h\left(x, \frac{x+y}{W}\right)$ *is strictly quasiconcave for any* $W > 0$, *and* $y \geq 0$;

(v) $\lim_{x \to +\infty} \frac{\partial h}{\partial x}(x, X) < 0$, $\lim_{X \to 0} \inf_{0 < x \leq X} \frac{X \frac{\partial h}{\partial X}(x,X)}{x \frac{\partial h}{\partial x}(x,X)} \geq -1$ *and* $\lim_{X \to +\infty} \frac{\partial h}{\partial x}(x, X) < 0$.

These assumptions are relatively mild and standard.[14] They guarantee that the game is well-defined. In particular, assumption (*ii*) guarantees that we focus on competition while assumption (*v*) guarantees that zero effort is not an equilibrium. Indeed, given (*v*), when aggregate effort is very low, agents have an incentive to increase their effort. Note, however, that this does not imply that all efforts are positive at a PCE. Note also that assumption (*v*) is also easy to check, as we will show below.

Theorem 1. *In a competitive aggregative game on a directed network* $(N, \mathbf{G})$*, there always exists a Perception-Consistent Equilibrium (PCE). Moreover, at any PCE* $\mathbf{x}^*$ *and for any pair of active agents* i, j*, we have* $x_i^* \geq x_j^*$ *if and only if* $\sum_{l \in \mathcal{N}_i} x_l^* \geq \sum_{k \in \mathcal{N}_j} x_k^*$*. Hence the efforts at a PCE represent an ordinal centrality measure on the sub-network of active agents.*

This result is very general and shows that there always exists a PCE if the conditions in Definition 2 are satisfied. More importantly, it shows that at any PCE, the efforts represent an *ordinal centrality measure* —a general class of centrality measures introduced by Sadler (2021).[15,] Many well-known network centrality measures, such as degree, eigenvector and

[14] We denote by $\partial h/\partial x$ the derivative of h with respect to its first argument and $\partial h/\partial X$ the derivative of h with respect to its second argument. Note that these assumptions are not equivalent to strategic substitutes, since it should involve the second derivative with respect to twice the second argument of h. For instance, in the case of Tullock, we have $\frac{\partial^2 h}{\partial x \partial X} = -V/X^2$, while $\frac{\partial^2 \pi_i}{\partial x_i \partial x_j} = -V/X^2 + 2Vx_i/X^3$.

[15] The function $i \in N \to x_i^*$ represents an ordinal centrality (Sadler, 2021) if and only if it satisfies *recursive monotonicity*. That is, for any i, j, if there exists an injective map $\phi : \mathcal{N}_j \to \mathcal{N}_i$ such that $x_{\phi(k)}^* \geq x_k^*$ for all $k \in \mathcal{N}_j$, then $x_i^* \geq x_j^*$. Observe that, at a PCE, we have $x_i^* \geq x_j^*$ if and only if $\sum_{l \in \mathcal{N}_i} x_l^* \geq \sum_{k \in \mathcal{N}_j} x_k^*$. Hence, recursive monotonicity clearly holds for $i \mapsto x_i^*$.

Katz-Bonacich centrality, belong to this class. Below, in Theorem 2, we show that for a more specific class of competitive aggregative games, the efforts at any PCE represent a specific ordinal centrality measure, namely, eigenvector centrality.

An important class of competitive aggregative games. The payoff function of both applications—Tullock and Cournot—share a common structure: h can be written as $h(x, X) = xf(X)$. In this case, the payoff function of agent i is given by

$$\pi_i(\mathbf{x}) = x_i f(X). \tag{8}$$

where $f = \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable on $]0, +\infty[$, strictly decreasing with $\lim_{X\to 0^+} f(X) > 0$, $\lim_{X\to+\infty} f(X) < 0$, such that $x \mapsto xf\left(\frac{x+y}{W}\right)$ is strictly quasiconcave, and $\lim_{X\to 0} \frac{Xf'(X)}{f(X)} \geq -1$. Note that the last requirement is automatically satisfied, as soon as f is continuously differentiable in 0. Such payoff functions satisfy the assumptions of Definition 2, and thus both Tullock and Cournot belong to the large class of competitive aggregative games covered by Definition 2.[16,17]

3 General analysis: Perception-consistent equilibria

In this section, we present a complete analysis of perception-consistent equilibria in competitive aggregative games with payoff functions given by Equation (8). We first provide a general algebraic characterization in Section 3.1. We show that PCE provides a microfoundation of *eigenvector centrality.* Note that eigenvector centrality is an ordinal centrality measure. Hence the result can be interpreted within the context of Theorem 1: in the subclass of competitive aggregative games defined by Equation (8), any PCE represents the eigenvector centrality on the sub-network of active agents. In Section 3.2, we apply our result to the Tullock and Cournot models. Sections 3.3 and 3.4 are devoted to providing an alternative characterization; we introduce the concept of community and show that carefully ordering the agents by community allows the easy identification of all PCEs in any network. Finally, we show in Section 3.5 that the set of PCEs can be refined to a unique *stable* PCE. Importantly, this particular PCE provides a microfoundation of eigenvector centrality **in the whole network**, and this holds true for any weakly connected network.

[16]Parts (i), (ii) and (iii) follow from the fact that $\frac{\partial^2 h}{\partial x^2} = 0$, $\frac{\partial h}{\partial X} = xf'(X) < 0$, and $\frac{\partial^2 h}{\partial x \partial X} = f'(x) < 0$, respectively. Finally $\frac{\partial h}{\partial x} = f(X)$ and thus assumptions (iv) and (v) follows from the assumptions on f.

[17]Indeed, for Tullock, $f(X) = V/X - c$. Hence $xf\left(\frac{x+y}{W}\right) = x\left(\frac{VW}{x+y} - c\right)$, which is strictly concave in x. Moreover $\frac{Xf'(X)}{f(X)} = \frac{-V}{V-cX} \to_{X\to 0} -1$ For Cournot, $f(X) = (\overline{\alpha} - X)_+ - c$, and $xf\left(\frac{x+y}{W}\right) = x\left((\overline{\alpha} - X)_+ - c\right)$, which is strictly quasiconcave. Moreover $\frac{Xf'(X)}{f(X)} = \frac{-X}{\overline{\alpha}-c-X} \to_{X\to 0} 0$.

3.1 General characterization of perception-consistent equilibria

We already established that the set of perception-consistent equilibria is nonempty. In this section, we show that an effort profile is a perception-consistent equilibrium if and only if it is a (properly normalized) nonnegative *eigenvector* of $\mathbf{G}$. let $N_+(\mathbf{x}^*)$ be the set of agents who are *active* at equilibrium $\mathbf{x}^*$. That is, $N_+(\mathbf{x}^*) = \{i \in N : x_i^* > 0\}$.

Theorem 2. *Let $(N, \mathbf{G})$ be a weakly-connected network and consider the competitive aggregative game in which the utility of each agent i is given by (8). Then, $\mathbf{x}^*$ is a perception-consistent equilibrium if and only if*

$$\mathbf{G}\mathbf{x}^* = \left(\frac{-f(X^*) - f'(X^*)X^*}{f(X^*)} \right) \mathbf{x}^*, \quad \text{and} \quad \mathbf{x}^* \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}. \tag{9}$$

Theorem 2 provides a *microfoundation of eigenvector centrality.* It is a particular case of Theorem 1, since eigenvector centrality is an ordinal centrality measure. In particular, it shows that, for any weakly-connected network, at any PCE, the effort of each agent is proportional to her eigenvector centrality, in the sub-network of active agents.[18] This result is driven by the consistency requirement (*ii*) of Definition 1 and also to the utility function (2) of the aggregative game. This is a new result, complementing that of Ballester et al. (2006), who show that, for any network, in a game with strategic complementarities, for each agent who chooses effort that maximizes a linear-quadratic utility function, her equilibrium effort is equal to her Katz-Bonacich centrality. Here, we show that, if each agent chooses effort that maximizes utility (2) based on the competitive aggregative game, then, at any PCE, her effort will be proportional to her eigenvector centrality.

Another key feature of PCEs introduced in Theorem 2 is that the set of active agents is a *closed set.* That is, $i \rightrightarrows j$ and $j \in N_+(\mathbf{x}^*) \implies i \in N_+(\mathbf{x}^*)$. Note that the closedness of the set of active agents cannot be guaranteed in the general class of competitive aggregative games covered in Theorem 1.

Remark 4. *In any strongly-connected networks, there is a unique perception-consistent equilibrium $\mathbf{x}^*$. In addition, for each $i \in N$, $x_i^* > 0$.*

3.2 Microfoundation of eigenvector centrality: Some applications

We illustrate our characterization results for our two main applications of aggregative games: Linear (Tullock) contest games and Cournot competition.

[18] Eigenvector centrality is usually defined for strongly-connected networks. Indeed, in this case, it is a well-defined measure of centrality captured by the Perron-Frobenius vector associated with the adjacency matrix (Jackson, 2008). In Section B.2 of Online Appendix B, we provide a more general definition of eigenvector centrality for networks that are not necessarily strongly connected.

Linear (Tullock) contest games. Let us show (9) emerges in the Linear (Tullock) contest games described in Section 2.3.1. First, given W_i, each agent i chooses her effort x_i^* that maximizes her perceived utility (5). This leads to:

$$\frac{W_i \sum_j g_{ij} x_j^*}{\left(x_i^* + \sum_j g_{ij} x_j^*\right)^2} V = c.$$

Combining this identity with consistency condition $W_i = \frac{x_i^* + \sum_j g_{ij} x_j^*}{X^*}$, we obtain:

$$\frac{\sum_j g_{ij} x_j^*}{x_i^* + \sum_j g_{ij} x_j^*} \frac{V}{X^*} = c.$$

By solving this equation, we get:

$$\sum_j g_{ij} x_j^* = \left(\frac{cX^*}{V - cX^*}\right) x_i^*,$$

In matrix form, we therefore obtain:

$$\mathbf{G}\mathbf{x}^* = \frac{cX^*}{V - cX^*} \mathbf{x}^*. \tag{10}$$

Cournot competition. Let us perform the same exercise for the standard homogeneous good Cournot oligopoly game of Section 2.3.2. First, given W_i, each firm i chooses a quantity x_i^* that maximizes her perceived utility (7). This leads to:

$$\bar{\alpha} - \frac{x_i^* + \sum_j g_{ij} x_j^*}{W_i} - \frac{x_i^*}{W_i} = c.$$

Combining this identity with consistency condition $W_i = \frac{x_i^* + \sum_j g_{ij} x_j^*}{X^*}$, we obtain:

$$\bar{\alpha} - X^* - \frac{x_i^* X^*}{x_i^* + \sum_j g_{ij} x_j^*} = c.$$

By solving this equation, we get:

$$\sum_j g_{ij} x_j^* = \left(\frac{2X^* - \bar{\alpha} + c}{\bar{\alpha} - c - X^*}\right) x_i^*,$$

In matrix form, we obtain:

$$\mathbf{G}\mathbf{x}^* = \left(\frac{2X^* - \bar{\alpha} + c}{\bar{\alpha} - c - X^*}\right) \mathbf{x}^*. \tag{11}$$

Let us emphasize one important difference between the Cournot competition and the Tullock contest game. Suppose that —dropping the no-isolation assumption—we allow an agent to be completely *invisible*, i.e., no other agent is aware of her. In the Tullock model, there must be at least two active agents at a PCE. This is due to the fact that the aggregator f is not regular at zero in the Tullock model, a consequence of which being that the quantity $-f(X^*) - f'(X^*)X^*$ cannot be equal to zero. Hence, the set of active agents at a PCE cannot be a network with largest eigenvalue 0 and thus it contains at least two elements. In contrast, this is not anymore true in the Cournot competition model, where there is a PCE in which the isolated agent is the only one who is active, and she optimally chooses quantity as if she were in a monopolistic situation. Thus, it chooses the quantity x_i^* that solves the equation $-f(X) - f'(X)X = 0$, i.e., $x_i^* = (\overline{\alpha} - c)/2$.

3.3 Communities

Let us start with some important definitions. For any subset of agents $M \subseteq N$, let $\mathbf{G}_M$ denote the restriction of matrix $\mathbf{G}$ to M.

Definition 3. *Given $M \subset N$, $(M, \mathbf{G}_M)$ is a **strongly-connected component** (or **community**)*[19] *of $(N, \mathbf{G})$ if:*

(i) it is a strongly-connected sub-network;

(ii) for each $I \subset N \setminus M$, $(M \cup I, \mathbf{G}_{M\cup I})$ is not a strongly-connected network.

Let $\mathcal{C}(\mathbf{G})$ denote the set of communities in $(N, \mathbf{G})$:

$$\mathcal{C}(\mathbf{G}) := \{M \subset N : (M, \mathbf{G}_M) \text{ is a community of } (N, \mathbf{G})\}.$$

An element of $\mathcal{C}(\mathbf{G})$ is a subset of agents of cardinal at least two, such that the corresponding sub-network is strongly connected. Note that under the no-isolation property, $\mathcal{C}(\mathbf{G})$ is never empty and communities form a partition of the set of agents.

To illustrate this definition, consider the network in Figure 1(a) with $N = \{1, 2, 3, 4, 5, 6\}$ and define $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$. We can see that both $(M_1, \mathbf{G}_{M_1})$ and $(M_2, \mathbf{G}_{M_2})$ are strongly-connected components: each is a strongly-connected sub-network, and it is not possible to enlarge any of these two sub-networks to form a larger strongly-connected network.

[19]Since we assumed the no-isolation property, a strongly-connected component contains at least two agents. However, we can still define communities in the same way even if there are isolated agents, since these agents are by themselves a strongly connected component. In that case, the characterization we establish in the next section still holds but might depend on the aggregator f.

Hence, $\mathcal{C}(\mathbf{G}) = \{M_1, M_2\}$. Note that the set of communities is unchanged in the network of Figure 1(b).

Definition 4. *Let $M \subset N$. Agent $i \in N$ is an **adjunct** to the sub-network $(M, \mathbf{G}_M)$ of $(N, \mathbf{G})$ if i is connected to some agent $j \in M$ through a path. The **adjunct set** of M, denoted by $\bar{M}$, is therefore defined as the set of all agents that are adjuncts to M, that is:*

$$\bar{M} = \{i \in N : \exists j \in M \text{ with } i \rightrightarrows j\}.$$

Given $M, M' \in \mathcal{C}(\mathbf{G})$, we say that M' is adjunct to M if $M' \subset \bar{M}$.

Note that the definition of the adjunct set of a community M is inclusive in the sense that M is also part of the adjunct set. In the network of Figure 1(a), $\bar{M}_1 = M_1 = \{1, 2, 3\}$ and $\bar{M}_2 = M_2 \cup M_1 = N = \{1, 2, 3, 4, 5, 6\}$. Importantly, this definition of adjunct sets induces a partial ordering $\succeq$ on the set of communities $\mathcal{C}(\mathbf{G})$.[20] The binary relation $\succeq$ is defined as follows:

$$M' \succeq M \text{ if and only if } M' \subset \bar{M}. \tag{12}$$

In other terms, $M' \succeq M$ if there exists a path from M' to M. As usual, if $M' \succeq M$ and $M' \neq M$, we write $M' \succ M$. A community M is $\succeq$-*maximal* if no community M' exists such that $M' \succ M$. That is, there is no M' that *is aware* of M. If not, we say that M is *hidden* from M'. Maximal elements with respect to this partial ordering are communities few agents are aware of. This "advantage" can be intuitively captured by making the following observation: given a PCE $\mathbf{x}^*$, we have

$$M \subset N_+(\mathbf{x}^*) \Rightarrow M' \subset N_+(\mathbf{x}^*), \ \forall M' \succeq M.$$

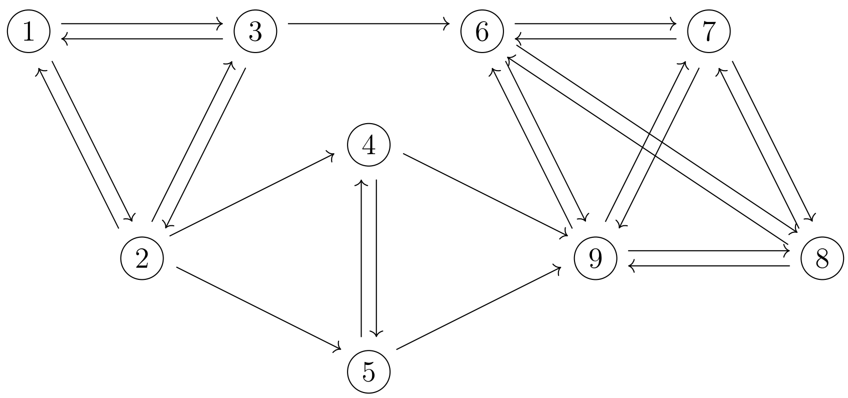
Let us now illustrate the concept of communities and the $\succeq$-ordering.

Example 1. Consider the network $(N, \mathbf{G})$ in Figure 1(a) with $N = \{1, 2, 3, 4, 5, 6\}$. As we saw above, there are two communities: $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$. Since there is a link from 1 to 6, for instance, we have $M_1 \succ M_2$, and, clearly, the community M_1 is $\succeq$-*maximal*.

Example 2. Consider now the network $(N, \mathbf{G})$ displayed in Figure 2 with $N = \{1, 2, \cdots, 9\}$. There are three communities in this network: $M_1 = \{1, 2, 3\}$, $M_2 = \{4, 5\}$, and $M_3 = \{6, 7, 8, 9\}$. To check that these three connected subnetworks satisfy (ii) in Definition 3, note that there is no path from M_3 to either M_1 or M_2, and that there is no path from M_2 to M_1. However, there is a link from 2 to 4, so that $M_1 \succ M_2$. There is also a link from 4 to 9, so that $M_2 \succ M_3$. Finally, we obtain $M_1 \succ M_2 \succ M_3$. Community M_1 is $\succeq$-*maximal*.

[20] A *partial ordering* is a reflexive, antisymmetric, and transitive binary relation.

Figure 2: Network structure in Example 2



In both examples, the ordering $\succeq$ is *complete* (every pair of communities is comparable as per the $\succeq$ ordering). This is not always the case. It is possible to have communities that are not comparable according to $\succeq$.

3.4 Characterization of perception-consistent equilibria

Thus far, we have shown that any weakly-connected network $(N, \mathbf{G})$ can be associated with an ordering in the set of strongly-connected components. In this section, we characterize active agents at a PCE in terms of their position in the network, along with the "density" of the community they belong to.

Definition 5. *For each community* $(M, \mathbf{G}_M)$*, we refer to its adjunct set* $\bar{M}$ *as a* candidate set with root M. *A perception-consistent equilibrium* $\mathbf{x}^*$ *of* $(N, \mathbf{G})$ *such that* $N_+(\mathbf{x}^*) = \bar{M}$ *for some community* M *is called an* equilibrium with root M.

A *candidate set* is a set of agents that could naturally be the set of active players at equilibrium. Indeed, if there is one agent i who is active in M, then all agents in its adjunct set $\bar{M}$ will be active. This is because all agents who are path-connected to i are necessarily active, since the set of active agents at a perception-consistent equilibrium is closed (see (**??**)). We obtain the following key result:

Proposition 1. *There is at most one perception-consistent equilibrium* $\mathbf{x}^*$ *with root* M*. It exists if and only if*

$$\rho(\mathbf{G}_M) > \max\left\{\rho(\mathbf{G}_{M'}) : \ M' \in \mathcal{C}(\mathbf{G}),\ M' \succ M\right\}. \tag{13}$$

In particular, for any $\succeq$*-maximal* M*, there always exists an equilibrium with root* M*.*

This proposition is very useful in terms of characterizing PCEs which admits a root. It states that, for any community M, there is *at most* one PCE with root M. This provides a necessary and sufficient condition in terms of the largest-eigenvalue comparisons for this equilibrium to exist. This condition is automatically satisfied for maximal communities, because the set $\{M' \in \mathcal{C}(\mathbf{G}) : M' \succ M\}$ is empty. For non-maximal elements of $\mathcal{C}(\mathbf{G})$, however, ascertaining if a PCE with root M exists requires checking the non-trivial inequality (13).

Let us now illustrate Proposition 1 for the linear (Tullock) contest games (Section 2.3.1) with the following examples.[21]

Examples 1 and 2: Perception-consistent equilibria

- Consider the network $(N, \mathbf{G})$ in Figure 1(a). As we saw above, there are two communities: $M_1 := \{1, 2, 3\}$ and $M_2 := \{4, 5, 6\}$, with $M_1 \succ M_2$. Hence there is an equilibrium with root M_1, where only agents $1, 2$, and 3 are active. Since $\rho(\mathbf{G}_{M_2}) = \rho(\mathbf{G}_{M_1})$ and there is no equilibrium with root M_2. As a result, there is a unique perception-consistent equilibrium such that $x_1^* = x_2^* = x_3^* = \frac{2V}{9c}$ and $x_4^* = x_5^* = x_6^* = 0$.
- Consider now the network $(N, \mathbf{G})$ displayed in Figure 1(b), a variation of Figure 1(a) in which we *deleted* the directed edges between agents 1 and 2, so that the strongly-connected component M_1 is now less dense and, thus, its spectral radius is smaller. The partial order is unchanged: $M_1 \succ M_2$. Hence there is again an equilibrium with root $M_1 = \{1, 2, 3\}$, where only agents $1, 2$ and 3 are active, with $x_1^* = x_2^* = \frac{1}{(1+\sqrt{2})^2} \frac{V}{c}$, $x_3^* = \frac{\sqrt{2}}{(1+\sqrt{2})^2} \frac{V}{c}$, and $x_4^* = x_5^* = x_6^* = 0$. Since $\rho(\mathbf{G}_{M_2}) = 2 > \sqrt{2} = \rho(\mathbf{G}_{M_1})$, there is now an equilibrium with root M_2, where all agents in the network are active. At such an equilibrium, note that efforts are not symmetric. Indeed, $x_1^* = \frac{V}{9c}$, $x_2^* = \frac{7V}{45c}$, $x_3^* = \frac{2V}{15c}$, and $x_i^* = \frac{4V}{45c}$ for each $i \in \{4, 5, 6\}$.
- Finally Consider the network displayed in Figure 2 with $M_1 = \{1, 2, 3\}$, $M_2 = \{4, 5\}$, and $M_3 = \{6, 7, 8, 9\}$, with $M_1 \succ M_2 \succ M_3$. Despite the existence of three strongly-connected components, let us use Proposition 1 to show that there are (only) two perception-consistent equilibria. Each community's subnetwork is complete, so that $\rho(\mathbf{G}_{M_2}) = 1 < \rho(\mathbf{G}_{M_1}) = 2 < \rho(\mathbf{G}_{M_3}) = 3$.

Since M_1 is $\succeq$-maximal, there is an equilibrium with root M_1 (with set of active agents $\bar{M}_1 = \{1, 2, 3\}$) which, using Theorem 2, is given by:

$$x_i^* = \frac{2V}{9c} \text{for } i \in \{1, 2, 3\},\ x_j^* = 0 \text{ for } j \in \{4, 5, 6, 7, 8, 9\}.$$

[21] We can perform the same exercise for Cournot Competition (Section 2.3.2) with utility function (7). For instance, in the network in Figure 1(a), there is a unique PCE where only firms 1, 2, 3 are active with $x_1^* = x_2^* = x_3^* = (\bar{\alpha} - c)/4$ and $X^* = 3(\bar{\alpha} - c)/4$. In the network in Figure 1(b), there are two PCE. The first one is such that only firms 1, 2, 3 are active with $x_1^* = x_2^* = (\bar{\alpha} - c) / \left[2\left(\sqrt{2} + 1\right)\right]$ and $x_3^* = (\bar{\alpha} - c) / \left[\sqrt{2}\left(\sqrt{2} + 1\right)\right]$, with $X^* = (\bar{\alpha} - c)/\sqrt{2}$. The other PCE is such that all firms are active with $\mathbf{x}^* = \frac{(\bar{\alpha} - c)}{40}(5, 7, 6, 4, 4, 4)$ and $X^* = 3(\bar{\alpha} - c)/4$.

Since $M_1 \succ M_2$ and $\rho(\mathbf{G}_{M_2}) < \rho(\mathbf{G}_{M_1})$, there is no equilibrium with root M_2.
Finally, Since $\rho(\mathbf{G}_{M_3}) > \max\{\rho(\mathbf{G}_{M_1}), \rho(\mathbf{G}_{M_2})\}$, there is an equilibrium with root M_3 such that the set of active individuals is $\bar{M}_3 = N$. This equilibrium is given by:

$$x_i^* = \frac{3V}{56c} \text{for } i \in \{1,4,5\},\ x_j^* = \frac{9V}{112c} \text{ for } j \in \{2,3\}, \text{ and } x_k^* = \frac{3V}{28c}, \text{ for } k \geq 6.$$

In summary, in the network in Figure 2, there are two PCEs: one in which only agents 1, 2, and 3 are active, and one in which all agents are active. $\diamond$

In order to complete our characterization of perception-consistent equilibria, let us now consider an interesting subset of weakly connected networks.

Definition 6. *A weakly-connected network* $(N, \mathbf{G})$ *is* ***simple*** *if for any* $M_1, M_2 \in \mathcal{C}$ *such that* $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = \rho$, *we have* $\max\{\rho(\mathbf{G}_M) :\ M \succ M_1 \text{ or } M \succ M_2\} \geq \rho$.

Hence, simple networks are such that, for any two distinct communities with the same spectral radius, a community must exist whose spectral radius is at least as large, and which is aware of one of them. In other words, (i) we exclude weakly-connected networks for which two PCEs with different roots have the same spectral radius, but (ii) we allow the existence of two communities with the same spectral radius if one of them is not part of a PCE. In particular, we exclude networks in which two $\succeq$-maximal communities have the same spectral radius. For example, the network displayed in Figure E1 in the Online Appendix E.2 is not simple because the two $\succeq$-maximal communities $M_1 = \{2,3\}$ and $M_2 = \{4,5\}$ have the same spectral radius (i.e., $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = 1$). On the other hand, the networks in Figures 1(a) and (b) (Example 1) and in Figure 2 (Example 2) are simple. Observe, in particular, that in the network displayed in Figure 1(a), the communities $M_1 = \{1,2,3\}$ and $M_2 = \{4,5,6\}$ have the same spectral radius (i.e., $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = 2$). However, because M_2 is not the root of a PCE, this network is simple.

We show that, in simple networks, perception-consistent equilibria always admit a root, meaning that Proposition 1 provides a full characterization of the set of equilibria in these networks.

Proposition 2. *Let* $(N, \mathbf{G})$ *be a simple network, and let* $\mathbf{x}^*$ *be a perception-consistent equilibrium of* $(N, \mathbf{G})$. *Then,* $\mathbf{x}^*$ *admits a root. Moreover, equilibrium efforts are proportional to eigenvector centrality in the sub-network of active players.*

The last statement of Proposition 2 must be understood as follows: if $\mathbf{x}^*$ is a PCE, then the effort of active agents is proportional to their eigenvector centrality *in the sub-network they generate.* It is important to understand that this result does not say anything about the eigenvector centrality of agents in the whole network, since inactive agents are not taken into account. A direct consequence of Proposition 2 is that there is a *finite* number of equilibria in simple networks because, for any community M, there is at most one PCE with root M. Actually, the set of perception-consistent equilibria is finite if and only if the network is simple. This is formally stated in Proposition D1 in the Online Appendix D.1. In Section E of the Online Appendix, we show that we can still describe the set of perception-consistent equilibria in a simple way when the network is no longer simple (see Proposition E6).

Remark 5 (Equilibrium payoffs)**.** *Pick any perception-consistent equilibrium* $\mathbf{x}^*$ *with root* M*. Then, the payoff of each active agent* i *is given by*

$$\pi_i(\mathbf{x}^*) = f(X^*)x_i^*, \tag{14}$$

Moreover, the sum of utilities of active agents at $\mathbf{x}^*$ *is given by:*

$$\sum_i \pi_i(\mathbf{x}^*) = f(X^*)X^* = \frac{-(X^*)^2 f'(X^*)}{(\rho(\mathbf{G}_M)+1)}, \tag{15}$$

since $(\rho(\mathbf{G}_M)+1)f(X^*) = -X^* f'(X^*)$

This remark shows that the equilibrium utility of each active agent is proportional to her equilibrium effort and thus the utility of active agents is proportional to their eigenvector centrality in the sub-network of active agents. This only informs us about the relative utility of active agents but does not tell us anything on the aggregate utility. However, for the Tullock context function, we have:

$$\sum_i \pi_i(x_i^*) = \frac{V}{\rho(\mathbf{G})+1},$$

while, for Cournot competition, we obtain:

$$\sum_i \pi_i(x_i^*) = \frac{(X^*)^2}{\rho(\mathbf{G}_M)+1} = (\bar{\alpha}-c)\frac{\rho(\mathbf{G}_M)+1}{(\rho(\mathbf{G}_M)+2)^2}.$$

In both examples (Tullock and Cournot), we can write the aggregate utility as an explicit function of $\rho(\mathbf{G}_M)$, the spectral radius of the network associated to the equilibrium. Hence, in these examples, the aggregate utility decreases when the network becomes denser. In the general case, as can be seen in equation (15), we cannot make such a statement because X^* depends on $\rho(\mathbf{G}_M)$.

3.5 Stability and eigenvector centrality

This section is devoted to refining the set of equilibria by characterizing those perception-consistent equilibria that are stable. Such an approach has two fundamental objectives. First, it allows us to identify which equilibria are robust to perturbations and provides a *dynamic microfoundation* to the concept of perception-consistent equilibrium. Second, refining the set of equilibria is necessary if one wants to extend the eigenvector centrality microfoundation to general networks. Indeed, as noted above, the efforts of active agents are proportional to their eigenvector centrality in the sub-network of active players. However, this raises a natural question: *Is there a link between eigenvector centrality in the whole network and PCE?* The answer is positive and we show that, in any simple network, exactly one PCE is proportional to the eigenvector centrality of the whole network, and it is precisely this PCE that we identify as the stable one.

As usual, stability of equilibria is defined through a meaningful dynamical system, the rest point of which is the equilibria we want to consider. A stable equilibrium is then defined as a stable rest point of the dynamics, that is, a rest point to which, starting from conditions close enough to it, the system asymptotically stabilizes. For this purpose, we introduce *perceived best-response dynamics*. This captures the idea that agents smoothly adapt their actions in the direction of their *best possible action*, given the information available to them.

3.5.1 The perceived best-response dynamics

We now present the continuous-time dynamics to which we characterize stability. Even though it is very close—in terms of interpretation—to the classical continuous-time best-response dynamics,[22] we explain how it is related to a simple discrete-time model.

Consider a discrete-time sequence of effort profiles in which, after observing their neighbors' effort level as well as W_i in the previous period, agents adapt their effort levels at each period of time. Specifically, before choosing her effort level at period t, agent i observes the effort of her neighbors $\mathbf{x}_{-i}^{t-1}$ as well as the realized shifter W_i^{t-1} at period $t-1$. She can then compute her optimal effort level with respect to quantity W_i^{t-1} by maximizing the map[23]

$$b_i \in [0, +\infty[\mapsto b_i \cdot f\left(\frac{b_i + (\mathbf{G}\mathbf{x}^{t-1})_i}{W_i^{t-1}}\right).$$

We denote by $Br_i(\mathbf{x}^{t-1})$ this maximizer. Since $W_i^{t-1} = \frac{x_i^{t-1}+(\mathbf{G}\mathbf{x}^{t-1})_i}{X^{t-1}}$, the map $Br_i(\cdot)$ is given

[22] See e.g., Gilboa and Matsui (1991), Matsui (1992), and, more recently, Bramoullé et al. (2014) and Bervoets and Faure (2019).

[23] Observe that $(\mathbf{G}\mathbf{x})_i = \sum_{j\in\mathcal{N}_i} x_j$. We use this more compact notation whenever it is convenient.

by

$$Br_i(\mathbf{x}) = Argmax_{b_i \geq 0}\, b_i \cdot f\left(\frac{X(b_i + (\mathbf{Gx})_i)}{x_i + (\mathbf{Gx})_i}\right).$$

Tullock model. In the linear Tullock contest model, we can compute explicitely the perceived best-response map:[24]

$$Br_i(\mathbf{x}) = \max\left\{-(\mathbf{Gx})_i + \left(\frac{V}{cX}(\mathbf{Gx})_i\,(x_i + (\mathbf{Gx})_i)\right)^{1/2}, 0\right\}. \tag{16}$$

Cournot competition. In the Cournot model, we can also compute explicitely the perceived best-response map:

$$Br_i(\mathbf{x}) = \frac{1}{2}\max\left\{-(\mathbf{Gx})_i + (\bar{\alpha} - c)\frac{x_i + (\mathbf{Gx})_i}{X}, 0\right\}. \tag{17}$$

Agent i chooses an effort level equal to a convex combination of her last effort level and the perceived best response based on what she observed at the last time period:

$$x_i^t = (1-\epsilon)x_i^{t-1} + \epsilon Br_i(\mathbf{x}^{t-1}) \tag{18}$$

When ϵ is small, the sequence generated by (18) is related to the solution curves of the continuous-time system

$$\dot{\mathbf{x}}(t) = \mathbf{B}(\mathbf{x}(t)), \tag{19}$$

where $B_i(\mathbf{x}) = -x_i + Br_i(\mathbf{x}),\ i = 1, ..., N$. Indeed, system (18) is a so-called *Cauchy-Euler scheme*, designed to approximate the solutions of (19) by choosing a small ϵ. In other words, system (19) can be interpreted as a smooth *limit* version of (18).

Choosing the appropriate state space, the stationary points of this ordinary differential equation are precisely the perception-consistent equilibria of our problem. We now consider the stability notion to be naturally associated to the dynamics (19). Stability for a given PCE $\mathbf{x}^*$ means that the solutions of (19) starting from initial conditions close enough to $\mathbf{x}^*$ converge back to $\mathbf{x}^*$. Formally:

Definition 7. *A perception-consistent equilibrium $\mathbf{x}^*$ is said to be* ***asymptotically stable*** *for (19) if there exists an open neighborhood U of $\mathbf{x}^*$ such that*

$$\lim_{t\to+\infty}\ \sup_{\mathbf{x}_0 \in U\cap\mathbf{S}} \|\phi(\mathbf{x}_0, t) - \mathbf{x}^*\| = 0,$$

where $\mathbf{S}$, defined in (23) in the proof of Theorem 3, contains all the relevant states of the

[24] This holds if and only if the action profile $\mathbf{x}$ is such that $(\mathbf{Gx})_i = 0 \Rightarrow x_i = 0$.

problem we consider, and $(\phi(\mathbf{x},t))_{\mathbf{x}\in\mathbf{S},t\geq 0}$ is the semi-flow associated to (19) on $\mathbf{S}$. Specifically, $\phi(\mathbf{x},t)$ is equal to the position of the (unique) solution of (19) starting at $\mathbf{x}$.

Definition 7 states that a PCE $\mathbf{x}^*$ is asymptotically stable if it *uniformly* attracts all solutions starting in an open neighborhood of itself. This is a standard concept of stability used in economics (Benaïm and Hirsch, 1999; Weibull, 2003), and in network games in particular (Bramoullé et al., 2016; Bervoets and Faure, 2019).

3.5.2 Stable PCE: A simple characterization

We now characterize the PCEs that are asymptotically stable with respect to the best-response dynamics (19). It turns out that being asymptotically stable depends entirely on the sub-network of active players in this PCE, in a very simple and intuitive way. Let $(N,\mathbf{G})$ be a simple network.[25] Given a PCE $\mathbf{x}^*$, we call $\rho(\mathbf{x}^*)$ the largest eigenvalue of the sub-network $(N_+(\mathbf{x}^*),\mathbf{G}_{N_+(\mathbf{x}^*)})$.

Theorem 3. *Let $(N,\mathbf{G})$ be a simple network. Then, there is a unique asymptotically stable equilibrium $\mathbf{x}^*$ such that $\rho(\mathbf{x}^*)=\rho(\mathbf{G})$. Moreover, agents' effort levels at the stable PCE are proportional to their eigenvector centrality in the whole network $(N,\mathbf{G})$.*

The intuition behind the characterization in terms of largest eigenvalues is as follows. Since the network is simple, there is exactly one PCE for which the largest eigenvalue of the set of active players is equal to $\rho(\mathbf{G})$. We must show that it is the only asymptotically stable equilibrium. Suppose that $\mathbf{x}^*$ is a PCE such that $\rho(\mathbf{x}^*)$ is strictly smaller than $\rho(\mathbf{G})$. Then, one can find a community M in which agents are inactive at $\mathbf{x}^*$, while having $\rho(\mathbf{G}_M)=\rho(\mathbf{G})$. Now, suppose that we slightly perturb $\mathbf{x}^*$ so that, instead of playing zero, agents in M play $\epsilon\mathbf{u}_i$, where $\mathbf{u}$ is the normalized positive eigenvector associated with $\rho(\mathbf{G})$. Since, for agents in M, this initial condition is associated with an eigenvalue that is strictly larger than the eigenvalue associated with $\mathbf{x}^*$, the agents in M will want to increase their effort and not come back to zero. Thus, it is clear that $\mathbf{x}^*$ cannot be stable.[26] We conclude the proof by showing that the (unique) PCE for which $\rho(\mathbf{x}^*)=\rho(\mathbf{G})$ is stable using standard methods. The last part of the theorem directly follows from the definition of eigenvector centrality.

Theorem 3 provides a simple and efficient analytic method for checking which PCEs are stable by looking for communities with the highest spectral radii. First, consider Example 1 with the networks displayed in Figure 1(a) and Figure 1(b). There are two communities: $M_1=\{1,2,3\}$ and $M_2=\{4,5,6\}$, with $M_1\succ M_2$ in both networks. The only difference

[25] Our main result (Theorem 3) holds under the less restrictive assumption that $(N,\mathbf{G})$ has a *unique dominant component*, as properly defined in condition (UDC) in Section B.2 of the Online Appendix B. In fact, eigenvector centrality is well defined if and only if the network satisfies the condition (UDC).

[26] For ease of presentation, *asymptotically stable* PCEs are referred to as *stable* PCEs.

between these two networks is that the one in Figure 1(a) has two extra links between agents 1 and 2 compared to the network in Figure 1(b). This is an important difference because the largest eigenvalue of the $\succeq$-maximal community, M_1, changes: it is equal to 2 in Figure 1(a), whereas it is equal to $\sqrt{2}$ in Figure 1(b). In Figure 1(a), there is a unique equilibrium that is clearly stable, in which only agents 1, 2, and 3 are active. In Figure 1(b), we have seen that there were two PCEs, one with root $M_1 = \{1, 2, 3\}$ and one with root $M_2 = \{4, 5, 6\}$. Since $\rho(\mathbf{G}_{M_1}) = \sqrt{2} < \rho(\mathbf{G}_{M_2}) = 2 = \rho(\mathbf{G})$, there is a unique stable PCE for which all agents are active. Thus, disconnecting agents 1 and 2 has a dramatic impact on the stable perception-consistent equilibria. The fact that the $\succeq$-maximal community in Figure 1(b) is less dense than in Figure 1(a) prevents agents 1, 2, and 3 from capturing the entire resource V and thus obliges them to share V with the other players in the PCE.

Second, consider Example 2 with the network depicted in Figure 2. We have seen that there were two PCE with roots $M_1 = \{1, 2, 3\}$ and $M_3 = \{6, 7, 8, 9\}$, respectively. Since $\rho(\mathbf{G}_{M_1}) = 2 < \rho(\mathbf{G}_{M_3}) = 3 = \rho(\mathbf{G})$, the only stable PCE is the equilibrium with root M_3, where all agents are active. Note that in both examples where there exists a perception-consistent equilibrium $\mathbf{x}^*$ with $N_+(x^*) = N$, $\mathbf{x}^*$ is the stable equilibrium. This is actually always true:

Corollary 1. *In a simple network, if there exists a perception-consistent equilibrium* $\mathbf{x}^*$ *with* $N_+(\mathbf{x}^*) = N$*, then* $\mathbf{x}^*$ *is the asymptotically stable PCE.*

In summary, for any simple network, we can determine the unique stable perception-consistent equilibrium. First, we establish the $\succeq$-ordering as defined in Section 3.3. Second, we determine the different perception-consistent equilibria by checking, for each community, if its spectral radius is strictly greater than that of the communities that dominate it as per the $\succeq$-ordering (Proposition 1). For each PCE, we can ascertain the effort of each agent, which is equal to her eigenvector centrality (Theorem 2) in the set of active agents. Finally, the unique stable perception-consistent equilibrium in the network is the PCE for which the corresponding root has the same largest eigenvalue as the whole network (Theorem 3).

4 Policy interventions

4.1 Adding links

We now consider the policy implications of our model. We start with the simplest intervention: *Given a network and its unique stable perception-consistent equilibrium, what would happen if we added a link between two agents?*

Consider networks with a unique dominant component. We only focus on *stable* perception-consistent equilibria, that is, equilibria for which the largest eigenvalue of the root is equal to that of the whole network (Theorem 3). We examine whether adding a link from individual i to individual j has an impact on individual efforts. If we do not make additional assumptions on the payoff structure, adding links does not have a clear impact on the aggregate equilibrium effort.[27] Hence we only obtain results on relative efforts in the general model:

Proposition 3. *Pick a simple network $(N, \mathbf{G})$ with $\mathbf{x}^*$ being the asymptotically stable perception-consistent equilibrium. Suppose that $i, j \in N_+(\mathbf{x}^*)$, and $g_{ij} = 0$. Let $\widehat{\mathbf{G}}$ be the network obtained from $\mathbf{G}$ by adding a link from i to j. Then, $\widehat{\mathbf{G}}$ admits an asymptotically stable perception-consistent equilibrium $\widehat{\mathbf{x}}^*$ that has the following properties:*

(*i*) $N_+(\widehat{\mathbf{x}}^*) \subseteq N_+(\mathbf{x}^*)$,

(*ii*) *for any $k \in N$, we have* $\frac{\widehat{x}_i^*}{x_i^*} > \frac{\widehat{x}_k^*}{x_k^*}$.

Since both players i and j initially belong to the set of active agents, there is no reason why adding a link between them should induce a positive effort from an initially inactive agent. Indeed, the spectral radius of the subgraph of inactive agents remains the same while the spectral radius of the set of initially active agents can only increase. In other words, $N_+(\widehat{\mathbf{x}}^*) \subseteq N_+(\mathbf{x}^*)$. This is part ($i$). Additionally, agent i becomes more central relatively to other agents when adding a link from i to j. This implies that the relative effort increase is maximal for agent i. This is captured by part (ii) of the proposition. Note that, if $f(\cdot)$ is such that X^* increases with the spectral radius of the graph, then part (ii) of Proposition 3 directly implies that $\hat{x}_i^* > x_i^*$.

4.2 Key players

Another possible intervention involves removing one agent as well as all links from the network. This is known as the *key-player* policy (Zenou, 2016) and is particularly relevant in the crime application (Ballester et al., 2006, 2010) but also in the conflict application (König et al., 2017), because governments want to target these individuals (the key players) in order to reduce total activity X (total crime or total conflict).

Because there is no clear relationship between network density (captured by the spectral radius $\rho(\mathbf{G})$) and total equilibrium effort X^* (see Remark 5), it is difficult to obtain general results of the key player policy. However, in specific cases, such as the Tullock contest function

[27] Observe that, in the general case, X^* is increasing with $\rho(\mathbf{G})$ if $f(X)(-f'(X) - Xf''(X)) + X(f'(X))^2 > 0$. A sufficient condition is that $Xf'(X)$ is non-increasing. However, it is not necessary. For example, in the linear Tullock model, $Xf'(X) = -V/X$. Nevertheless, $f(X)(-f'(X) - Xf''(X)) + X(f'(X))^2) = cV/X^2 > 0$

model,[28] we can derive some results. Proposition D2 in the Online Appendix D.2.1 shows that, in the Tullock model, when removing a player, total effort will never increase. This is because the largest eigenvalue either stays the same or is reduced; the latter decreases total effort. However, the distribution of efforts may be greatly altered, as shown in the following example.

Example 3. Key players and the spread of efforts across neighborhoods for the (linear) Tullock context function
Consider the network displayed in Figure 1(a) (Example 1). We have shown that there is a unique stable perception-consistent equilibrium where the only active agents belong to $\succeq$-maximal community $M_1 = \{1, 2, 3\}$ with $x_1^* = x_2^* = x_3^* = \frac{2V}{9c}$ and thus the total effort is $X^* = \frac{2V}{3c}$.

Let us now remove the active agent 1 from the network as well as all of her links. It is easily verified that the unique stable PCE $\mathbf{x}^{[-1]*}$ is such that *now* $\{2, 3, 4, 5, 6\} \subseteq N_+(\mathbf{x}^*)$, even though the total effort remains the same at $\frac{2V}{3c}$. Indeed, by removing agent 1, the spectral radius of $M_1 = \{1, 2, 3\}$ decreases from 2 to 1 and becomes strictly smaller than the spectral radius of $M_2 = \{4, 5, 6\}$, which is equal to 2. As a result, the only stable PCE is now such that agents 2, 3, 4, 5, and 6 are active. Removing an agent can thus have the *counter-productive effect* of making inactive agents active. In the standard key-player policy (Zenou, 2016), this is not possible since total effort always decreases as *all* agents reduce their individual effort. $\diamond$

4.3 Social mixing

We conclude this section with a brief look at the issue of *social mixing*. To address this issue, we need to depart slightly from our initial model in which there was one (simple) network. Suppose, instead, that we start with two disconnected (simple) networks $(N^1, \mathbf{G}^1)$ and $(N^2, \mathbf{G}^2)$, each of which has a unique stable PCE. As above, we can obtain results only for specific cases. We consider here social mixing in the Tullock contest function model.[29] Think of social mixing as starting with two fully segregated neighborhoods, each endowed with their own resources V^1 and V^2. The key question for the planner is whether merging these two neighborhoods (social mixing) into a connected network $(N, \mathbf{G})$, with $N = N^1 + N^2$, $V = V^1 + V^2$, leads to an increase in total activity and resources.

Proposition D3 in the Online Appendix D.2.2 shows that the total effort in any new stable PCE of the connected network $(N, \mathbf{G})$ is higher than the sum of total efforts in each disconnected neighborhood. Hence, linking the two neighborhoods is beneficial to aggregate effort. On the other hand, the distribution of resources between agents in N^1 and N^2 is less

[28] The same is true for the Cournot model.
[29] Similar results can be obtained for the Cournot model.

clear. Indeed, distribution in the new equilibrium depends on the specific connections that are formed between the two groups. It is therefore possible to have some agents who are worse off following the mixing of the two neighborhoods.

5 Economic implications of our model

5.1 The concept of perceived competition

We would now like to illustrate our results and to highlight our concept of "perceived" competition. Understanding how community ordering works in combination with community densities is crucial to detect who is active in a network as well as how much effort agents exert when they are active. However, these two questions need to be answered separately.

Let us first focus on the question: *Who is active in a network?* Note that, given a community M, either all agents in M are active in the stable equilibrium, or none of them are. In the former case, we will say that the community is active (at the stable equilibrium). For a given community, being active or not will be determined by a combination of the two following ingredients: (i) its relative position with respect to other communities in the network (indeed, the relation $\succeq$ translates into an advantage in terms of competition); and (ii) its density, in terms of the largest eigenvalue of the corresponding sub-network.

Indeed, according to Proposition 1 and Theorem 3, in order to be active at the stable equilibrium, a community must satisfy (at least) *one of the two* following conditions:

(a) it must exhibit the largest spectral radius among all communities and be "hidden" from all other communities, if any, that have the same property;[30]

(b) it must be aware of all communities with the largest spectral radius.[31]

For obvious reasons, being "denser" makes it more likely for a community to satisfy condition (a), while having a better relative position as per $\succeq$-ordering makes condition (b) more likely to hold.

Once the set of active players is established, we can turn to the second question: *How active is an agent among the set of active agents?* Here, the answer is simpler, since the effort level of an active player is determined by her relative position in the sub-network of active players, which is fully captured by her eigenvector centrality. Consequently, the more aware of other active players an agent is, the more active she is. Also, the more "hidden" from other active agents she is, the more active she is. However, this does not mean that removing

[30]This means that a path must not exist from another community with the same spectral radius.
[31]This means a path exists from this community to every community with the largest spectral radius.

links from other active agents will necessarily increase her effort level because, in doing so, it might be that the community is no longer dense enough, rendering this community inactive in equilibrium.

5.2 Dynamic competition: Numerical simulations for the (linear) Tullock contest function

In this section, we illustrate how the dynamics of our game works and how it converges to the unique stable PCE for the Tullock contest function.[32]

To understand the underlying dynamics of our model, consider the network displayed in Figure 1(a). As stated above, each agent i best replies to the observed W_i^{t-1}. When $\epsilon = 0.1$, this is described by the following equation (see (18)):

$$x_i^t = \frac{9}{10} x_i^{t-1} + \frac{1}{10} \epsilon Br_i(\mathbf{x}^{t-1}), \tag{20}$$

Indeed, at time t, each agent i does not know V or the efforts of all agents in the network. She only observes W_i^{t-1}, her own payoff from the previous period, and her neighbors' effort.

By taking the initial conditions $x_i^0 = 0.1$ for all $i = 1, \dots, 6$, we obtain Figure 3.[33] The

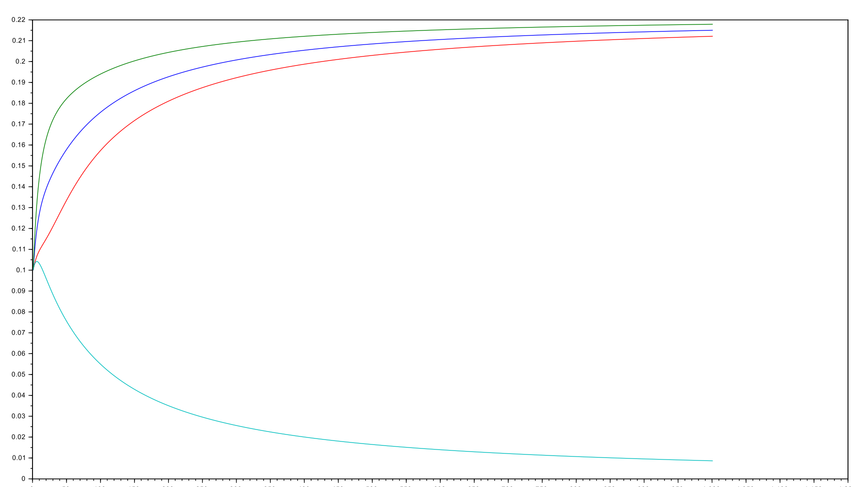


Figure 3: Convergence to the unique stable PCE in the network of Figure 1(a)

green, blue, and red curves correspond to the effort of agents 2, 1, and 3, respectively, while

[32] A similar exercise can be done for the Cournot competition model.

[33] For simplicity, in all numerical simulations, we take $V/c = 1$.

the cyan curve corresponds to that of agent 4, 5, or 6. By using (20), agents best respond to their neighbors' efforts and their perceived resources up to the point when this dynamic process converges to the unique stable PCE in which agents 1, 2, and 3 make $2V/9c = 2/9$ effort while agents 4, 5, and 6 exert zero effort.[34] At this equilibrium, $W_i = \frac{x_i^* + \sum_{j \in \mathcal{N}_i} x_j^*}{\sum_{j \in N} x_j^*}$, for each agent $i \in N$. In other words, perceived and real resources are equal. Observe that agents 4, 5, and 6 start with the wrong perception that there are large local resources; over time, they observe that these resources decrease and thus reduce their effort until it reaches zero, since there are no resources left to grab.

If we now turn to the network in Figure 1(b) and consider the same dynamical system given by (20), starting with the same initial conditions, we obtain Figure 4. Here, agents

Figure 4: Convergence to the unique stable PCE in the network of Figure 1(b)

adjust their effort and reach the interior equilibrium in which $x_1^* = 1/9$ (blue), $x_2^* = 7/45$ (green), $x_3^* = 2/15$ (red), and $x_i^* = 4/45$ for $i = 4, 5, 6$ (cyan). Agents 4, 5, and 6 do not end up making zero effort because W_i^0; their perception of local resources at $t = 0$ is not too far from the reality and thus they slightly change their effort over time.

These two examples illustrate the fact that local perceived resources vary over time, which makes agents change their effort in order to best reply to what they observe (that is, their local resources and their neighbors' effort from the previous period) at each period of time.

In Section 3.4, we showed that, in the network of Figure 1(b), there were two PCEs: one in which all agents are active and one in which only agents 1, 2, and 3 are active, with

[34]This is independent of the initial conditions.

$x_1^* = x_2^* \sim 0.1715$ and $x_3^* \sim 0.2426$. We showed that the latter was asymptotically *unstable*. To understand this, let us start with initial conditions very close to this equilibrium, that is, $x_1^0 = x_2^0 = 0.1715$, $x_3^0 = 0.2426$, and $x_i^0 = 0.001$. Figure 5 shows the dynamics of the system and the convergence to the unique stable PCE in which all agents are active. This clearly illustrates that the equilibrium in which agents 4, 5, and 6 are inactive is unstable. Moreover, since the initial conditions are very close from another equilibrium, it takes some time for agents to adjust their effort and to converge back to the unique stable PCE discussed above. The fact that this equilibrium is unstable does not change the fact that the drift is very small around it and, thus, the dynamical system moves very slowly.

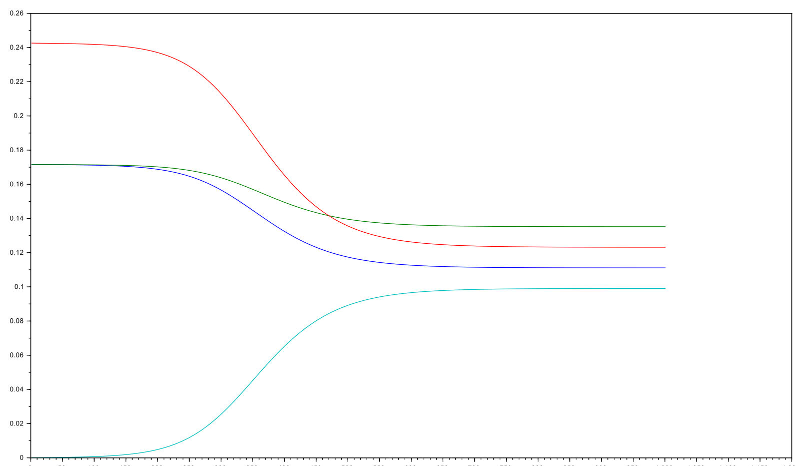


Figure 5: Convergence to the unique stable PCE in the network of Figure 1(b) when starting from the unstable PCE

6 Conclusion

In this paper, we consider a competitive aggregative game in which agents have an imperfect knowledge of their competitors. We model this imperfect knowledge by a network by assuming that each agent only has information on the activities of their *perceived* competitors. We develop a new concept of equilibrium, which we refer to as perception-consistent equilibrium (PCE). Each agent chooses an effort level that maximizes her perceived utility. However, at the PCE, effort levels of all agents have to be consistent: for each agent, her *perceived subjective* utility has to be equal to her *objective* payoff.

First, for a large class of competitive aggregative games, we show that there always exists a PCE and that agents' efforts represent an ordinal centrality. Second, for a specific class of aggregative games, we demonstrate that, at any PCE, the effort of an active agent is proportional to her eigenvector centrality. We then introduce the concept of community: within each community, all agents have the same propensity to exert positive effort. We construct an ordering of the communities in terms of active agents. Agents in the better ranked communities are more likely to be active because few agents are "aware" of them, and agents in these communities are more likely to be active. Then, we determine all perception-consistent equilibria by comparing the spectral radius of these communities and that of their adjunct set (i.e., agents that can reach them through a path) in the whole network. We show that, to be active in equilibrium, one needs to belong either to a $\succeq$-maximal community (because few agents are aware of these individuals) or to communities of large size. Finally, we demonstrate that there is a unique stable PCE in each network. This PCE corresponds to the community that has the largest spectral radius in the network. We illustrate all our results with two well-known applications of aggregative games: Tullock contest function and Cournot oligopoly competition.

Lastly, we study the policy implications of our model. We show that adding a link can reduce the number of active agents in the network because it creates a new path that makes some agents more likely to be reached; in turn, this may lower their status in terms of community. We also study the key-player policy and show that, by removing an agent from the network, we may make several inactive agents active. Further, we examine social mixing by merging two different disconnected networks, highlighting that total activity is higher than the sum of total activity in each network.

In many real-world situations, agents are not aware of the full set of agents with whom they are in competition and thus only take into account their perceived competitors when deciding which actions to take. In this paper, we shed some light on this issue by foregrounding the importance of individual network position and the community to which each agent belongs. More generally, we believe that the concept of "perceived" competition is important to understand and explain many situations in which competition is not perceived as reciprocal and agents only care about their local competitors, even though competition is global.

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Appendix: Proofs of all results in the main text

Proof of Theorem 1. If the game is a competitive aggregative game as per Definition 2, the map $Br_i : \mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\} \to \mathbb{R}_+$ given by

$$Br_i(\mathbf{x}) := Argmax_{b_i \geq 0}\ h\left(b_i, \frac{X}{x_i + \sum_j g_{ij}x_j}\left(b_i + \sum_j g_{ij}x_j\right)\right)$$

is single-valued. A PCE is a fixed point of the map

$$Br : \mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\} \mapsto (Br_1(\mathbf{x}), ..., Br_n(\mathbf{x}))\,.$$

However, there is a regularity issue in zero, which prevents us from directly using a fixed point theorem on the action profiles space (it is not closed).
Let $\overline{X}$ be the unique positive real number such that $\frac{\partial h}{\partial x}(0, \overline{X}) = 0$. Pick $\delta < 1$ and choose $\underline{X} > 0$ small enough so that, for any $0 < x \leq \underline{X}$, we have $\frac{-X\frac{\partial h}{\partial X}(x,\underline{X}))}{x\frac{\partial h}{\partial x}(x,\underline{X})} < 1 + \delta$. Let $K := \left\{\mathbf{x} :\ X \in [\underline{X}, \overline{X}]\right\}$. This set is compact and convex. Moreover Br is continuous on K. Hence $Br : K \to K$ admits a fixed point $\mathbf{x}^*$. We now need to check that $\mathbf{x}^*$ is a PCE.

First, note that we have $\frac{\partial h}{\partial x}(x_i, \overline{X}) \leq \frac{\partial h}{\partial x}(0, \overline{X}) = 0$. Consequently

$$\frac{\partial h}{\partial x}(x_i, X) + \frac{X}{x_i + (\mathbf{G}\mathbf{x})_i}\frac{\partial h}{\partial X}(x_i, X) < 0, \text{ if } \mathbf{x} \text{ is s.t. } \ X = \overline{X}.$$

As a result, $X^* < \overline{X}$.
Moreover, by construction of $\underline{X}$, we have that $\frac{\partial h}{\partial x}(x, \underline{X}) + \frac{\underline{X}}{(1+\delta)x}\frac{\partial h}{\partial X}(x, \underline{X}) > 0$ for any $0 < x \leq \underline{X}$. Now suppose that $\mathbf{x}^*$ is such that $X^* = \underline{X}$. Note that, if $(\mathbf{G}\mathbf{x}^*)_i > 0$ then $x_i^* > 0$, because

$$\frac{\partial h}{\partial x}(x_i, \underline{X}) + \frac{X^*}{(x_i + (\mathbf{G}\mathbf{x}^*)_i}\frac{\partial h}{\partial X}(x_i, \underline{X}) \geq \frac{\partial h}{\partial x}(x_i, \underline{X}) + \frac{\underline{X}}{(1+\delta)x_i}\frac{\partial h}{\partial X}(x_i, \underline{X}) > 0$$

for small enough x_i. Let now i be such that $(\mathbf{G}\mathbf{x}^*)_i \geq x_i^* > 0$ (such an index i exists by assumption). We have

$$\frac{\partial h}{\partial x}(x_i^*, \underline{X}) + \frac{\underline{X}}{x_i + (\mathbf{G}\mathbf{x}^*)_i}\frac{\partial h}{\partial X}(x_i^*, \underline{X}) \quad > \quad \frac{\partial h}{\partial x}(x_i^*, \underline{X}) + \frac{\underline{X}}{(1+\delta)x_i^*}\frac{\partial h}{\partial X}(x_i^*, \underline{X}) > 0.$$

This contradicts the fact that $x_i^* = Br_i(\mathbf{x}^*)$. Hence $X^* > \underline{X}$. Finally $X^* \in Int\,(K)$.

If $x_i^* > 0$, then we necessarily have $\frac{\partial h}{\partial x}(x_i^*, X^*) + \frac{X^*}{x_i^* + (\mathbf{G}\mathbf{x}^*)_i}\frac{\partial h}{\partial X}(x_i^*, X^*) = 0$, because there exists $\epsilon > 0$ such that $(x_i, x_{-i}^*) \in K$ for all $x_i \in]x_i^* - \epsilon, x_i^{+}\epsilon[$. Hence $x_i^* = Br_i(\mathbf{x}^*)$. Let now i be such that $x_i^* = 0$. Then, $\frac{\partial h}{\partial x}(0, X^*) + \frac{X^*}{(\mathbf{G}\mathbf{x}^*)_i}\frac{\partial h}{\partial X}(0, X^*) \leq 0$ and $0 = Br_i(\mathbf{x}^*)$.

Let now $\mathbf{x}^*$ be a PCE such that $x_i^* \geq x_j^* > 0$. Then,

$$\frac{\partial h}{\partial x}(x, X^*) + \frac{X^*}{x + (\mathbf{G}\mathbf{x}^*)_i} \frac{\partial h}{\partial X}(x, X^*) > 0 \text{ for any } x < x_i^*.$$

On the other hand, we have

$$\frac{\partial h}{\partial x}(x_j^*, X^*) + \frac{X^*}{x_j^* + (\mathbf{G}\mathbf{x}^*)_j} \frac{\partial h}{\partial X}(x_j^*, X^*) = 0$$

and thus

$$\frac{X^*}{x_j^* + (\mathbf{G}\mathbf{x}^*)_j} \frac{\partial h}{\partial X}(x_j^*, X^*) \leq \frac{X^*}{x_j^* + (\mathbf{G}\mathbf{x}^*)_i} \frac{\partial h}{\partial X}(x_j^*, X^*).$$

Since $\frac{\partial h}{\partial X} < 0$, this implies that $(\mathbf{G}\mathbf{x}^*)_j \leq (\mathbf{G}\mathbf{x}^*)_i$, with equality if and only if $x_i^* = x_j^*$. This proves that $x_i^* \geq x_j^*$ if and only if $\sum_{l \in \mathcal{N}_i} x_l^* \geq \sum_{k \in \mathcal{N}_j} x_k^*$. □

Proof of Theorem 2.

• We first show that, if $\mathbf{x}^*$ is a PCE, then

$$\mathbf{G}\mathbf{x}^* = \frac{-f(X^*) - f'(X^*)X^*}{f(X^*)} \mathbf{x}^* \tag{21}$$

Since $X^* > 0$ there exists i with $x_i^* > 0$. Using the assumptions on f, it necessarily implies that x_i^* satisfies the first-order condition

$$f(X^*) + \frac{x_i^*}{x_i^* + (\mathbf{G}\mathbf{x}^*)_i} f'(X^*) X^* = 0.$$

Since $f'(X^*)X^* < 0$, we necessarily have $f(X^*) > 0$, and thus

$$x_i^* > 0 \text{ implies that } (\mathbf{G}\mathbf{x}^*)_i = \left(\frac{-f(X^*) - f'(X^*)X^*}{f(X^*)} \right) x_i^*. \tag{22}$$

Recall that, if $(\mathbf{G}\mathbf{x}^*)_i > 0$, then $x_i^* > 0$. Let $\lambda(X) := \frac{-f(X) - f'(X)X}{f(X)}$. Note that we necessarily have $\lambda(X^*) \geq 0$. We distinguish two cases:

a) Suppose that $\lambda(X^*) = 0$. If $x_i^* = 0$ then $(\mathbf{G}\mathbf{x}^*)_i = 0$. Moreover, if $x_i^* > 0$ then equality (22) shows that $(\mathbf{G}\mathbf{x}^*)_i = 0$ still holds. Hence (21) holds

b) Suppose now that $\lambda(X^*) > 0$. If $x_i^* = 0$ we necessarily have $(\mathbf{G}\mathbf{x}^*)_i = 0$ and (21). holds.

• We now prove the reverse implication. Suppose that $\mathbf{x}^* \in \mathbb{R}_+^n$ is different from zero and satisfies identity (21), with $\lambda(X^*) > 0$. For each agent i for whom $(\mathbf{G}\mathbf{x}^*)_i > 0$, x_i^* satisfies the first-order condition associated to the maximization problem (i) of the PCE definition, with $W_i = \frac{x_i^* + (\mathbf{G}\mathbf{x}^*)_i}{X^*}$. Meanwhile, for each agent i for whom $(\mathbf{G}\mathbf{x}^*)_i = 0$, $x_i^* = 0$ solves the optimization problem (i) of the PCE definition, with $W_i = 0$.

Now, if $\lambda(X^*) = 0$ then identity (21) means that $(\mathbf{G}\mathbf{x}^*)_i = 0$ for all i. Take any profile with X^* such that $\lambda(X^*) = 0$ and $(\mathbf{G}\mathbf{x}^*)_i = 0$ for all i. Then, for any i and any $x_i^* \geq 0$, the map $b_i \mapsto b_i f\left(\frac{b_i}{x_i^*} X^*\right)$ is maximal in $b_i = x_i^*$. Hence $\mathbf{x}^*$ is a PCE. This proves the reverse implication. $\square$

Proof of Remark 4. Since Remark 4 is a special case of Theorem 2, we will prove Remark 4 as the following corollary of Theorem 2.

Corollary 2. *Let $(N, \mathbf{G})$ be a strongly connected network. Then, there exists a unique Perception-Consistent Equilibrium.*

Proof. Suppose that $(N, \mathbf{G})$ is a strongly connected network. Then $\mathbf{G}$ is irreducible and, by Perron-Frobenius Theorem, there exists a positive eigenvector $\mathbf{y}$ associated to $\rho(\mathbf{G})$. Moreover any non-negative eigenvector of $\mathbf{G}$ is a multiple of $\mathbf{y}$. By Theorem 2, $\mathbf{x}^*$ is a PCE if and only if it is a non-negative eigenvector of $\mathbf{G}$, associated to eigenvalue $\frac{-f(X^*) - f'(X^*)X^*}{f(X^*)}$. Hence $\mathbf{x}^*$ is a PCE if and only if $\mathbf{x}^*$ is a multiple of $\mathbf{y}$ and $\rho(\mathbf{G}) = \frac{-f(X^*) - f'(X^*)X^*}{f(X^*)}$. Such a vector exists and is uniquely defined. $\square$

Proof of Proposition 1. First note that if $\mathbf{x}$ is a PCE with root M then its restriction to $\bar{M}$ is a positive eigenvector of $\mathbf{G}_{\bar{M}}$. By definition of $\bar{M}$, the matrix $\mathbf{G}_{\bar{M}}$ admits a Frobenius normal form as follows:

$$\mathbf{G}_{\bar{M}} = \begin{bmatrix} A_1 & A_{12} & ... & ... & A_{1s+1} \\ 0 & A_2 & A_{23} & ... & A_{2s+1} \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} \\ 0 & ... & ... & 0 & A_{s+1} \end{bmatrix}, \text{ with } A_{s+1} = \mathbf{G}_M.$$

Note that the set $V := \cup_{i=1}^s V_i$ is closed and, by definition of $\bar{M}$ we necessarily have $\{M' \in \mathcal{C}(\mathbf{G}) : M' \subset V\} = \{M' \in \mathcal{C}(\mathbf{G}) : M' \succ M\}$. Hence

$$\rho(\mathbf{G}_V) = \max_{i=1,\dots,s} \rho(A_i) = \max_{M' \in \mathcal{C}(\mathbf{G}):\, M' \succ M} \rho(\mathbf{G}_{M'}).$$

If $\rho(\mathbf{G}_M) > \max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M} \rho(\mathbf{G}_{M'})$ then

$$\rho(A_{s+1}) = \rho(\mathbf{G}_M) > \max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M} \rho(\mathbf{G}_{M'}) = \max_{i=1,\dots,s} \rho(A_i),$$

Consequently, we are in the conditions of Lemma B7. Thus $\mathbf{G}_{\bar{M}}$ then admits a unique positive eigenvector $\mathbf{y} = (y_i)_{i\in\bar{M}}$, such that $\sum_{i\in\bar{M}} y_i$ solves the equation $\frac{-f(X)-Xf'(X)}{f(X)} = \rho$. Let then $\mathbf{x}$ be defined as $x_i = y_i$ if $i \in \bar{M}$ and $x_i = 0$ if $i \in N \setminus \bar{M}$. By construction, $\mathbf{x}$ is a PCE with root M and there can be no other one.
Now suppose that $\rho(\mathbf{G}_M) \le \max_{M'\in\mathcal{C}(\mathbf{G}):M'\succ M} \rho(\mathbf{G}_{M'})$. Then

$$\rho(A_{s+1}) = \rho(\mathbf{G}_M) \le \max_{M'\in\mathcal{C}(\mathbf{G}):M'\succ M} \rho(\mathbf{G}_{M'}) = \max_{i=1,\dots,s} \rho(A_i),$$

meaning that $\mathbf{G}_{\bar{M}}$ admits no positive eigenvector, by Lemma B4. This concludes the proof. □

Proof of Proposition 2. Suppose that $\mathbf{x}$ does not admit a root. Then, by Propositions E4 and E5, we have $N_+(\mathbf{x}) = \cup_{i=1}^n \bar{M}_i$, with $n \ge 2$, $M_1, \dots, M_n$ being distinct elements of $\mathcal{C}(\mathbf{G})$ and

$$\rho\left(\mathbf{G}_{M_i}\right) = \rho > \max_{M'\in\mathcal{C}(\mathbf{G}):M'\succ M_i} \rho(\mathbf{G}_{M'}),\ \ \forall i = 1, \dots, n,$$

which contradicts the fact that $\mathbf{G}$ is simple. □

Proof of Theorem 3. Let us first show that the system (19) is well-behaved on the set

$$\mathbf{S} := \{\mathbf{x} \ne \mathbf{0} :\ x_i \ge 0\ \forall i,\ f(X) > 0\} \tag{23}$$

in the sense that, for any initial condition in $\mathbf{S}$, there exists a unique solution $(\mathbf{x}(t))_{t\ge 0}$ which forever remains in $\mathbf{S}$.

Given a PCE $\mathbf{x}^*$, we have $\mathbf{x}^* \in \mathbf{S}$, because the expression $\frac{-f(X)-Xf'(X)}{f(X)}$ is either undefined or negative when $f(X) \le 0$. As a consequence $\mathbf{S}$ contains all the relevant states of the problem we consider. We denote by $(\phi(\mathbf{x}, t))_{\mathbf{x}\in\mathbf{S}, t\ge 0}$ the semi-flow associated to (19) on $\mathbf{S}$. Namely $\phi(\mathbf{x}, t)$ is equal to the position of the (unique) solution of (19) starting in $\mathbf{x}$.

Lemma 1. *System (19) induces a semiflow on* $\mathbf{S}$.

Proof. We need to check that the vector field B points inward on the boundary of $\mathbf{S}$. Suppose that $\mathbf{x} \in \mathbf{S}$, with $f(X) = 0$. Then $Br_i(x_{-i}) = Argmax_{b_i\ge 0}\, b_i f\left(X \frac{b_i+(\mathbf{Gx})_i}{x_i+(\mathbf{Gx})_i}\right)$. Since $f(X) = 0$, this map is equal to zero in $b_i = x_i$, negative when $b_i > x_i$ and positive when $b_i < x_i$. Hence $Br_i(x_{-i}) < x_i$ for all i such that $x_i > 0$ and $\dot{X} < 0$ in $\mathbf{x}$. ■

For the Tullock model, on the positively invariant set $\mathbf{S}$, system (19) writes

$$\dot{x}_i(t) = -x_i(t) - (\mathbf{Gx})_i(t) + \left(\frac{V}{cX(t)}(\mathbf{Gx})_i(t)\,(x_i(t) + (\mathbf{Gx})_i(t))\right)^{1/2} \text{ for } i = 1, \dots, N.$$

For the Cournot model we have

$$\dot{x}_i(t) = -x_i(t) - \frac{1}{2}(\mathbf{G}\mathbf{x})_i(t) + \frac{1}{2}(\bar{\alpha} - c)\frac{x_i(t) + (\mathbf{G}\mathbf{x})_i(t)}{X(t)} \text{ for } i = 1, ..., N.$$

The following result will be useful to prove that a point is not asymptotically stable. It directly follows from the definition of asymptotic stability.

Lemma 2. *Suppose that there exists an open neighborhood U_0 of $\mathbf{x}^*$ with the property that, for any open neighborhood U of $\mathbf{x}^*$ and any $T > 0$, there exists $\mathbf{x} \in U$ such that $\phi(\mathbf{x}, t) \notin U_0$, for any $t \geq T$. Then $\mathbf{x}^*$ is not asymptotically stable.*

Lemma 3. *Let $\mathbf{x}^*$ be a PCE such that $\rho(\mathbf{x}^*) < \rho(\mathbf{G})$. Then $\mathbf{x}^*$ is not asymptotically stable.*

Proof. Recall that $\mathbf{x}^*$ is an eigenvector of $\mathbf{G}$, associated to eigenvalue $\rho(\mathbf{x}^*)$, given by

$$\rho(\mathbf{x}^*) = \frac{-f(X^*) - f'(X^*)X^*}{f(X^*)}.$$

In what follows, let $\rho^* := \rho(\mathbf{x}^*)$ and $\rho := \rho(\mathbf{G})$.

Let $M^* \in \mathcal{C}$ be the root of $\mathbf{x}^*$. For any $M \succeq M^*$ we necessarily have $\rho(\mathbf{G}_M) < \rho^*$. Let $C := N \setminus \bar{M}^*$. By construction, $\mathbf{G}_C$ is a nonnegative matrix with largest eigenvalue ρ, and we call $\mathbf{u}$ the eigenvector associated to ρ, whose components sum to one.
Define $\mathbf{a}^\epsilon$ as follows:

$$\mathbf{a}_i^\epsilon = \epsilon u_i \; \forall i \in C, \quad \text{and} \quad \mathbf{a}_i^\epsilon = x_i^* \; \forall i \in \bar{M}^*,$$

where ϵ is a positive number. We claim that, for any $i \in C$, $B_i(\mathbf{a}^\epsilon) > 0$. By definition of C, we have $g_{ij} = 0$ for any $i \in C$ and any $j \in \bar{M}^*$. Consequently

$$(\mathbf{G}\mathbf{a}\epsilon)_i = \sum_{j \in C} g_{ij}\mathbf{a}_j^\epsilon = (\mathbf{G}_C\mathbf{a}^\epsilon)_i = \rho\epsilon u_i.$$

Define, for $i \in C$ and $b_i > 0$,

$$H_i^\epsilon(b_i) := g\left(\frac{A^\epsilon(b_i + \rho\epsilon u_i)}{(1+\rho)\epsilon u_i}\right) + \frac{b_i A^\epsilon}{(1+\rho)\epsilon u_i} g'\left(\frac{A^\epsilon(b_i + \rho\epsilon u_i)}{(1+\rho)\epsilon u_i}\right) - c.$$

where $A(\epsilon) = \sum_i a(\epsilon)_i = X^* + \epsilon$. Then $Br_i(\mathbf{a}^\epsilon)$ is the unique zero of Br_i, and $Br_i(b_i) < 0 \; \forall b_i > Br_i(\mathbf{a}^\epsilon)$ (resp. $Br_i(b_i) > 0 \; \forall b_i < Br_i(\mathbf{a}^\epsilon)$). We have

$$H_i^\epsilon(a_i^\epsilon) = f(A^\epsilon) + \frac{A^\epsilon}{1+\rho} f'(A^\epsilon) - c.$$

Since $\rho > \rho^* = \frac{c-f(X^*)-f'(X^*)X^*}{f(X^*)-c}$, we have

$$f(X^*) + \frac{X^*}{1+\rho} f'(X^*) - c > 0.$$

Hence, for small enough $\epsilon > 0$, we have $Br_i^\epsilon(a_i^\epsilon) > 0$, and thus $\mathbf{a}_i^\epsilon < Br_i(\mathbf{a}^\epsilon)$, i.e. $B_i(\mathbf{a}^\epsilon) > 0$. This concludes the proof that $\mathbf{x}^*$ is not asymptotically stable for dynamics (19).

We now prove the following lemma, which completes the proof of Theorem 3:

Lemma 4. *Let $\mathbf{x}^*$ be a PCE such that $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$. Then $\mathbf{x}^*$ is asymptotically stable.*

Proof. We prove this lemma in the particular case of Tullock contest. The proof is similar in the case of Cournot contest. However, writing a general proof without relying on the explicit formulation of map $f(\cdot)$ would be extremely tedious and lengthy. We believe that illustrating the spirit of the proof on a concrete example is more illminating. If $\mathbf{B}(.)$ in (19) is differentiable in an open neighborhood of a PCE, then a simple sufficient condition for an interior equilibrium to be asymptotically stable is that the eigenvalues of the Jacobian matrix of $B(.)$, evaluated at $\mathbf{x}^*$, have negative real parts. Unfortunately the map $\mathbf{B}$ is not differentiable at a non-interior PCE, and we then cannot use this result. However we can compute the directional derivatives of $\mathbf{B}$ at any PCE: let $\mathbf{u} \neq \mathbf{0}$ be such that $u_i \geq 0\ \forall i$. Then the directional derivative of $\mathbf{B}$ in $\mathbf{x}^*$ along $\mathbf{u}$, namely the quantity

$$D_{\mathbf{u}}\mathbf{B}(\mathbf{x}^*) := \lim_{h \to 0, h>0} \frac{\mathbf{B}(\mathbf{x}^* + h\mathbf{u})}{h}$$

exists, and we can compute it: given $h > 0$,

$$B_i(\mathbf{x}^*+h\mathbf{u}) = -(\mathbf{x}_i^*+hu_i+(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)+\left(\frac{V}{c(X^*+hU)}(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)(x_i^*+hu_i+(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)\right)^{1/2}$$

The term in the square root can be written

$$\begin{aligned}
& \frac{V}{cX^*}\left(1 - h\frac{U}{X^*}\right)\left[(\mathbf{G}\mathbf{x}^*)_i(x_i^* + (\mathbf{G}\mathbf{x}^*)_i) + h\left[(\mathbf{G}\mathbf{x}^*)_i(u_i + (\mathbf{G}\mathbf{u})_i) + (\mathbf{G}\mathbf{u})_i(x_i^* + (\mathbf{G}\mathbf{x}^*)_i)\right]\right] + \mathcal{O}(h^2) \\
= & \frac{V}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^* + (\mathbf{G}\mathbf{x}^*)_i)\left(1 - h\frac{U}{X^*}\right)\left[1 + h\left[\frac{u_i + (\mathbf{G}\mathbf{u})_i}{x_i^* + (\mathbf{G}\mathbf{x}^*)_i} + \frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right] + \mathcal{O}(h^2) \\
= & \frac{V}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^* + (\mathbf{G}\mathbf{x}^*)_i)\left[1 + h\left[-\frac{U}{X^*} + \frac{u_i + (\mathbf{G}\mathbf{u})_i}{x_i^* + (\mathbf{G}\mathbf{x}^*)_i} + \frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right] + \mathcal{O}(h^2)
\end{aligned}$$

Observing that $(\frac{V}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^* + (\mathbf{G}\mathbf{x}^*)_i))^{1/2} = x_i^* + (\mathbf{G}\mathbf{x}^*)_i$, the square root of the above quantity is equal to

$$(x_i^* + (\mathbf{G}\mathbf{x}^*)_i)\left[1 + \frac{h}{2}\left[\frac{-U}{X^*} + \frac{u_i + (\mathbf{G}\mathbf{u})_i}{x_i^* + (\mathbf{G}\mathbf{x}^*)_i} + \frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right] + \mathcal{O}(h^2)$$

Hence, since $(x_i^* + (\mathbf{G}\mathbf{x}^*)_i) = \frac{V}{V-cX^*}x_i^*$, we obtain

$$\begin{aligned} B_i(\mathbf{x}^* + h\mathbf{u}) &= -(hu_i + h(\mathbf{G}\mathbf{u})_i) + \frac{h}{2}\left[\frac{-UV}{X^*(V-cX^*)}x_i^* + (u_i + (\mathbf{G}\mathbf{u})_i) + \frac{V}{cX^*}(\mathbf{G}\mathbf{u})_i\right] + \mathcal{O}(h^2) \\ &= \frac{h}{2}\left[\frac{-UV}{X^*(V-cX^*)}x_i^* - u_i + \frac{V-cX^*}{cX^*}(\mathbf{G}\mathbf{u})_i\right] + \mathcal{O}(h^2) \end{aligned}$$

Consequently

$$\lim_{h\to+\infty, h>0} \frac{B_i(\mathbf{x}^* + h\mathbf{u})}{h} = \frac{1}{2}\left[\frac{-UV}{X^*(V-cX^*)}x_i^* - u_i + \frac{V-cX^*}{cX^*}(\mathbf{G}\mathbf{u})_i\right] = \frac{1}{2}\left(\mathbf{D}F(\mathbf{x}^*)\mathbf{u}\right)_i,$$

which proves that

$$D_{\mathbf{u}}\mathbf{B}(\mathbf{x}^*) = \frac{1}{2}\left(-I_N + \frac{1+\rho(\mathbf{x}^*)}{X^*}L(\mathbf{x}^*) + \frac{1}{\rho(\mathbf{x}^*)}\mathbf{G}\right)\cdot\mathbf{u},$$

where $\mathbf{L}(\mathbf{x}^*)$ is the matrix where every column is equal to $\mathbf{x}^*$.

Let $\mathbf{D}(\mathbf{x}^*) := \frac{1}{2}\left(-I_N + \frac{1+\rho(\mathbf{x}^*)}{X^*}L(\mathbf{x}^*) + \frac{1}{\rho(\mathbf{x}^*)}\mathbf{G}\right)$. We first show that all eigenvalues of $\mathbf{D}(\mathbf{x}^*)$ have a negative real part. Suppose that $\mathbf{D}(\mathbf{x}^*)\cdot\mathbf{u} = \lambda\cdot\mathbf{u}$, with $\mathbf{u}\neq 0$. Call $U := \sum_{i\in N} u_i$. Then we have

$$-\mathbf{u} - \frac{1+\rho}{X^*}U\mathbf{x}^* + \frac{1}{\rho}\mathbf{G}\mathbf{u} = 2\lambda\mathbf{u}$$

which gives

$$\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)\mathbf{u} = -\frac{1+\rho}{X^*(1+2\lambda)}U\mathbf{x}^*.$$

Suppose that $Re(\lambda) > 0$ or that λ is pure imaginary. Then $|1+\lambda| > 1$ and the matrix $\mathbf{G}/(\rho(1+2\lambda))$' spectral radius is strictly smaller than one. As a consequence $\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}$ is invertible and

$$\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)^{-1} = \sum_{p=0}^{+\infty}\frac{1}{\rho^p(1+2\lambda)^p}\mathbf{G}^p.$$

Consequently

$$\begin{aligned} \mathbf{u} &= -\frac{1+\rho}{X^*(1+2\lambda)}U\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)^{-1}\mathbf{x}^* \\ &= -\frac{1+\rho}{X^*(1+2\lambda)}U\sum_{p=0}^{+\infty}\frac{1}{\rho^p(1+2\lambda)^p}\mathbf{G}^p\mathbf{x}^* \\ &= -\frac{1+\rho}{X^*(1+2\lambda)}U\sum_{p=0}^{+\infty}\frac{1}{(1+2\lambda)^p}\mathbf{x}^* \\ &= -\frac{1+\rho}{2X^*\lambda}U\mathbf{x}^* \end{aligned}$$

Since $\mathbf{u} \neq 0$, this equality implies that $U \neq 0$ and summing the coordinates of $\mathbf{u}$ we obtain that $2\lambda = -(1+\rho) < 0$, a contradiction.
Suppose now that $\lambda = 0$. Then we have

$$\left(\mathbf{I}_N - \frac{1}{\rho}\mathbf{G}\right)\mathbf{u} = -\frac{1+\rho}{X^*}U\mathbf{x}^*.$$

Suppose that $U \neq 0$. Then, multiplying both sides of the equality by $\sum_{k=0}^{K} \frac{1}{\rho^k}\mathbf{G}^k$, we obtain the identity

$$\left(\mathbf{I}_N - \frac{1}{\rho^{K+1}}\mathbf{G}^{K+1}\right)\mathbf{u} = -\frac{1+\rho}{X^*}U\sum_{k=0}^{K}\frac{1}{\rho^k}\mathbf{G}^k\mathbf{x}^* = -\frac{1+\rho}{X^*}UK\mathbf{x}^*$$

The modulus of the left-hand is bounded above by $2|\mathbf{u}|$, while the modulus of the right-hand side term grows to infinity with K, which is a contradiction. Hence $U = 0$. This means that

$$\mathbf{G}\mathbf{u} = \rho\mathbf{u},$$

i.e. that $\mathbf{u}$ is in fact an eigenvector associated to the largest eigenvalue of $\mathbf{G}$. Since $\sum_i u_i = 0$, this contradicts the fact that $(N, \mathbf{G})$ is a simple network.

We proved that the real part of every eigenvalue of $\mathbf{D}F(x^*)$ is strictly negative.

As we proved above, for $\mathbf{x} \in \mathbf{X}$, we have

$$\mathbf{B}(\mathbf{x}) = \mathbf{D}(\mathbf{x}^*) \cdot (\mathbf{x} - \mathbf{x}^*) + \|\mathbf{x} - \mathbf{x}^*\|^2 g(\|\mathbf{x} - \mathbf{x}^*\|)$$

Denote by $(\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_P, ..., \lambda_P)$ the eigenvalues of $\mathbf{D}(\mathbf{x}^*)$, and call n_p the multiplicity of eigenvalue λ_p. Let us first put $\mathbf{D}(\mathbf{x}^*)$ in its Jordan form:

$$\mathbf{D}F(\mathbf{x}^*) = \mathbf{P}\mathbf{J}\mathbf{P}^{-1},$$

where $\mathbf{J}$ is diagonal by blocks, i.e.

$$\mathbf{J} = Diag\left(\mathbf{J}_1, ..., \mathbf{J}_P\right) := \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & ... & \mathbf{0} \\ ... & ... & ... & ... \\ \mathbf{0} & ... & \mathbf{0} & \mathbf{J}_P \end{pmatrix}, \quad \text{with } \mathbf{J}_p = \begin{pmatrix} \lambda_p & 1 & 0 & ... & 0 \\ 0 & \lambda_p & 1 & ... & 0 \\ ... & ... & ... & ... & \\ 0 & ... & 0 & \lambda_p & 1 \\ 0 & ... & ... & 0 & \lambda_p \end{pmatrix}$$

Define now $\mathbf{Q} := Diag\,(\mathbf{Q}_1, ..., \mathbf{Q}_P)$, with $\mathbf{Q}_p = Diag(1, \epsilon, ..., \epsilon^{n_p-1})$. We then have

$$\mathbf{Q}_p^{-1}\mathbf{J}_p\mathbf{Q}_p = \begin{pmatrix} \lambda_p & \epsilon & 0 & ... & 0 \\ 0 & \lambda_p & \epsilon & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & \lambda_p & \epsilon \\ 0 & ... & ... & 0 & \lambda_p \end{pmatrix}$$

Thus, defining $\mathbf{R} := \mathbf{P}\mathbf{Q}$ we obtain

$$\mathbf{R}^{-1}\mathbf{D}(\mathbf{x}^*)\mathbf{R} = \mathbf{Q}^{-1}\mathbf{J}\mathbf{Q} = \mathbf{D}(\lambda) + \epsilon\mathbf{B},$$

where $\mathbf{D}(\lambda)$ is the diagonal matrix filled with the eigenvalues of $\mathbf{D}(\mathbf{x}^*)$.
Now define $V : \mathbf{S} \to \mathbb{R}^+$ as follows:

$$V(\mathbf{x}) := \left|\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*)\right|^2 = \left\langle \mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*)\right\rangle$$

We have

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \left\langle \mathbf{R}^{-1}\dot{\mathbf{x}} \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*)\right\rangle + \left\langle \overline{\mathbf{R}^{-1}}\dot{\mathbf{x}} \mid \mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*)\right\rangle \\ &= \left\langle (\mathbf{D}(\lambda) + \epsilon\mathbf{B})\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*)\right\rangle + \left\langle (\overline{\mathbf{D}(\lambda)} + \epsilon\overline{\mathbf{B}})\overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*) \mid \mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*)\right\rangle \\ &+ \|\mathbf{x} - \mathbf{x}^*\|^2 h(\|\mathbf{x} - \mathbf{x}^*\|), \end{aligned}$$

where $h(a) \to_{a\to 0} 0$. Hence we have

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \left\langle (\mathbf{D}(\lambda) + \overline{\mathbf{D}(\lambda)})\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*)\right\rangle + 2\epsilon Re\left(\left\langle \mathbf{B}\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*)\right\rangle\right) \\ &+ \|\mathbf{x} - \mathbf{x}^*\|^2 h(\|\mathbf{x} - \mathbf{x}^*\|) \end{aligned}$$

Let $\alpha := \max_{p=1,\dots,P} Re(\lambda_p) < 0$. We have

$$\left\langle (\mathbf{D}(\lambda) + \overline{\mathbf{D}(\lambda)})\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*)\right\rangle \leq 2\alpha|\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*)|^2 = 2\alpha V(\mathbf{x}).$$

As a consequence, choosing ϵ small enough and $\mathbf{x}$ close enough of $\mathbf{x}^*$ we obtain that

$$\dot{V}(\mathbf{x}) \leq \alpha V(\mathbf{x}),$$

which proves that $V(\mathbf{x}(t))$ goes to zero exponentially fast, as t goes to infinity, and this concludes the proof. □

Proof of Proposition 3. Let M be the root of $\mathbf{x}^*$, meaning that $N_+(\mathbf{x}^*) = \bar{M}$, and $\rho := \rho(\mathbf{G}_{\bar{M}}) = \rho(\mathbf{G})$. The network $\hat{\mathbf{G}}$ also has a unique dominant component, $\hat{M}$. Either $\hat{M} = M$[35], or $\hat{M}$ is a community which did not exist in $\mathbf{G}$. When it is the case, we have $i, j \in \hat{M}$, $\hat{M} \subset \bar{M}$ and $\rho(\mathbf{G}_{\hat{M}}) \geq \rho$. Hence there is a unique stable equilibrium $\hat{\mathbf{x}}$ (with root $\hat{M}$) in $\hat{\mathbf{G}}$, $N_+(\hat{\mathbf{x}}) \subset N_+(\mathbf{x})$ and $\hat{\rho} := \rho(\hat{\mathbf{G}}_{\hat{M}}) = \rho(\hat{\mathbf{G}})$.

We now prove that (ii) holds. Note that $\hat{\mathbf{x}}^* \neq \mathbf{x}^*$: suppose by contradiction that $\mathbf{x}^* = \hat{\mathbf{x}}^*$. Let $k \neq i$ with $k \in N^+(\mathbf{x}^*)$. Then $\rho x_k^* = (\mathbf{G}\mathbf{x}^*)_k = (\hat{\mathbf{G}}\mathbf{x}^*)_k = \hat{\rho}\hat{x}_v^*$, implying that $\rho = \hat{\rho}$. Thus $\rho x_i^* = (\mathbf{G}\mathbf{x}^*)_i = (\hat{\mathbf{G}}\mathbf{x}^*)_i - x_j^* = \rho x_i^* - x_j^*$, a contradiction. Consider the following subsets of agents:

$$K_+ := \left\{ k \in \hat{M} : \frac{\hat{x}_k^*}{x_k^*} \geq \frac{\hat{x}_l^*}{x_l^*} \ \forall l \in \hat{M} \right\}, \ K_- := \left\{ k \in \hat{M} : \frac{\hat{x}_k^*}{x_k^*} \leq \frac{\hat{x}_l^*}{x_l^*} \ l \in \hat{M} \right\}.$$

We actually prove a stronger property, namely that $i \in K_+$. Note that if $k \neq i$ and $k \in K_+$ then $\frac{\hat{x}_k^*}{x_k^*} = \frac{\sum_{w \in \mathcal{N}_k} \hat{x}_w^*}{\sum_{w \in \mathcal{N}_k} x_w^*}$. Hence $w \in K_+$ for all $w \in \mathcal{N}_k$. By a recursive argument this implies that, if k is connected to w through a path then $w \in K_+$. The same property also holds for K_-. As a consequence $i \in K_+ \cup K_-$. If this were not the case there would exist two nodes $k_+ \neq i$ and $k_- \neq i$ such that $k_+ \in K_+$ and $k_- \in K_-$, which would imply that elements of M belong to both K_+ and K_-, a contradiction.
Suppose first that we are in the case where $\hat{\rho} > \rho$, and let $k \neq i$. Suppose that $k \in K_+$. Then

$$\frac{1}{\hat{\rho}} = \frac{\hat{x}_k^*}{(\hat{\mathbf{G}}\hat{\mathbf{x}}^*)_k} = \frac{\hat{x}_k^*}{\sum_{w \in \mathcal{N}_k} \hat{x}_w^*} \geq \frac{x_k^*}{\sum_{w \in \mathcal{N}_k} x_w^*} = \frac{x_k^*}{(\mathbf{G}\mathbf{x}^*)_k} = \frac{1}{\rho},$$

a contradiction. Hence $K_+ = \{i\}$.
Suppose now that $\hat{\rho} = \rho$. Showing that $i \in K_+$ is equivalent to showing that $i \notin K_-$. Suppose by contradiction that $i \in K_-$. Then

$$\frac{1}{\rho} = \frac{\hat{x}_i^*}{(\hat{\mathbf{G}}\hat{\mathbf{x}}^*)_i} = \frac{\hat{x}_i^*}{\sum_{w \in \mathcal{N}_i} \hat{x}_w^* + \hat{x}_j^*} < \frac{\hat{x}_i^*}{\sum_{w \in \mathcal{N}_i} \hat{x}_w^*} \leq \frac{x_i^*}{\sum_{w \in \mathcal{N}_i} x_w^*} = \frac{x_i^*}{(\mathbf{G}\mathbf{x}^*)_i} = \frac{1}{\rho},$$

where the strict inequality follows from the fact that $j \in N_+(\hat{\mathbf{x}})$ (see above). This is a contradiction. Thus $i \in K_+$. $\square$

[35] If, for instance there is no path from j to i.

(Non-for-Publication) Online Appendix

Perceived competition in networks

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A Descriptive evidence

We would now like to provide some simple descriptive evidence of networks with *perceived* competition, which implies that networks are directed and weakly connected. Daniele et al. (2021)[5] observed (roughly) 700 street-food vendors in Kolkata (India), which were distributed across 10 areas in the city. Within each area, there are blocks of streets of 8-12 vendors where all vendors are very close geographically to each other. Vendors are sellers of food in kiosks and sell different types of food or beverage. Competition is rather fierce. Among many other variables, the authors collected information about the network of these vendors, assuming that each block is a network. By showing the names and pictures of all vendors in the same block, they asked each vendor which vendors he knew in the block. This is a good way of measuring competition, since knowing a vendor is equivalent knowing the competition. In Figures A1 and A2, we display two of these networks.[6]

Each vendor is represented by a circle whose color indicates which type of food or beverage they sell. Consider, first, the network displayed in Figure A1. There are eight vendors who sell different types of food and beverage. We see that the network is *directed*. For example, the "red" vendor on the left is not aware of any other vendors in his block but the "red" vendor next to him is aware of him. In other words, the "red" vendor, who sells drinks, perceives that he is not in competition with any other vendors, but the "red" vendor next to him, who also sells drinks, perceives him as a competitor. The same is true for the top "blue" (selling vegetarian meals). We also see that the network is *weakly connected* since, for example, there is no path between the "red" vendor on the left and any other vendors in the network.

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[5]Daniele, G., Mookerjee, S., and Tommasi, D. (2021). Informational shocks and food safety: A field study on street vendors in urban India. *Review of Economics and Statistics*, forthcoming.

[6]There are 100 networks/blocks in the dataset. We thank Denni Tommasi for giving us access to this dataset.

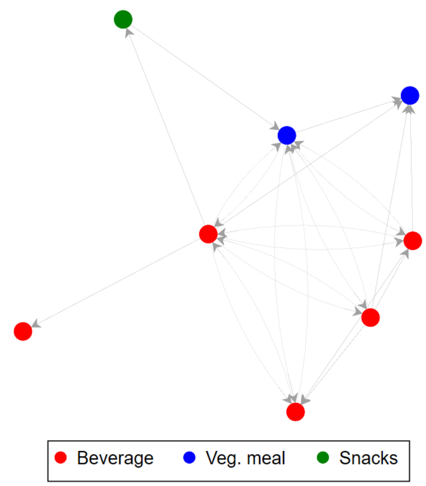

Figure A1: Perceived competition in India: Network 1 with 8 vendors *Source:* Own calculation based on the dataset from Daniele et al. (2021)

If we now consider the network in Figure A2 with nine vendors, a similar pattern emerges. For example, nobody is aware of the "blue" vendor on the left whereas he can reach all other vendors. Similarly, nobody is aware of the "blue" vendor on the right while, apart from the "green" vendor on the left, he can reach all other vendors. Similarly, nobody is aware of the "orange" vendor on the right but he can reach all vendors apart from the "blue" and the "red" vendors on the right. Also, even though some vendors perceive that competition is local, it is in fact global at the block level. For example, the "blue" vendor on the left who sells vegetarian meals is not aware that he is in competition with the "green" vendor who also sells vegetarian meals and with all the "orange" vendors who sell snacks.

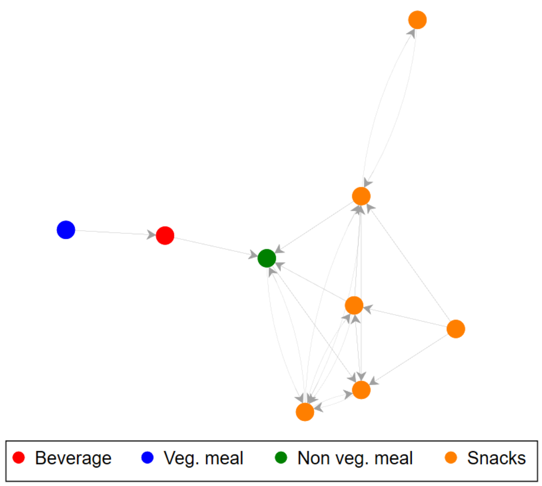

Figure A2: Perceived competition in India: Network 2 with 9 vendors. *Source:* Own calculation based on the dataset from Daniele et al. (2021)

B Non-negative matrices and eigenvector centrality

B.1 The Frobenius normal form

A matrix is called **nonnegative** if all its elements are nonnegative. Here we consider only nonnegative square matrices of order n, i.e., matrices that have n rows and n columns. A non-negative matrix A is called **irreducible** if the associated directed graph is strongly connected. For convenience any one-by-one matrix is regarded as irreducible.

Lemma B1. *(Perron-Frobenius Theorem) Let* $\mathbf{A}$ *be an irreducible matrix. Then*

(i) $\mathbf{A}$ *has a positive eigenvalue* $\rho(\mathbf{A})$ *such that the value of* $\rho(\mathbf{A})$ *is not less than the absolute value of any other eigenvalue of* $\mathbf{A}$*;*

(ii) the eigenvalue $\rho(\mathbf{A})$*is simple, and corresponds to a positive eigenvector* $\mathbf{x}(\mathbf{A})$*;*

(iii) any non-negative eigenvector is a multiple of $\mathbf{x}(\mathbf{A})$.

The vector $\mathbf{x}(\mathbf{A})$ *and the number* $\rho(\mathbf{A})$ *that appear in this lemma are called the* ***Perron-Frobenius vector*** *and the* ***Perron-Frobenius eigenvalue*** *of* $\mathbf{A}$*, respectively.*

The following lemma extends some conclusions of the Perron-Frobenius Theorem to nonnegative matrices (not necessarily irreducible).

Lemma B2. *Let* $\mathbf{A}$ *be a nonnegative matrix; then*

a) $\mathbf{A}$ *has a nonnegative eigenvalue* $\rho(\mathbf{A})$ *such that the value of* $\rho(\mathbf{A})$ *is not less than the absolute value of any other eigenvalue of* $\mathbf{A}$.

b) To eigenvalue $\rho(\mathbf{A})$ *corresponds a nonnegative eigenvector* $\mathbf{x}(\mathbf{A})$.

c) If there exists a positive eigenvector, then it is necessarily associated to eigenvalue $\rho(\mathbf{A})$.

Note that if $\mathbf{x}$ is a non-negative eigenvector of $\mathbf{A}$, $\mathbf{x}$ is not necessarily associated with $\rho(\mathbf{A})$. Also there could exist eigenvectors with both negative and positive entries, associated to $\rho(\mathbf{A})$.

Lemma B3. *Any nonnegative matrix* $\mathbf{A}$ *can be put in an upper-triangular block form as* *follows:*[7]

$$\mathbf{A} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1r} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2r} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{sr} \\ 0 & ... & ... & 0 & A_{s+1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & A_{r-1} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & A_r \end{bmatrix} \tag{B.1}$$

such that:

(*i*) *each block matrix A_i is square and irreducible;*

(*ii*) *for any $i = 1, ...s$, there exists $j \in \{i+1, ..., r\}$ such that the block matrix A_{ij} is not zero.*

This upper triangular block form is known as the **Frobenius normal form**. It is unique up to a permutation. We have $\rho(\mathbf{A}) = \max_{i=1...r} \rho(A_r)$. We call V_i the set of nodes corresponding to the block matrix A_i.

Definition B1. *A nonnegative matrix $\mathbf{A}$ is* ***strongly nonnegative*** *if we have*

$$\rho(A_r) = \rho(A_{r-1}) = ... = \rho(A_{s+1}) > \max_{i=1,...,s} \{\rho(A_i)\}$$

Obviously, any irreducible matrix is strictly nonegative because the Frobenius normal form then consists of one block. The next results can be found in Rothblum (2014)[8] or Hu and Qi (2016).[9]

Lemma B4. *A nonnegative matrix $\mathbf{A}$ admits a positive eigenvector if and only if $\mathbf{A}$ is strongly nonnegative.*

Note that, if $\mathbf{A}$ is an irreducible nonnegative matrix, then the conclusion of Lemma B4 directly implies point (*ii*) of Lemma B1, i.e., the Perron Frobenius Theorem.

We illustrate the Frobenius normal form for network $(N, \mathbf{G})$ displayed in Figure 2, with $N = \{1, 2, \cdots, 10\}$ and with three communities: $M_1 = \{2, 3, 4\}$, $M_2 = \{5, 6\}$, and $M_3 = \{7, 8, 9, 10\}$.

[7]Up to a permutation of indices.

[8]Rothblum, U. (2014). Nonnegative matrices and stochastic matrices. In: L. Hogben (Ed.), *Handbook of Linear Algebra, Second Edition*, Chap. 10. CRC Press, pages 1–26.

[9]Hu, S. and Qi, L. (2016). A necessary and sufficient condition for existence of a positive Perron vector. *SIAM Journal of Matrix Analysis and Applications* 37(4), 1747–1770.

Let $\mathbf{C}(m)$ be the adjacency matrix of the complete m-agents network.[10] Keeping the indexing of agents as it is, we have

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & A_2 & A_{23} & A_{24} \\ 0 & 0 & A_3 & A_{34} \\ 0 & 0 & 0 & A_4 \end{bmatrix},$$

where $A_1 = 0$, $A_2 = \mathbf{G}_{M_1} = \mathbf{C}(3)$, $A_3 = \mathbf{G}_{M_2} = \mathbf{C}(2)$, and $A_4 = \mathbf{G}_{M_3} = \mathbf{C}(4)$, while $A_{12} = [0\,0\,1]$, $A_{13} = [0\,0]$, $A_{14} = [1\,0\,0\,0]$, etc. In particular, A_{ij} is distinct from the null matrix, except for A_{13} (there is no link from group 1, i.e., agent 1, to community M_2, i.e., agents $\{5, 6\}$). Consequently we have $s = 3$ and $r = 4$ and $\rho(A_4) = 3$ while $\rho(A_1) = 0, \rho(A_2) = 2$ and $\rho(A_3) = 1$. Hence, $\mathbf{G}$ is strongly nonnegative and thus admits a positive eigenvector. Now, remove agent 9 from this network. Then, the Frobenius normal form has the same structure, except that $\rho(A_4) = 2 = \max_{i=1,...,3} \rho(A_i)$. Hence, the matrix is no longer strictly nonnegative and, thus, there is no positive eigenvector.

It might be useful to clarify the relationship between the Frobenius normal form and the $\succeq$-ordering on communities. In the Frobenius normal form of $\mathbf{G}$, any A_i corresponds to the submatrix of a strongly connected component, which can either be a community, or a singleton. Note that, by the no-isolation assumption, A_i cannot be a size one matrix for $i = s+1, ..., r$; it then necessarily corresponds to a community for these indexes. If $M' \succ M$, then there exists some i, i' such that $i' < i$, $\mathbf{G}_M = A_i$ and $\mathbf{G}_{M'} = A_{i'}$. In other words the indexes in the Frobenius normal form are inversely ordered in accordance with the $\succeq$ ordering.

The Frobenius normal form does not help us to characterize the perception-consistent equilibria (i.e., which agents are active and which are not) but will be very useful for some of our proofs because of Lemma B4, which can be applied to any closed set, as we will see in the proof section.

[10] That is, $C(m)_{ii} = 0$, $C(m)_{ij} = 1$ for $i \neq j$

B.2 Eigenvector centrality in weakly connected networks

Eigenvector centrality has been informally introduced by Bonacich (1972)[11] to measure popularity in friendship networks. Given a weighted network $(N, \mathbf{G})$, it was originally defined as any non-negative vector $\mathbf{e}$ having the property that the centrality of agent i is proportional to the average centrality of her neighbors:

$$\lambda e_i = \sum_j \mathbf{G}_{ij} e_j, \ \forall i. \tag{B.2}$$

In the particular case of strongly connected networks, this vector is well-defined because there is a unique solution to the system (B.2), given by the eigenvector associated to the largest eigenvalue λ of $\mathbf{G}$. More generally, there is a consensus consisting in regarding eigenvector centrality as being the normalized eigenvector associated to the largest eigenvalue of the network (see e.g., Jackson (2008)).[12]

In weakly connected networks, however, eigenvector centrality cannot be defined in the same way because the largest eigenvalue of a weakly connected network is not always simple. For instance, consider the network in Figure E3 in Appendix E, where $\rho(\mathbf{G}) = 1$. The eigenspace associated to $\rho(\mathbf{G})$ is generated by normalized vectors $(1/3, 1/3, 1/3, 0, 0)$ and $(1/3, 0, 0, 1/3, 1/3)$. Hence, any convex combination of these two vectors is a non-negative eigenvector, which means that eigenvector centrality is not defined for this network.

Consequently, we focus on an (arguably large) subset of weakly connected graphs, in which the notion of eigenvector centrality can be naturally extended. A weakly connected network has *a unique dominant component* if

$$\forall M, M' \in \mathcal{C}(\mathbf{G}), \ \ \rho(\mathbf{G}_M) = \rho(\mathbf{G}_{M'}) = \rho(\mathbf{G}) \Rightarrow M \succeq M' \text{ or } \ M' \succeq M. \tag{UDC}$$

Obviously any *simple network* has a unique dominant component. A simple adaptation of the proof of Proposition D1 shows that a weakly connected network admits a unique normalized eigenvector associated to $\rho(\mathbf{G})$ if and only if it has a unique dominant component.

Definition B2 (Eigenvector centrality)**.** *Suppose that* $(N, \mathbf{G})$ *has a unique dominant component. Then, the eigenvector centrality of agent* i *is the* i*-th component of the normalized eigenvector associated to* $\rho(\mathbf{G})$*.*

In some networks, it may be the case that some agents in the network exhibit a null eigenvector centrality, and one may wonder what it means, and whether or not this definition

[11] Bonacich, P. (1972). Factoring and weighting approaches to status scores and clique identification. *Journal of Mathematical Sociology* 2(1), 113–120.

[12] meaning the eigenvector whose components sum to one.

makes sense when this happens. As we show now, this definition is indeed meaningful, because our definition of eigenvector centrality is robust to any small perturbations of the network, in the following sense:

Lemma B5. *Suppose that $(N, \mathbf{G})$ has a unique dominant component and call $\mathbf{e}$ the normalized eigenvector associated to $\rho(\mathbf{G})$. Let $(\mathbf{G}^n)_n$ be a sequence of irreducible matrices such that $\lim_{n\to+\infty} \mathbf{G}^n_{ij} = \mathbf{G}_{ij}$. Then $\mathbf{e}^n \to \mathbf{e}$, where $\mathbf{e}^n$ is the normalized eigenvector associated to $\rho(\mathbf{G}^n)$.*

In other words, the sequence of centrality measures always converge to the same vector, regardless of *how* $\mathbf{G}^n$ converges to $\mathbf{G}$. The implication of this observation is that eigenvector centrality is unambiguously defined in networks having a unique dominant component.

Observe that the network $(N, \mathbf{G})$ depicted in Figure E3 in Appendix E does not exhibit such a property; thus, defining an eigenvector centrality for such a network would imply making an arbitrary choice. Indeed, it can be shown that, for any $\lambda \in [0, 1]$, one can find a sequence of strongly connected weighted networks $(N, \mathbf{G}^n)$ such that $\mathbf{e}^n$ converges to $\frac{1}{3}(1, \lambda, \lambda, 1-\lambda, 1-\lambda)$.

B.3 Relationship between the $\succeq$ ordering and the Frobenius normal form

Lemma B6. *Let $(N, \mathbf{G})$ be a weakly connected network. Consider its Frobenius normal form (B.1). For any $i = 1, ..., r$ either $|V_i| = 1$ or $V_i \in \mathcal{C}(\mathbf{G})$. As a consequence*

$$\rho(\mathbf{G}) = \max_{i=1,\ldots,r} \rho(A_i) = \max_{M\in\mathcal{C}(\mathbf{G})} \rho\left(\mathbf{G}_M\right) \tag{B.3}$$

Proof. Suppose that $|V_i| > 1$. By construction of the Frobenius normal form, (V_i, A_i) is a strongly connected component of $(N, \mathbf{G})$. Hence V_i belongs to the set of communities $\mathcal{C}(\mathbf{G})$. Since $\rho(\mathbf{G}) = \max_{i=1...r} \rho(A_i)$ and $\rho(A_i) = 0$ if $|V_i| = 1$ this concludes the proof of (B.3). □

For any closed set $N' \subset N$, note that $\mathcal{C}(\mathbf{G}_{N'}) = \{M \in \mathcal{C}(\mathbf{G}) :\ M \subset N'\}$. Hence we have

$$\rho(\mathbf{G}_{N'}) = \max_{M'\in\mathcal{C}(\mathbf{G}):\, M\subseteq N'} \rho(\mathbf{G}_{M'}) \tag{B.4}$$

Lemma B7. *Suppose that $\mathbf{A}$ is a nonnegative matrix that admits a Frobenius normal form (B.1) with $r = s + 1$ and $\rho(A_{s+1}) > \max_{i=1,\ldots,s} \{\rho(A_i)\}$. Then $\mathbf{A}$ admits a* ***unique*** *positive eigenvector.*[13]

[13]Uniqueness is up to multiplication by a constant.

Proof. We only need to show that, if $\mathbf{x}$ and $\mathbf{y}$ are two positive eigenvector of $\mathbf{A}$ then $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha > 0$. We can write $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{bmatrix} A' & B \\ 0 & A_{s+1} \end{bmatrix}, \text{ where } A' = \begin{bmatrix} A_1 & A_{12} & ... & ... & A_{1s} \\ 0 & A_2 & A_{23} & ... & A_{2s} \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & ... & ... & 0 & A_s \end{bmatrix} \text{ and } B = \begin{bmatrix} A_{1s+1} \\ A_{2s+1} \\ ... \\ ... \\ A_{ss+1} \end{bmatrix}.$$

Let us write $\mathbf{x}$ as $(\mathbf{x}', \mathbf{x}_{[s+1]})$, according to the decomposition of $\mathbf{A}$ we just wrote and let $\rho := \rho(A_{s+1}) = \rho(\mathbf{A})$. We have

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{x}_{[s+1]} \end{bmatrix} = \rho^{-1} \begin{bmatrix} \mathbf{A}' \cdot \mathbf{x}' + \mathbf{B} \cdot \mathbf{x}_{[s+1]} \\ \mathbf{A}_{s+1} \cdot \mathbf{x}_{[s+1]} \end{bmatrix},$$

so that, in particular, $\left(\mathbf{I} - \rho^{-1}\mathbf{A}'\right)\mathbf{x}' = \rho^{-1}\mathbf{B}\mathbf{x}_{[s+1]}$. Since $\rho(\mathbf{A}') < \rho$ by construction, the matrix $\mathbf{I} - \rho^{-1}\mathbf{A}'$ is invertible and we have

$$\mathbf{x}' = \rho^{-1}\left(\mathbf{I} - \rho^{-1}\mathbf{A}'\right)^{-1}\mathbf{B}\mathbf{x}_{[s+1]} \tag{B.5}$$

Now the matrix $\mathbf{A}_{s+1}$ being irreducible and $\mathbf{x}_{[s+1]}, \mathbf{y}_{[s+1]}$ both being positive eigenvectors of $\mathbf{A}_{s+1}$ we must have $\mathbf{x}_{[s+1]} = \alpha\mathbf{y}_{[s+1]}$ Since identity (B.5) holds for both $\mathbf{x}$ and $\mathbf{y}$, we obtain that $\mathbf{x}' = \alpha\mathbf{y}'$, concluding the proof. $\square$

C Cournot with non-linear demand

Consider a standard homogeneous good Cournot oligopoly game on a network with n firms competing in quantities, but with non-linear demand:

$$p = (\bar{\alpha} - k(X))_+ ,$$

where $k : [0, +\infty($ is non-decreasing, such that $h(0) = 0$, $\lim_{x \to +\infty} k(x) \geq \overline{\alpha}$, and $x \in]0, +\infty[\mapsto xk\left(\frac{x+z}{W}\right)$ is quasi-concave for any $z \geq 0, W > 0$. Hence, firm i's perceived utility can be written as

$$u_i(x_i, \mathbf{x}_{-i}; W_i) = \left[\bar{\alpha} - k\left(\frac{x_i + \sum_j g_{ij} x_j}{W_i}\right)\right] x_i - c x_i.$$

Again, the map f satisfies the required assumptions, that is, $f(0) = \overline{\alpha} > c$, $\lim_{y \to +\infty} f(y) = 0 < c$ and $x \mapsto x\left(\overline{\alpha} - h\left(\frac{x+z}{W}\right)\right)_+$ is quasiconcave. Finally, $\lim_{x \to 0^+} x f(x) = 0$.

First, given W_i, each firm i chooses quantity x_i^* that maximizes her perceived utility. This leads to:

$$\bar{\alpha} - k\left(\frac{x_i + \sum_j g_{ij} x_j}{W_i}\right) - \frac{x_i}{W_i} k'\left(\frac{x_i + \sum_j g_{ij} x_j}{W_i}\right) = c.$$

Second, Definition 1(ii) requires that quantity choices are consistent at a PCE by imposing that

$$W_i = \frac{x_i^* + \sum_j g_{ij} x_j^*}{\sum_j x_j^*}.$$

By plugging this value in the FOC above, we obtain:

$$\bar{\alpha} - k(X^*) - \frac{x_i^* X^*}{x_i^* + \sum_j g_{ij} x_j^*} k'(X^*) = c.$$

or equivalently

$$\sum_j g_{ij} x_j^* = \left(\frac{k(X^*) + X^* k'(X^*) - \bar{\alpha} + c}{\bar{\alpha} - c - k(X^*)}\right) x_i^*.$$

In matrix form, we have

$$\mathbf{G}\mathbf{x}^* = \left(\frac{k(X^*) + X^* k'(X^*) - \bar{\alpha} + c}{\bar{\alpha} - c - k(X^*)}\right) \mathbf{x}^*, \quad \text{and} \quad \mathbf{x}^* \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}.$$

It is easily verified that, when $k(X) = X$, we obtain the result with linear-demand (equation (11)).

D Additional results

D.1 Peer-confirming equilibria

Corollary D1. *If the network $(N, \mathbf{G})$ is a semi-connected network then there are at most n perception-consistent equilibria, where n is the number of communities.*

Proof of Corollary D1. If the network is semi-connected then the communities are totally ordered: $M_1 \succ M_2 \succ ... \succ M_n$. Hence the number of PCE is equal to

$$Card\left\{s = 1, ..., n:\ \rho(\mathbf{G}_{M_s}) > \max_{k=1,...,s-1} \rho(\mathbf{G}_{M_k})\right\}.$$

$\square$

Proposition D1. *Let $(N, \mathbf{G})$ be a weakly-connected network. The following are equivalent:*

(i) *The set of perception-consistent equilibria is finite.*

(ii) *For any pair $\left(\mathbf{x}^{1*}, \mathbf{x}^{2*}\right)$ of perception-consistent equilibria, $\rho\left(\mathbf{G}_{N_+(\mathbf{x}^{1*})}\right) \neq \rho\left(\mathbf{G}_{N_+(\mathbf{x}^{2*})}\right)$.*

(iii) *$(N, \mathbf{G})$ is a simple network.*

Proof of Proposition D1. $(i) \Rightarrow (ii)$: suppose that (ii) does not hold. Then there exists two PCE $\mathbf{x}_1, \mathbf{x}_2$ such that $\rho(\mathbf{x}_1) = \rho(\mathbf{x}_2) =: \rho$. For $\lambda \in [0, 1]$ and define $\mathbf{x}^\lambda := \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$. Then $X^\lambda = X_1 = X_2$. Hence

$$\mathbf{G}\mathbf{x}^\lambda = \lambda\mathbf{G}\mathbf{x}_1 + (1-\lambda)\mathbf{G}\mathbf{x}_2 = \lambda\rho\mathbf{x}_1 + (1-\lambda)\rho\mathbf{x}_2 = \rho\mathbf{x}^\lambda,$$

and $\mathbf{x}^\lambda$ is a PCE. Thus there is a continuum of PCE, contradicting (i).

$(ii) \Rightarrow (i)$: this implication follows from the fact that the set of eigenvalues of subgraphs of $\mathbf{G}$ is finite.

$(ii) \Rightarrow (iii)$: Suppose that (iii) does not hold. Then there exists M_1, M_2 such that $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_1'})$, $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_1} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_1})$ and $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_2} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_2})$. The last two strict inequalities mean that there exists a PCE with root M_1, and a PCE with root M_2, contradicting (ii).

$(iii) \Rightarrow (ii)$: Assume that (ii) does not hold, and let M_1 (resp. M_2) be the root of $\mathbf{x}_1$ (resp. $\mathbf{x}_2$). Being both PCE, it follows that we have $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_1} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_1})$ and $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_2} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_2})$, contradicting (iii).

Finally we obtain $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and the proof is complete. $\square$

D.2 Policy interventions

D.2.1 Key players

Proposition D2. *Consider the (linear) Tullock contest game. Let* $\mathbf{x}^*$ *be the (unique) asymptotically stable equilibrium of the simple network* $(N, \mathbf{G})$ *and* $\widehat{\mathbf{x}}^*$ *the (unique) asymptotically stable equilibrium of the simple network* $(N\backslash\{i\}, \mathbf{G}_{N\backslash\{i\}})$*. Then,* $\widehat{X}^* \leq X^*$*.*

Proof of Proposition D2. We have $X^* = \frac{V\rho(\mathbf{G})}{c[1+\rho(\mathbf{G})]}$ and $\widehat{X}^* \leq \frac{V\rho(\mathbf{G}_{N\backslash\{i\}})}{c[1+\rho(\mathbf{G}_{N\backslash\{i\}}]}$. By standard results, $\rho(\mathbf{G}) \geq \rho(\mathbf{G}_{N\backslash\{i\}})$. Hence $\widehat{X}^* \leq X^*$. □

D.2.2 Social mixing

Proposition D3. *Consider the (linear) Tullock contest game. Let* $(N^1, \mathbf{G^1})$ *and* $(N^2, \mathbf{G}^2)$ *be two simple networks endowed with resources equal to* V_1 *and* V_2*, respectively. Let* $\mathbf{x}^{1*}$ *(resp.* $\mathbf{x}^{2*}$*) be the unique stable PCE of* $(N^1, \mathbf{G^1})$ *(resp.* $(N^2, \mathbf{G}^2)$*), with root* M_1 *(resp.* M_2*).Let also* $(N, \mathbf{G})$ *be the network obtained from* $(N^1, \mathbf{G^1})$ *and* $(N^2, \mathbf{G}^2)$ *in which* $N = N^1 \cup N^2$*,* $V = V^1 + V^2$*, with* $g_{ij} = 1$ *and* $g_{k\ell} = 1$ *for some* $(i, \ell) \in M_1$*,* $(j, k) \in M_2$*. Then, there is a unique stable PCE* $\mathbf{x}^*$ *of* $(N, \mathbf{G})$ *satisfying* $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$*, and* $X^* > X^{1*} + X^{2*}$*.*

Proof of Proposition D3. We have

$$X^1 = \frac{V^1}{c}\frac{\rho(\mathbf{G}^1)}{\rho(\mathbf{G}^1)+1}; \; X^2 = \frac{V^2}{c}\frac{\rho(\mathbf{G}^2)}{\rho(\mathbf{G}^2)+1}; \; X = \frac{V^1+V^2}{c}\frac{\rho(\mathbf{G})}{\rho(\mathbf{G})+1}$$

We have $\rho(\mathbf{G}) = \rho(M_1 \cup M_1) > \max\left\{\rho(\mathbf{G}^1), \rho(\mathbf{G}^2)\right\}$. Hence

$$X^1 + X^2 = \frac{V^1}{c}\frac{\rho(\mathbf{G}^1)}{\rho(\mathbf{G}^1)+1} + \frac{V^2}{c}\frac{\rho(\mathbf{G}^2)}{\rho(\mathbf{G}^2)+1}\frac{V^1+V^2}{c} < \frac{\rho(\mathbf{G})}{\rho(\mathbf{G})+1} = X.$$

□

E Beyond simple networks

In this section we assume that $\mathbf{G}$ is a weakly connected directed graph satisfying the no isolation assumption. As mentioned in the text, the PCE set can be infinite in non-simple networks.

E.1 Structure of the equilibrium set

We say that two distinct communities M_1 and M_2 are **disconnected** if neither $M_1 \succ M_2$ nor $M_2 \succ M_1$.

Proposition E4. *Let $\mathbf{x}^*$ be a PCE. Then, there exists a family of pairwise disconnected communities $\{M_i\}_{i=1,...,n}$ such that*

$$N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i. \tag{E.1}$$

Proof of Proposition E4. Since $N_+(\mathbf{x}^*)$ is a closed set of $\mathbf{G}$, we have that $\mathbf{x}^*$ is a positive eigenvector of $\mathbf{G}_{N_+(\mathbf{x})}$, associated to eigenvalue $\rho > 0$. By Lemma B4, that implies that $\mathbf{G}_{N_+(\mathbf{x})}$ is strongly nonnegative, and thus can be written

$$\mathbf{G}_{N_+(\mathbf{x})} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1r} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2r} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{sr} \\ 0 & ... & ... & 0 & A_{s+1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & A_{r-1} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & A_r \end{bmatrix} \tag{E.2}$$

where $r > s$, $\rho(A_r) = ... = \rho(A_{s+1}) = \rho$, and $\rho(A_i) < \rho$ for $i = 1, ..., s$. Each A_{s+i} being such that $|V_{s+i}| \geq 2$ for $i = 1, ..., r-s$, we have $V_{s+i} \in \mathcal{C}(\mathbf{G})$. Hence, taking $n := r - s$, there exists $M_1, ..., M_n \in \mathcal{C}(\mathbf{G})$ such that $A_{s+i} = \mathbf{G}_{M_i}$ for $i = 1, ..., n$.
We now show that $N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i$. Since $N_+(\mathbf{x}^*)$ is closed and $M_i \subset N_+(\mathbf{x})$ we have $\bar{M}_i \subset N_+(\mathbf{x}^*)$. Hence $\cup_{i=1}^n \bar{M}_i \subset N_+(\mathbf{x})$. Now pick $j \in N_+(\mathbf{x}^*)$. By property (ii) of the Frobenius normal form (see Definition B3), there exists some $i \in \{1, ..., n\}$ such that $j \rightrightarrows M_i$, meaning that $j \in \bar{M}_i$. This concludes the proof. $\square$

Proposition E5. *Let $(M_i)_{i=1,...,n}$ be a family of pairwise disconnected communities. There* *exists a perception-consistent equilibrium (PCE)* $\mathbf{x}^*$ *with* $N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i$ *if and only if*

$$\rho\left(\mathbf{G}_{M_1}\right) = ... = \rho\left(\mathbf{G}_{M_n}\right) > \max_{i=1,...,n} \max_{M' \in \mathcal{C}(\mathbf{G}):\, M' \succ M_i} \rho\left(\mathbf{G}_{M'}\right). \tag{E.3}$$

Proof of Proposition E5. The Frobenius normal form of $\mathbf{G}_{\cup_{i=1}^n \bar{M}_i}$ can be written as

$$\mathbf{G}_{\cup_{i=1}^n \bar{M}_i} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1s+n} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2s+n} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{ss+n} \\ 0 & ... & ... & 0 & \mathbf{G}_{M_1} & 0 & ... & 0 \\ 0 & ... & ... & .... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & \mathbf{G}_{M_{n-1}} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & \mathbf{G}_{M_n} \end{bmatrix}. \tag{E.4}$$

By Lemma B4, this matrix admits a positive eigenvector (and therefore there exists a PCE $\mathbf{x}^*$ such that $N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i$) if and only if

$$\rho\left(\mathbf{G}_{M_1}\right) = ... = \rho\left(\mathbf{G}_{M_n}\right) > \max_{i=1,...,s} \rho(A_i).$$

Note that $V = \cup_{i=1,...,s} V_i$ is a closed set and thus

$$\{M' \in \mathcal{C}(\mathbf{G}) :\ M' \subset V\} = \{M' \in \mathcal{C}(\mathbf{G}) : M' \succ M_i \text{ for some } i = 1, ..., n\}.$$

Hence

$$\max_{i=1,...,s} \rho(A_i) = \rho\left(\mathbf{G}_V\right) = \max_{i=1,...,n} \max_{M' \in \mathcal{C}(\mathbf{G}):\, M' > M_i} \rho\left(\mathbf{G}_{M'}\right)$$

This concludes the proof. □

In full generality, even if the set of peer-confirming equilibria is no longer finite, we can still describe it in a simple way; it is always a finite union of convex sets. Recall that the set of equilibria which admit a root is finite: there is at most one PCE with root M, for $M \in \mathcal{C}$. Let $\{\rho_1, ..., \rho_P\}$ be the set of positive eigenvalues of $\mathbf{G}$. The set of equilibria with a root can be written as

$$\bigcup_{p=1}^{P} S_p, \quad \text{where } S_p := \{\mathbf{x}^* :\ \mathbf{x}^* \text{ is a PCE with root } M \text{ such that } \rho(\mathbf{G}_M) = \rho_p\},$$

Proposition E6. *Given any network* $\mathbf{G}$ *the set of perception-consistent equilibria can be* *written as*

$$PCE = \bigcup_{p=1}^{P} \Lambda_p,$$

where Λ_p *is the convex polytope generated by* S_p*:* $\Lambda_p = Conv\left(S_p\right)$.

Proof of Proposition E6. We first show that $\bigcup_{p=1}^{P} \Lambda_p \subset PCE$. It amounts to showing that, if $S_p = \left\{\mathbf{x}^1, ..., \mathbf{x}^n\right\}$, and $\lambda_1, ..., \lambda_p$ are nonnegative numbers that sum to one then $\mathbf{x} := \sum_{j=1} \lambda_j \mathbf{x}^j$ is a PCE. We have

$$\mathbf{G}\mathbf{x} = \sum_{j=1}^{n} \lambda_j \mathbf{G}\mathbf{x}^j = \sum_{j=1}^{p} \lambda_j \rho_p \mathbf{x}^j = \rho_p \mathbf{x}.$$

and this concludes this implication.

We now turn to the other inculsion. Let $\mathbf{x}$ be a PCE. Then, by Proposition E5, there exists $p \in \{1, ..., P\}$ and a family of pairwise disconnected communities $\{M_i\}_{i=1,...,n}$ such that $N_+(\mathbf{x}) = \cup_{i=1}^{n} \bar{M}_i$, and $\rho\left(\mathbf{G}_{M_i}\right) = \rho_p > \max_{M' \succ M_i} \rho\left(\mathbf{G}_{M'}\right),\ \forall i = 1, ..., n$. Call $\mathbf{x}^i$ the equilibrium with root M_i, for $i = 1, ..., n$. We first define the following objects:

$$\tilde{M}_i := \bar{M}_i \setminus \left(\cup_{j \neq i} \bar{M}_j\right);\ \ \tilde{M} := \cup_{i=1}^{n} \bar{M}_i \setminus \left(\cup_{i=1}^{n} \tilde{M}_i\right);\ \ \lambda_i := \frac{\sum_{j \in \tilde{M}_i} x_j}{\sum_{i \in \tilde{M}_i} x_j^i}.$$

Note that, by construction, the family $\left\{\tilde{M}, \tilde{M}_1, ..., \tilde{M}_n\right\}$ constitutes a partition of $\cup_{i=1}^{n} \bar{M}_i$. Call $\mathbf{A}_i := \mathbf{G}_{\tilde{M}_i}$ and $\mathbf{A} := \mathbf{G}_{\tilde{M}}$. Then we can write

$$\mathbf{G}_{\cup_{i=1}^{n} \bar{M}_i} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & ... & ... & \mathbf{B}_n \\ 0 & \mathbf{A}_1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & \mathbf{A}_{n-1} & 0 \\ 0 & ... & ... & 0 & \mathbf{A}_n \end{bmatrix}$$

Be aware that this is not a Frobenius normal form because matrices $\mathbf{A}$ and $\mathbf{A}_i$ are in general not irreducible. However we know the following: $\rho(\mathbf{A}_i) = \rho_p$ for $i = 1, ..., n$ and $\rho(\mathbf{A}) < \rho_p$. Moreover, for $j = 1, ..., p$, $\mathbf{x}^i_{|\tilde{M}_i}$ is, by definition, a positive eigenvector of matrix $\mathbf{A}_i$. This is also true for $\mathbf{x}_{|\tilde{M}_i}$. The Frobenius normal form of $\mathbf{A}_i$ verifies the conditions of Lemma B7, (A_{s+1} corresponding here to M_i). As a result $\mathbf{x}_{|\tilde{M}_i}$ and $\mathbf{x}^i_{|\tilde{M}_i}$ are proportionnal:

$$\mathbf{x}_{|\tilde{M}_i} = \alpha_i \mathbf{x}^i_{|\tilde{M}_i}. \tag{E.5}$$

Since $\mathbf{x}_{|\cup_{i=1}^{n}\bar{M}_i}$ is an eigenvector of $\mathbf{G}_{\cup_{i=1}^{n}\bar{M}_i}$ associated to ρ_p we have

$$\rho_p \mathbf{x}_{|\tilde{M}} = \mathbf{A}\mathbf{x}_{|\tilde{M}} + \sum_{i=1}^{n} \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i},$$

and thus, since $\mathbf{I} - \rho_p^{-1}\mathbf{A}$ is invertible,

$$\rho_p \mathbf{x}_{|\tilde{M}} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \sum_{i=1}^{n} \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \sum_{i=1}^{n} \alpha_i \mathbf{B}_i \mathbf{x}^i_{|\tilde{M}_i}.$$

On the other hand $\mathbf{x}^i_{|\tilde{M}\cup\tilde{M}_j}$ is an eigenvector of $\mathbf{G}_{|\tilde{M}\cup\tilde{M}_j}$ associated to ρ_p. Hence

$$\rho_p \mathbf{x}^i_{|\tilde{M}} = \mathbf{A}\mathbf{x}^i_{|\tilde{M}} + \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i},$$

that is,

$$\rho_p \mathbf{x}^i_{|\tilde{M}} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i}.$$

Finally we get

$$\rho_p \mathbf{x}_{|\tilde{M}} = \sum_{i=1}^{n} \alpha_i \rho_p \mathbf{x}^i_{|\tilde{M}_i},$$

i.e. $\mathbf{x}_{|\tilde{M}} = \sum_{i=1}^{n} \alpha_i \mathbf{x}^i_{|\tilde{M}_i}$. Combining this equality with (E.5) and the fact that $\mathbf{x}^m_{|\tilde{M}_i} = 0$ when $i \neq m$, we obtain that

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{x}^i$$

Now $\mathbf{x}$ and $\mathbf{x}^i$ being all associated to the same eigenvalue ρ_p we necessarily have $X = X^i$ for $i = 1, ..., n$. As a result $\sum_{i=1}^{n} \alpha_i = 1$ and this concludes the proof. □

Remark E1. *When* $\mathbf{G}$ *is a simple network, then every component is degenerate, i.e., they reduce to a singleton. In full generality, in a given component, the largest eigenvalue of the subgraph of active players is invariant.*

E.2 Example

We illustrate this in the following example for the linear Tullock contest game.

Example E1. Non-finiteness of equilibria

Consider the network $(N, \mathbf{G})$ in Figure E3 with $N = \{1, 2, \cdots, 5\}$. Both $M_1 = \{2, 3\}$, and $M_2 = \{4, 5\}$ are $\succ$-maximal communities. Moreover we have $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = 1$. Consequently, the set of perception-consistent equilibria is not finite since the network is not simple.

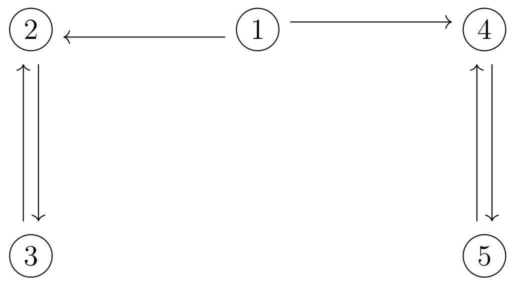

Figure E3: Infinite set of PCE in a non-simple network

More precisely:

$$PCE = \left\{ \frac{V}{12c}(1, \lambda, \lambda, 1-\lambda, 1-\lambda) : \ \lambda \in [0,1] \right\}.$$

◇