The Macroeconomics of Partial Irreversibility*

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Abstract

We investigate the macroeconomic effects of partially irreversible investment arising from a wedge between the buying and selling price of capital. We derive sufficient statistics that characterize the implications of irreversibility for three long-run macroeconomic outcomes—capital allocation, capital valuation, and capital fluctuations. Measuring these statistics with investment microdata, we find that irreversibility distorts the allocation of capital, lowers capital valuation, and increases the persistence of capital fluctuations. Corporate income tax cuts, by lowering the deductibility of capital losses due to the price wedge, effectively increase irreversibility and structurally change long-run capital behavior.

JEL: D30, D80, E20, E30

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1 Introduction

Investing in physical capital is far from frictionless. Building capital stock and setting it up for productive purposes entails adjustment costs that reflect a variety of frictions involved in the investment process (Caballero and Engel, 1999), such as searching for suppliers, disrupting production, restructuring plants, and retraining workers. Disinvesting is also costly, as resources are needed to uninstall the capital goods, search for buyers of the used capital, and conduct all necessary transactions. Besides the adjustment costs, the market for physical capital is characterized by a significant *wedge* between the buying and selling prices, which renders investment partially irreversible (Bertola and Caballero, 1994; Abel and Eberly, 1996; Lanteri, 2018).[1]

Exposed to a price wedge, firms operate with caution. In times of high productivity, firms do not scale up their capital stock as quickly because they fear an adverse shock that would force them to sell their capital at a lower price; in times of low productivity, firms prefer to hold onto their capital or only sell modestly to avoid the price penalty. Due to the potential capital losses, investment becomes less responsive to productivity shocks, and capital reallocation slows. What is the role of price wedges in shaping the allocation of capital? What are their implications for firms' market value and the transmission of aggregate shocks? And finally, are the macroeconomic implications of price wedges different from those of other adjustment frictions?

To answer these questions, we develop a parsimonious investment model with idiosyncratic productivity shocks and two frictions: a fixed adjustment cost and a price wedge. In this setting, we (i) analytically characterize the long-run properties of aggregate capital, including its allocation, valuation, and fluctuations; (ii) measure the aggregate effects of irreversibility and disentangle them from the effects of fixed adjustment costs using investment microdata; and (iii) study the effects of corporate tax reforms with the new notion of "after-tax" frictions, which reduces the complex interactions of corporate taxes and investment frictions to the simple rescaling of the frictions.

Our contribution is to demonstrate—theoretically and quantitatively—that partial irreversibility is crucial in understanding firm-level investment, aggregate capital behavior, and the effects of corporate tax changes. Concretely, we show that ignoring price wedges and wrongly attributing all lumpy behavior to fixed adjustment costs results in overestimating the value of capital and underestimating the persistence of aggregate fluctuations. Additionally, to the extent that the tax code allows for deductions of capital losses, price wedges change the elasticity of aggregate outcomes to corporate income taxes. These results highlight the importance of measuring and modeling partial irreversibility in capital markets—and its macroeconomic implications—in designing growth and stabilization policies.

[1] Ramey and Shapiro (2001) measure a price wedge above 70% for aerospace manufacturing equipment, and new evidence by Kermani and Ma (2020) shows that price wedges are quite sizeable across sectors and types of capital, with an industry-wide price wedge of 65% for plant, property, and equipment. We calculate price wedges as 1 minus the recovery rate, defined as the liquidation value over replacement cost net of depreciation.

Let us explain the key features of an environment with irreversibility and its challenges. To guide the discussion, consider a firm's capital to productivity ratio. In a frictionless world, capital tracks productivity and their ratio is constant across time and firms. With only fixed adjustment costs, this ratio moves with idiosyncratic productivity during periods of inaction, which generates heterogeneity across firms. However, upon adjustment, all firms reset to the same ratio, fully cleansing the history of shocks. The *unique reset point* significantly simplifies aggregation as firms become identical conditional on adjustment. With a price wedge, there are *two reset points*, each corresponding to the decision to upsize the capital stock at the buying price or to downsize it at the selling price. The investment policy now depends not only on the history of idiosyncratic shocks but also on the past reset point. This micro-history dependence—a Markovian structure for the sign of consecutive adjustments—adds a new layer of heterogeneity across firms and significantly complicates aggregation. We solve this challenge by providing unique steady-state cross-sectional statistics, easily computed in the microdata, that fully encode the history dependence generated by irreversibility for each macroeconomic outcome.

The first outcome is capital valuation, as measured by the weighted average of firm-level marginal q's—the value of an additional unit of capital relative to its replacement cost (Tobin, 1969; Abel, 1979; Hayashi, 1982; Abel and Eberly, 1996). Aggregate q summarizes an economy's benefits and costs of accumulating capital: the stream of future production, the user cost of capital and, naturally, the capital losses accrued due to the price wedge. We characterize the effect of irreversibility on q with two insights: First, the capital losses of adjusters can be "amortized" across the periods in which they were inactive and transitioning from being buyers to sellers and vice versa; and second, the time spent in a particular state during the transition is proportional to the mass of firms occupying that state. Taken together, these results imply that the economy-wide capital losses are equal to the average local drift of capital losses in the cross-section. Using Chilean microdata, we find that irreversibility reduces q by approximately 3% (from 1.09 to 1.06); but since q is close to unity, investment frictions have a moderate effect on capital valuation and primarily operate by reducing the stock of capital.

The second outcome is capital fluctuations around the steady state, as measured by the cumulative impulse response (CIR) of average capital to an unanticipated small shock to aggregate productivity. Along the transition path, we assume that firms follow their steady-state decision rules and thus focus on the partial equilibrium response. The CIR summarizes both the impact and the persistence of the capital response to the aggregate productivity shock. With one reset point, Álvarez, Le Bihan and Lippi (2016) characterized the CIR by keeping track of firms only until their first adjustment, and in Baley and Blanco (2021), we derived two sufficient statistics for the CIR of aggregate capital: the dispersion of capital-productivity ratios and their covariance with the time elapsed since their last adjustment. With two reset points, to our surprise, these sufficient statistics remain valid but with higher values due to irreversibility. Additionally, irreversibility introduces a new term that reflects how aggregate productivity shocks change the mass of adjusters across the two reset points relative to the steady state. We obtain a CIR of 3.1, which means that a 1% reduction in aggregate productivity generates a total deviation in capital-to-productivity ratios of 3.1% above steady state along the transition path. Our estimates show that irreversibility accounts for at least 25% of the CIR, which suggests that price wedges are an important source of persistence behind aggregate fluctuations.

Our characterization of aggregate outcomes highlights the importance of price wedges. A natural question arises: Can we precisely identify the aggregate effects of price wedges vis-à-vis other investment frictions? In particular, both price wedges and fixed adjustment costs may appear observationally equivalent under the lens of a particular set of data moments, for example, the rate of inaction and the dispersion of the investment rates. Nevertheless, these two frictions have very different implications for aggregate outcomes. Addressing the identification challenge is fundamental to correctly measure and assess the implications of investment frictions.

We proceed in two steps, which require investment microdata and a price wedge as inputs. In the first stage, we recover the two reset points from the firms' Euler equations constructed with the first-order conditions and the envelope theorem for arbitrary choice sets (Milgrom and Segal, 2002). In the second stage, with the reset points in hand, we derive mappings from the joint distribution of investment and duration of inaction *conditional on the sign of the last adjustment* to the cross-sectional steady-state moments that characterize capital allocation, valuation, and fluctuations, including the new sufficient statistics that capture the role of irreversibility. The premise behind these mappings is that, conditional on the sign of the last adjustment, the only remaining source of heterogeneity is the history of productivity shocks received during periods of inaction, which can be summarized by the size and timing of investment. Our estimates suggest that half of the difference between reset points comes from the endogenous response to the exogenous price wedge, which confirms irreversibility's large effects for firm-level investment.

As a concrete application of our framework, we examine the macroeconomic effects of corporate income tax cuts. The novelty of our analysis is a parsimonious modeling of the intricacies of the tax code regarding the deductions of capital losses and the capitalization of adjustment costs. Specifically, because capital losses are deducted, the corporate income tax effectively reduces the price wedge. To our knowledge, this is the first analysis of the macroeconomic consequences of capital loss deductions.

We introduce a corporate tax schedule (Summers, 1981; Abel, 1982) and show analytically that *after-tax investment frictions*—the fixed adjustment cost relative to the after-tax frictionless profits and the price wedge relative to the after-tax frictionless profit-capital ratio—are the key objects that affect dynamic investment decisions. We then examine a regime shift from a high to a low corporate income tax rate, comparing the macroeconomic outcomes across steady states. There are two opposing effects. On the one hand, a lower tax rate decreases the after-tax fixed adjustment costs. On the other hand, a lower tax rate raises the after-tax price wedge because capital loss deductions fall. The calibrated model suggests that, other things equal, a corporate income tax cut from 42% to 25%—corresponding to the median decrease in OECD countries between 1980 and 2020—improves the allocation of capital, decreases capital valuation (q falls), and causes fluctuations to be more persistent (CIR rises) at the new steady state.

Contributions to the literature. First, we contribute to the irreversibility literature (Bertola and Caballero, 1994; Dixit and Pindyck, 1994; Abel and Eberly, 1996; Ramey and Shapiro, 2001; Veracierto, 2002; Lanteri, 2018) by deriving sufficient statistics that capture the role of irreversibility for aggregate capital's allocation, valuation, and fluctuations. To do this, we extend the sufficient statistics approach of Álvarez, Le Bihan and Lippi (2016) and Baley and Blanco (2021) to environments with *multiple reset points*. This extension proved to be not trivial, as it required deriving new theoretical results to deal with micro-history dependence and to back out from the microdata the reset points, and their distance, in order to compute moments conditional on the last adjustment that determine the sufficient statistics.

Second, our work directly speaks to the quantitative investment literature. Following Cooper and Haltiwanger (2006), the literature has targeted a certain set of moments of the distribution of investment computed at plant level, including investment spikes and the autocorrelation of investment rates, to assess the nature of investment frictions and derive conclusions about their role for capital allocation (Asker, Collard-Wexler and De Loecker, 2014) and capital fluctuations (Khan and Thomas, 2008, 2013; Bachmann, Caballero and Engel, 2013; Bachmann and Bayer, 2014; Winberry, 2021). Our analysis shows that, while partially informative, widely used targets do not fully reflect the effects of irreversibility because they do not consider the Markovian structure that arises. We provide researchers with the key moments of the joint distribution of investment and duration of inaction that fully encode the role of irreversibility by conditioning on the sign of the last adjustment. Our approach complements other direct methods designed to assess the magnitude of frictions, such as examining the stationary distribution of marginal products of capital (Hsieh and Klenow, 2009; Restuccia and Rogerson, 2013) or conditional transitions of marginal products (Caballero, Engel and Haltiwanger, 1997; Lanteri, Medina and Tan, 2020).

Lastly, we contribute to the literature by studying the interaction of corporate taxes with investment frictions. Early work focused on the user-cost channel of taxation in frictionless environments. Subsequent work incorporated firm heterogeneity and non-convex adjustment costs to investigate the frictional channel of taxation (Miao, 2019; Gourio and Miao, 2010; Miao and Wang, 2014). We show how to reduce the complex interactions between corporate taxes and investment frictions to a rescaling of the appropriate friction. This idea considerably simplifies the analysis and highlights the channels through which corporate tax reforms affect private investment.

2 Investment with a Fixed Cost and a Price Wedge

In this section, we develop a parsimonious investment model with the following features: idiosyncratic productivity shocks, fixed capital adjustment costs, a positive wedge between the purchase and resale prices of capital, and a constant interest rate.

2.1 A firm's problem

Time is continuous, extends forever, and is denoted by s. The future is discounted at rate $\rho > 0$. For any stochastic process x_s, we use the notation $x_{s^-} \equiv \lim_{r \uparrow s} x_r$ to denote the limit from the left. We first present the problem of an individual firm and then consider a continuum of ex ante identical firms to characterize the aggregate behavior of the economy.

Technology and shocks. The firm produces output y_s using capital k_s according to a production function with decreasing returns to scale

$$y_s = u_s^{1-\alpha} k_s^{\alpha}, \quad \alpha < 1. \tag{1}$$

Flow profits are equal to $\pi_s \equiv A y_s$, where $A > 0$ is a profitability parameter. Idiosyncratic productivity u_s follows a geometric Brownian motion with drift $\mu > 0$ and volatility $\sigma > 0$,

$$\log u_s = \log u_0 + \mu s + \sigma W_s, \qquad W_s \sim Wiener. \tag{2}$$

The capital stock, if uncontrolled, depreciates at a constant rate $\xi^k > 0$.

Investment frictions. The firm can control its capital stock through buying and selling investment goods. For every active adjustment, $i_s \equiv \Delta k_s = k_s - k_{s^-}$, the firm must pay a fixed cost proportional to its productivity

$$\theta_s = \theta u_s, \tag{3}$$

where $\theta > 0$ is measured in output units. Capital is bought at price $p^{\text{buy}} = p$ and sold at price $p^{\text{sell}} = p(1-\omega)$, where ω is the price wedge. Alternatively, $1-\omega$ is the recovery rate. To simplify notation, we define the price function as

$$p(i_s) \equiv p^{\text{buy}} \mathbb{1}_{\{i_s > 0\}} + p^{\text{sell}} \mathbb{1}_{\{i_s < 0\}}. \tag{4}$$

Investment problem. Let $V(k,u)$ denote the value of a firm with capital stock k and productivity u. Given initial conditions (k_0, u_0), the firm chooses a sequence of adjustment dates $\{T_h\}_{h=1}^{\infty}$

and investments $\{i_{T_h}\}_{h=1}^{\infty}$, where h counts the number of adjustments, to maximize its expected discounted stream of profits. The sequential problem is

$$V(k_0, u_0) = \max_{\{T_h, i_{T_h}\}_{h=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} e^{-\rho s}\pi_s \,\mathrm{d}s - \sum_{h=1}^{\infty} e^{-\rho T_h}\left(\theta_{T_h} + p\left(i_{T_h}\right) i_{T_h}\right)\right], \tag{5}$$

subject to the production technology (1), the idiosyncratic productivity shocks (2), the fixed cost (3), the investment price function (4), and the law of motion for the capital stock

$$\log k_s = \log k_0 - \xi^k s + \sum_{h: T_h \leq s} \left(1 + i_{T_h}/k_{T_h^-}\right), \tag{6}$$

which describes a period's capital as a function of its initial value k_0, the physical depreciation rate ξ^k, and the sum of all adjustments made at prior adjustment dates.

2.2 Capital-productivity ratios $\hat{k}$

To characterize the investment decision, it is convenient to reduce the state space and recast the firm's problem using a new state variable, the log capital-productivity ratio:

$$\hat{k}_s \equiv \log\left(k_s/u_s\right). \tag{7}$$

The problem admits this reformulation because the production function is homothetic, the adjustment costs are proportional to productivity, and idiosyncratic shocks follow a Brownian motion with drift.[2] Note that in the absence of investment frictions, $\hat{k}_s$ is a constant. With investment frictions, between any two consecutive adjustment dates $[T_{h-1}, T_h]$, the capital-productivity ratio $\hat{k}$ follows a Brownian motion

$$\mathrm{d}\hat{k}_s = -\nu \,\mathrm{d}s + \sigma \,\mathrm{d}W_s, \tag{8}$$

where the drift $\nu \equiv \xi^k + \mu$ reflects the depreciation rate and productivity growth rate. At any adjustment date T_h, the log capital-productivity ratio changes by the amount

$$\Delta\hat{k}_{T_h} = \log\left(1 + i_{T_h}/k_{T_h^-}\right). \tag{9}$$

Using the principle of optimality, Lemma 1 rewrites the sequential problem in (5) as a recursive stopping-time problem. It also shows that the value of the firm equals a function of the log capital-productivity ratio $\hat{k}$ that scales with productivity; that is, $V(k, u) = uv(\hat{k})$. Since $\Delta\hat{k}_s$ and i_s have

[2] We can also reformulate the problem in terms of capital-productivity ratios assuming that adjustment costs scale with output or with the capital stock.

the same sign, we write the investment price as $p(\Delta\hat{k})$. All proofs appear in Appendix A.

Lemma 1. *Let $r \equiv \rho - \mu - \sigma^2/2$ be the adjusted discount factor and let $v(\hat{k}) : \mathbb{R} \to \mathbb{R}$ be a function of the log capital-productivity ratio equal to*

$$(10)\quad v(\hat{k}) = \max_{\tau,\Delta\hat{k}} \mathbb{E}\Big[\int_0^{\tau} Ae^{-rs+\alpha\hat{k}_s}\, \mathrm{d}s + e^{-r\tau}\Big(-\theta - p(\Delta\hat{k})(e^{\hat{k}_\tau+\Delta\hat{k}} - e^{\hat{k}_\tau}) + v(\hat{k}_\tau + \Delta\hat{k})\Big)\Big|\hat{k}_0 = \hat{k}\Big].$$

Then the firm value equals $V(k,u) = uv(\hat{k})$.

2.3 Optimal investment policy

The optimal investment policy is characterized by four numbers, $\mathcal{K} \equiv \{\hat{k}^- \leq \hat{k}^{*-} \leq \hat{k}^{*+} \leq k^+\}$, which correspond to the lower and upper borders of the inaction region

$$(11)\qquad \mathcal{R} = \Big\{\hat{k} : \hat{k}^- < \hat{k} < \hat{k}^+\Big\},$$

and two reset points $\hat{k}^{*-} < \hat{k}^{*+}$. A firm adjusts if and only if its log capital-productivity ratio falls outside the inaction region, that is, $\hat{k}_s \notin \mathcal{R}$. Conditional on adjusting, the firm purchases capital to bring its state up to $\hat{k}^{*-}$ if it hits the lower border $\hat{k}^-$, and sells capital to bring its state down to $\hat{k}^{*+}$ if it hits the upper border $\hat{k}^+$. Given $\mathcal{R}$, the optimal adjustment dates are

$$(12)\qquad T_h = \inf\Big\{s \geq T_{h-1} : \hat{k}_s \notin \mathcal{R}\Big\} \quad \text{with} \quad T_0 = 0.$$

The duration of a complete inaction spell τ_h and the time elapsed since the last adjustment a_s (or the age of the capital-productivity ratio) are

$$(13)\qquad \tau_h = T_h - T_{h-1},$$

$$(14)\qquad a_s = s - \max\{T_h : T_h \leq s\}.$$

To save on notation, we write the reset points and the stopped capitals (an instant before adjustment) as functions of the sign of adjustment:

$$(15)\qquad \hat{k}^*(\Delta\hat{k}) = \begin{cases} \hat{k}^{*-} & \text{if} \quad \Delta\hat{k} > 0 \\ \hat{k}^{*+} & \text{if} \quad \Delta\hat{k} < 0, \end{cases}$$

$$(16)\qquad \hat{k}_\tau(\Delta\hat{k}) = \hat{k}^*(\Delta\hat{k}) - \Delta\hat{k}.$$

Lemma 2 characterizes the value function and the optimal investment policy through the standard sufficient optimality conditions. The firm value and the policy must satisfy (i) the Hamilton-

Jacobi-Bellman equation in (17), which describes the evolution of the firm's value during periods of inaction, (ii) two value-matching conditions in (18) and (19), which set the value of adjusting equal to the value of not adjusting at the borders of the inaction region, and (iii) two smooth-pasting and optimality conditions in (20) and (21), which ensure the differentiability of the value function at the borders of inaction and the two reset points.

Lemma 2. *The value function $v(\hat{k})$ and the optimal policy $\mathcal{K} \equiv \{\hat{k}^{-}, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^{+}\}$ satisfy:*

(i) In the inaction region $\mathcal{R}$, $v(\hat{k})$ solves the HJB equation:

$$rv(\hat{k}) = Ae^{\alpha\hat{k}} - \nu v'(\hat{k}) + \frac{\sigma^2}{2}v''(\hat{k}). \tag{17}$$

(ii) At the borders of the inaction region, $v(\hat{k})$ satisfies the value-matching conditions:

$$v(\hat{k}^{-}) = v(\hat{k}^{*-}) - \theta - p^{buy}(e^{\hat{k}^{*-}} - e^{\hat{k}^{-}}), \tag{18}$$

$$v(\hat{k}^{+}) = v(\hat{k}^{*+}) - \theta + p^{sell}(e^{\hat{k}^{+}} - e^{\hat{k}^{*+}}). \tag{19}$$

(iii) At the borders of the inaction region and the two reset states, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$v'(\hat{k}) = p^{buy}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{-}, \hat{k}^{*-}\right\}, \tag{20}$$

$$v'(\hat{k}) = p^{sell}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^{+}\right\}. \tag{21}$$

The optimal policy in terms of capital is recovered as $\{k^{-}, k^{*-}, k^{*+}, k^{+}\} = u \times \exp\{\hat{k}^{-}, \hat{k}^{*-}, k^{*+}, k^{+}\}$.

2.4 Tobin's marginal q

Next, we express the optimal investment decision using Tobin's marginal q—the shadow value of installed capital. We identify a firm's marginal q as the marginal valuation of an extra unit of installed capital relative to the replacement cost of capital (the purchase price p):[3]

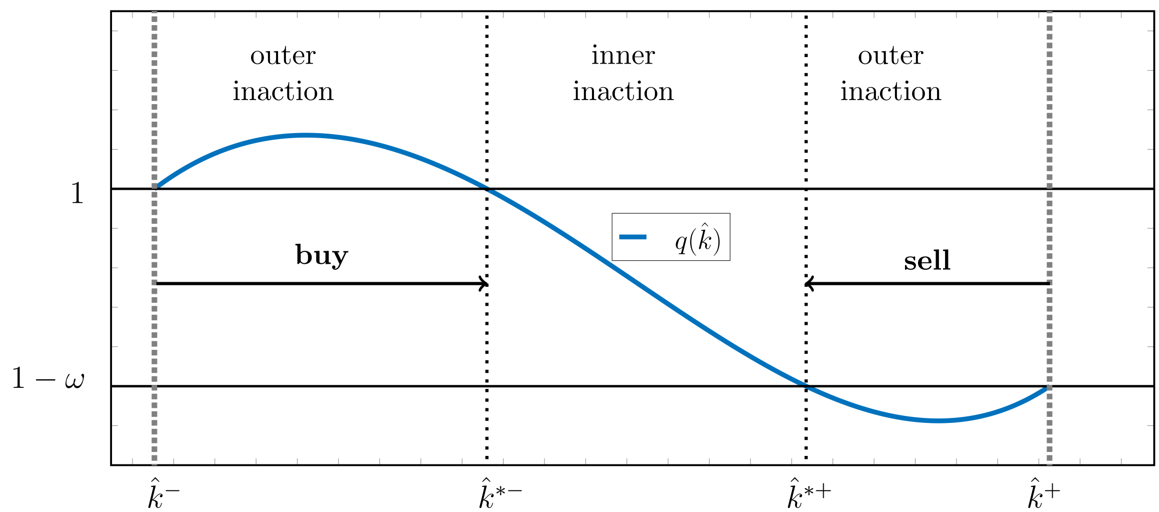
$$q(\hat{k}) \equiv \frac{1}{p}\frac{\partial V(k,u)}{\partial k} = \frac{v'(\hat{k})e^{-\hat{k}}}{p}. \tag{22}$$

Figure I describes the optimal investment policy using $q(\hat{k})$. We use this diagram to describe how each investment friction affects the firm's optimal policy. Let's consider first an environment

[3]Note that we define q relative to the purchase price $p^{buy} = p$. Alternatively, we could define q using the average investment price in the economy. These two formulations are proportional to each other.

with partial irreversibility and no fixed costs. Without the fixed adjustment cost ($\theta = 0$), a firm purchases capital if $q(\hat{k}) \geq 1$ (or $\hat{k} \leq \hat{k}^{*-}$) and sells capital if $q(\hat{k}) \leq 1 - \omega$ (or $\hat{k} \geq \hat{k}^{*+}$) without any delay. When $q(\hat{k})$ lies between the two prices (or the state between the two reset points), it is optimal to remain inactive. At that productivity level, it is too expensive to purchase capital and too cheap to sell it. This gives rise to an "inner" inaction region $[\hat{k}^{*-}, \hat{k}^{*+}]$ due exclusively to partial irreversibility. Next, let's consider an environment with fixed costs and no partial irreversibility. Without a price wedge ($\omega = 0$), the "inner" inaction region collapses to a unique reset point k^*. However, the fixed adjustment cost generates an "outer" inaction region $[\hat{k}^-, \hat{k}^+]$ that prevents firms from adjusting, even if $q(\hat{k})$ lies above or below the investment price. When both frictions are active, the policy features both "outer" and "inner" inaction regions and two reset points.

Figure I – Optimal Investment Policy



Notes: This figures illustrates a firm's marginal $q(\hat{k}) = v'(\hat{k})/pe^{\hat{k}}$ and its investment policy $\mathcal{K} = \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$.

The interaction of the investment frictions generates two interesting features in the optimal investment behavior. First, as argued by Caballero and Leahy (1996), individual $q(\hat{k})$ is *not monotonic* in $\hat{k}$. Without fixed costs, $q(\hat{k})$ monotonically decreases with $\hat{k}$ due to decreasing returns to scale $\alpha < 1$. With fixed costs, firms anticipate large adjustments when approaching the inaction thresholds. As $\hat{k}$ approaches the lower threshold $\hat{k}^-$, firms anticipate that a future tiny change in the state $\mathrm{d}\hat{k} < 0$ will trigger a large positive adjustment $\Delta\hat{k} > 0$. The future positive investment lowers future $q(\hat{k})$ and feeds back into lower current $q(\hat{k})$, bending the function down. A reverse argument explains why $q(\hat{k})$ bends up as $\hat{k}$ approaches the upper threshold $\hat{k}^+$. As a result, individual $q(\hat{k})$ is *not a sufficient statistic for individual investment*, in contrast to the postulate in Tobin (1969).

Second, optimal investment features an *endogenous positive serial correlation in the sign of adjustments*. A firm is more likely to buy capital if it bought capital recently, and it is more likely to sell capital if it sold capital recently. This correlation arises because the inner inaction region generated by the price wedge widens the distance between the two borders of inaction but shortens the distance between each border of inaction and its corresponding reset point. Thus, it is more likely to reach $\hat{k}^-$ from the nearby $\hat{k}^{*-}$ than from the further $\hat{k}^{*+}$. The serial correlation in adjustment sign—which we label micro-history dependence—brings technical challenges for characterizing aggregate outcomes and for identifying the role of irreversibility with data. Later in the paper, we explain how to handle micro-history dependence conditioning behavior on the last reset point.

2.5 Frictionless and frictional investment

Next, Proposition 1 shows that the investment policy can be separated into a static frictionless component and a dynamic frictional component, where we characterize the latter introducing the notion of effective investment frictions. From a firm's perspective, what matters for investment decisions is the fixed adjustment cost relative to frictionless profits and the price wedge relative to the frictionless profits-capital ratio, respectively. To simplify notation, we recall the definition of the discount r and define the user cost of capital $\mathcal{U}$:

$$r \equiv \rho - \mu - \frac{\sigma^2}{2}, \tag{23}$$

$$\mathcal{U} \equiv \rho + \xi^k - \sigma^2. \tag{24}$$

For the problem to be well defined, we assume $r > 0$ and $\mathcal{U} > 0$.

Proposition 1. *Let $\mathcal{K} \equiv \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ denote the firms' optimal investment policy characterized in Lemma 2. The optimal policy can be decomposed as the sum of a static and a dynamic component $\mathcal{K} = \hat{k}^{ss} + \mathcal{X}$, where $\hat{k}^{ss}$ is the static log capital-productivity ratio $\hat{k}^{ss}$ that firms would set in the absence of frictions and equal to*

$$\hat{k}^{ss} = \frac{1}{1-\alpha} \log\left(\frac{\alpha A}{p\mathcal{U}}\right), \tag{25}$$

and $\mathcal{X} \equiv \{x^-, x^{*-}, x^{*+}, x^+\}$ solves the following stopping problem for the normalized capital-productivity ratio $x \equiv \hat{k} - \hat{k}^{ss}$:

$$\mathcal{V}(x) = \max_{\tau, \Delta x} \mathbb{E}\left[\int_0^\tau e^{-rs}\left(e^{\alpha x_s} - \alpha e^{x_s}\right) \mathrm{d}s + e^{r\tau}\left(-\tilde{\theta} - \tilde{\omega}\mathbb{1}_{\{\Delta x<0\}}(e^{x_\tau} - e^{x_\tau + \Delta x}) + \mathcal{V}(x_\tau + \Delta x)\right) \Big| x_0 = x\right], \tag{26}$$

$$\mathrm{d}x_t = -\nu\, \mathrm{d}t + \sigma\, \mathrm{d}W_t. \tag{27}$$

The effective fixed cost $\tilde{\theta}$ and the effective wedge $\tilde{\omega}$ are defined as

$$\tilde{\theta} \equiv \frac{\theta}{Ae^{\alpha\hat{k}^{ss}}} = \left(\frac{1}{A}\right)^{\frac{1}{1-\alpha}} \left(\frac{p\mathcal{U}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} \theta \tag{28}$$

$$\tilde{\omega} \equiv \frac{p\omega}{Ae^{(\alpha-1)\hat{k}^{ss}}} = \frac{\alpha}{\mathcal{U}}\omega. \tag{29}$$

Proposition 1 provides several insights. The static optimal policy $\hat{k}^{ss}$ in (25) sets the capital-productivity ratio to a constant, and its value reflects profitability αA, the average user cost of capital $\mathcal{U}$, and the investment price p. By definition, investment frictions do not affect the static choice $\hat{k}^{ss}$. In contrast, the dynamic policy $\mathcal{X}$ characterized by (26) and (27) takes into account the fixed cost and the price wedge, but these frictions enter scaled by static profits or by the profit-capital ratio. Moreover, the flow payoff in the dynamic problem $e^{\alpha x_s} - \alpha e^{x_s}$ only depends on the curvature of the profit function α, and thus is invariant to frictions. Finally, any price can be used to construct k^{ss}, because $\mathcal{X}$ moves accordingly so that $\mathcal{K}$ is invariant to the price.

The dynamic policy $\mathcal{X}$ that solves the stopping-time problem above closely resembles the price-setting and investment problems with fixed costs, analyzed first by Barro (1972); Sheshinski and Weiss (1977); and Dixit (1991), but with the addition of a price wedge. We will leverage on this literature and our Proposition 1 to characterize analytically the effect of frictions (and their interaction with corporate taxes) on aggregate outcomes in the next sections, extending previous results to include partial irreversibility.

A remark on the fixed adjustment cost. For simplicity, we specify the fixed cost parameter θ as deterministic, symmetric for positive and negative investments, and equal across firms. However, we prove all results for the generalized hazard model proposed by Caballero and Engel (1999, 2007) and examined in contemporaneous work by Álvarez, Lippi and Oskolkov (2020). This generalized hazard model accommodates asymmetric fixed costs, and random fixed costs, as well as time-dependent adjustments that can be motivated by information frictions (Verona, 2014) or search frictions (Kurmann and Petrosky-Nadeau, 2007; Ottonello, 2018). Appendix B presents the generalized hazard model. Also see Baley and Blanco (2021) for ex ante heterogeneity in firms' production and adjustment technologies.

A remark on the price wedge. Our preferred interpretation of the wedge ω is a gap between the buying and selling prices, which may reflect asymmetric information about the capital's quality (Akerlof, 1970; Kurlat, 2013; Bigio, 2015); imperfect substitutability (Lanteri, 2018); obsolescence (Caunedo and Keller, 2020); intermediary fees (Nosal and Rocheteau, 2011); tax credits (Altug, Demers and Demers, 2009); or VAT taxes (Chen *et al.*, 2019). However, to the extent that ω is a linear and asymmetric adjustment cost, it allows for alternative interpretations besides a price gap, such as installation or transaction costs that scale with investment (Cooper and Haltiwanger, 2006; Fang, 2021). Moreover, setting $\omega = 1$ delivers the extreme irreversibility case that eliminates the possibility of disinvesting (Sargent, 1980; Veracierto, 2002).

3 Three macroeconomic outcomes

This section investigates how investment frictions shape aggregate capital's allocation, valuation, and fluctuations. We consider an economy populated by a continuum of ex ante identical firms that face the investment problem described above. Idiosyncratic shocks W_s are independent across firms, and as a result the economy features a stationary cross-sectional joint distribution of capital-productivity ratios $g(\hat{k})$. It solves a KFE that describes the evolution of capital-productivity ratios inside the inaction region, excluding the two reset points (where it has kinks), together with continuity, border, and resetting conditions. It is plotted in Panel A of Figure II. Next, we define and characterize the macroeconomic outcomes using steady-state moments of $g(\hat{k})$.[4]

3.1 Capital allocation

We measure capital allocation by the cross-sectional variance of the log marginal revenue product of capital. In our model, all firms produce the same good and the output price is normalized to one. Therefore, we measure instead the variance of marginal products. From the production function (1), the log of the marginal product of capital is collinear to the capital-productivity ratio $\hat{k}$, that is, $\log mpk_s = \log \alpha - (1-\alpha)\hat{k}_s$, which implies that

$$\mathbb{V}ar[\log mpk] \;=\; (1-\alpha)^2 \mathbb{V}ar[\hat{k}]. \tag{30}$$

In a frictionless environment, $\hat{k}_s$ is constant and $\mathbb{V}ar[\log mpk] = 0$. With investment frictions, however, dispersion in the marginal product of capital arises. Given the collinear relationship in (30), we use both $\mathbb{V}ar[\hat{k}]$ and $\mathbb{V}ar[\log mpk]$ to refer to the allocation of capital. In what follows, we show that the allocation of capital is a common driver of capital valuation and capital fluctuations.

3.2 Capital valuation

We measure capital's valuation by the weighted average of individual marginal $q(\hat{k})$ in (22), with weights equal to $\phi(\hat{k}) \equiv e^{\hat{k}}/\hat{K}$:

$$q \;\equiv\; \int_{\hat{k}^-}^{\hat{k}^+} q(\hat{k})\phi(\hat{k})g(\hat{k})\, \mathrm{d}\hat{k} \;=\; \frac{\mathbb{E}[v'(\hat{k})]}{p\hat{K}}. \tag{31}$$

Our definition of q—the weighted average of marginal q's across firms—is an adequate measure of aggregate capital valuation as it reflects the average of the true shadow value of capital in the

[4] According to Proposition 1, we can alternatively define aggregate outcomes in terms of normalized ratios x.

economy, in contrast to the average of average q's that is used in empirical analysis and is proxied by the ratio of firms' market value to the book value of their capital.

Without frictions, there is a unique investment price, and optimality implies that $q = 1$ at all times. Any changes in the costs or benefits of investing are immediately passed through to the capital stock, eliminating any possibility of q deviating from unity. With investment frictions, there is an incomplete passthrough from changes in the costs or the benefits of investing to the capital stock. Deviations of q from unity reflect the shadow value of eliminating the frictions.

Proposition 2 characterizes aggregate q in terms of steady-state moments of $\hat{k}$. The proof combines the HJB equation for $v'(\hat{k})$, which specifies firms' optimal behavior, with the KFE satisfied by $g(\hat{k})$, which describes the evolution of firms through the cross-sectional distribution, into a single "master equation." Then we integrate to eliminate idiosyncratic noise and recover q.

Proposition 2. *Consider the weights $\phi(\hat{k}) \equiv e^{\hat{k}}/\hat{K}$. Then aggregate q equals*

$$q \;=\; \frac{1}{r}\left(\underbrace{\frac{\alpha A\hat{Y}}{p\hat{K}} \;+\; \frac{\sigma^2}{2} - \nu}_{productivity} \;+\; \underbrace{\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{P}(\hat{k}_s)\phi(\hat{k}_s))\right]\right]}_{irreversibility\;<\;0}\right), \tag{32}$$

where $\hat{Y}/\hat{K}$ is equal, up to second order, to

$$\frac{\hat{Y}}{\hat{K}} \;=\; \frac{\mathbb{E}[e^{\alpha\hat{k}}]}{\mathbb{E}[e^{\hat{k}}]} \;=\; \exp\left\{-(1-\alpha)\left(\mathbb{E}[\hat{k}] + \frac{\alpha}{2}\mathbb{V}ar[\hat{k}]\right)\right\} \;+\; o(\hat{k}^3), \tag{33}$$

and $\mathcal{P}(\hat{k}) \in \mathbb{C}^2$ is any twice continuously differentiable function in the entire domain $[\hat{k}^+, \hat{k}^-]$ that takes the following two values in the outer inaction regions:

$$\mathcal{P}(\hat{k}) \;\equiv\; \begin{cases} 0 & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}], \\ -\omega & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+]. \end{cases} \tag{34}$$

Aggregate q in (32) depends on three terms: the average output-productivity ratio divided by the average capital-productivity ratio $\hat{Y}/\hat{K}$; the expected change in the average capital-productivity ratio $(\sigma^2/2 - \nu)$; and an irreversibility term.

The term $\hat{Y}/\hat{K}$ is further characterized in (33) in terms of the average and variance of $\hat{k}$.[5] Because of decreasing returns to scale ($\alpha < 1$), this term in decreasing in both the average $\mathbb{E}[\hat{k}]$ and the dispersion $\mathbb{V}ar[\hat{k}]$ of capital-productivity ratios. Consequently, aggregate q is also decreasing in these two cross-sectional moments, capturing the indirect effect of fixed costs and the price wedge on q through this channel.

The term $(\sigma^2/2 - \nu)$ reflects the expected change in the average capital-productivity ratio. Since

[5] Aggregate productivity differs from the average output-capital ratio $\mathbb{E}[y/k] = \mathbb{E}[e^{(\alpha-1)\hat{k}}]$ due to heterogeneity.

firms can upsize to exploit good outcomes and downsize to insure against bad outcomes, they are effectively risk loving (Oi, 1961; Hartman, 1972; Abel, 1983). Thus, an increase in idiosyncratic risk σ^2 directly increases q. At the same time, an increase in idiosyncratic risk indirectly affects q by increasing $\mathbb{V}ar[\hat{k}]$ and lowering the aggregate output-capital ratio in (33). The overall effect of risk on q depends on the relative strength of these two opposing forces.

The average "amortized" capital loss due to inaction represents irreversibility's direct negative effect on q. While irreversibility also indirectly affects q through decreasing productivity, this direct effect contains distinct information on distribution of capital losses. This effect, characterized by the function $\mathcal{P}(\hat{k}_s)\phi(\hat{k}_s)$, equals the average local drift of non-adjusters. By the optimal sampling theorem and the renewal principle, this term also equals the capital losses of adjusting firms.

Individual vs. aggregate q. In Section 2.4 we showed that individual $q(\hat{k})$ is a non monotonic function of $\hat{k}$. This observation has led some economists to argue that the individual non monotonicity translates into aggregate non-monotonicity, discarding q as a sufficient statistic for aggregate investment. Expression (32) shows that this argument is flawed. Fixed adjustment costs and partial irreversibility do not break the decreasing relation between aggregate q and aggregate $\hat{K}$. Although this result is counterintuitive, it is a natural consequence of aggregating the behavior of individual firms. The anticipatory effects that bend individual $q(\hat{k})$ in the vicinity of the borders of the inaction region disappear when aggregating over the cross-section, because the positive and negative stances of expected changes in $q(\hat{k})$ cancel each other out in the aggregate. As a result, aggregate q *is a sufficient statistic for aggregate investment.*

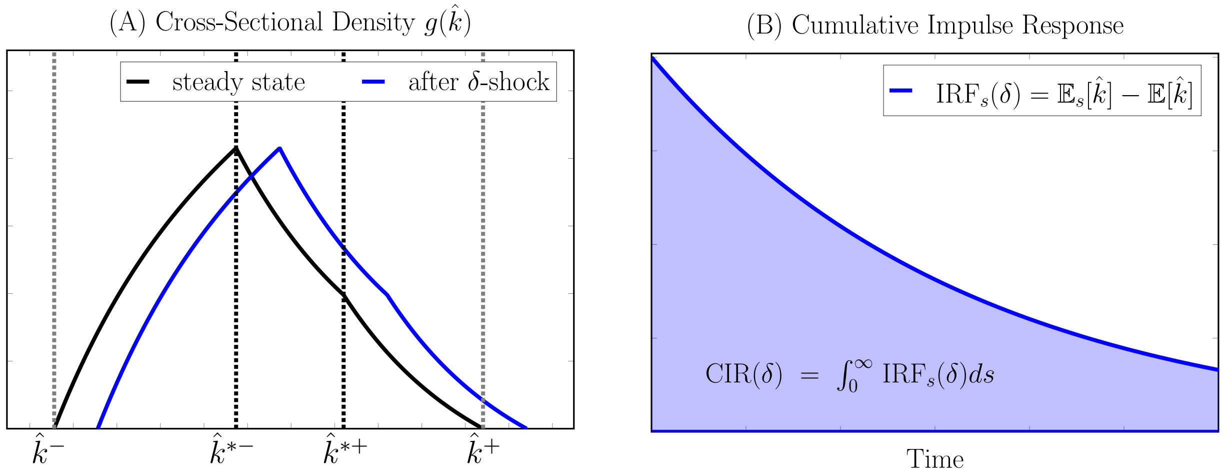
3.3 Capital fluctuations

We measure capital fluctuations by the transitional dynamics of aggregate capital following an aggregate productivity shock. Starting from the steady state at date $s = 0$, we introduce a small, permanent, and unanticipated decrease in the (log) level of productivity of size $\delta > 0$ to all firms. All firms' productivity and capital-productivity ratios change to

$$\log(u_0) = \log(u_{0^-}) - \delta; \qquad \log(\hat{k}_0) = \log(\hat{k}_{0^-}) + \delta. \tag{35}$$

Panel A of Figure II plots the initial density following the δ productivity shock (blue line) next to the steady-state density $g(\hat{k})$ (black line). The new distribution displaces horizontally to the right relative to the steady-state distribution. Our exercise consists of tracking the mean $\mathbb{E}_s[\hat{k}]$ as it makes its way back to its steady-state value $\mathbb{E}[\hat{k}]$. By assuming a constant interest rate, investment policies do not respond to changes in the distribution and remain fixed along the transition path.

Figure II – Distribution Dynamics and Cumulative Impulse Response



Notes: This figure illustrates the effects of an aggregate shock. Panel A shows the steady-state distribution $g(\hat{k})$ (blue line) and the initial distribution following a productivity shock (blue line). Panel B shows the $\mathrm{IRF}(\delta, s)$ (solid blue line) and the CIR (area).

Thus, our analysis measures the strength of the partial equilibrium response to aggregate shock.[6]

We define the impulse-response function, denoted by $\mathrm{IRF}(\delta, s)$, measured s periods after an aggregate productivity shock of size δ, as follows:

$$\mathrm{IRF}(\delta, s) \;\equiv\; \mathbb{E}_s[\hat{k}] - \mathbb{E}[\hat{k}], \tag{36}$$

where $\mathbb{E}_s[\cdot]$ denotes expectations with the time-s distribution. We define the cumulative impulse response $\mathrm{CIR}(\delta)$ as the area under the $\mathrm{IRF}_s(\delta)$ function across all dates $s \in (0, \infty)$:

$$\mathrm{CIR}(\delta) \;\equiv\; \int_0^\infty \mathrm{IRF}_s(\delta) \;\mathrm{d}s. \tag{37}$$

Panel B in Figure II plots these two objects. The solid line is the impulse-response function $\mathrm{IRF}(\delta, s)$, and the area underneath it is the cumulative impulse response function $\mathrm{CIR}(\delta)$. The CIR is a useful metric. It summarizes both the impact and persistence of the response in one scalar and eases comparison across different models.[7] Without frictions, firms respond instantly to the aggregate shock and the CIR is zero. With frictions, the larger the CIR, and the longer it takes firms to respond to the aggregate shock and the slower the transitional dynamics.

Proposition 3 characterizes the CIR as a function of cross-sectional moments of $\hat{k}$. To char-

[6]While assuming a constant interest rate (and investment policies) along the transition is an extreme assumption, Winberry (2021) shows that the interest-rate response to aggregate productivity shocks is small and even countercyclical. Appendix C relaxes this assumption and presents a general equilibrium model that delivers constant prices as an equilibrium outcome.

[7]Álvarez, Le Bihan and Lippi (2016), Baley and Blanco (2019); Álvarez, Lippi and Oskolkov (2020); and Alexandrov (2021) use the CIR in the context of price-setting models to assess the effects of monetary shocks.

acterize the role of irreversibility, we use a strategy analogous to the one we employed above to characterize aggregate q through an average local drift.

Proposition 3. *The CIR of the average log capital-productivity ratio $\mathbb{E}[\hat{k}]$ following a marginal aggregate productivity shock of size $\delta > 0$ is equal, up to first order, to*

$$\text{(38)} \qquad \frac{CIR(\delta)}{\delta} = \underbrace{\frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} + \frac{\nu \mathbb{C}ov[\hat{k}, a]}{\sigma^2}}_{responsiveness} + \underbrace{\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)]\right]}_{irreversibility} + o(\delta),$$

where $\mathcal{M}(\hat{k}) \in \mathbb{C}^2$ is any twice continuously differentiable function in the domain $[\hat{k}^+, \hat{k}^-]$ that takes the following two values in the outer inaction regions:

$$\text{(39)} \qquad \mathcal{M}(\hat{k}) \equiv \begin{cases} \mathcal{M}^{buy} & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ \mathcal{M}^{sell} & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+], \end{cases}$$

and $\mathcal{M}^{buy} < 0 < \mathcal{M}^{sell}$ are two numbers that measure the expected cumulative deviation of the capital-productivity ratio relative to the mean $\mathbb{E}[\hat{k}]$, conditional on the sign of the last adjustment.

According to (38), the CIR equals a linear combination of two steady-state moments and an irreversibility term. The moments are the cross-sectional variance of capital-productivity ratios $\mathbb{V}ar[\hat{k}]$ and the covariance of capital-productivity ratios $\hat{k}$ with the time elapsed since the last adjustment $\mathbb{C}ov[\hat{k}, a]$. These steady-state moments are informative about transitional dynamics because aggregate shocks δ and idiosyncratic shocks u enter symmetrically into $\hat{k}$ and, as a consequence, how firms respond to idiosyncratic shocks inform how they respond to aggregate shocks. Specifically, the variance $\mathbb{V}ar[\hat{k}]$ reflects insensitivity to idiosyncratic shocks, while the covariance $\mathbb{C}ov[\hat{k}, a]$ reflects the asymmetric costs of downsizing vs. upsizing. Baley and Blanco (2021) established the relationship between the CIR and these two steady-state moments in environments with drift, asymmetric fixed costs, and random opportunities of free adjustment, but without partial irreversibility. In all of those environments, the irreversibility term equals zero.

Irreversibility down the propagation of aggregate shocks through two channels. First, it has an indirect effect on the CIR by increasing the two cross-sectional moments $\mathbb{V}ar[\hat{k}]$ and $\mathbb{C}ov[\hat{k}, a]$. Second, it has a direct effect that reflects how the aggregate shock δ changes the mass of adjusters across the two reset points relative to the steady state. In principle, because of the correlated adjustments that arise with irreversibility, one should keep track of firms for a long time. Nevertheless, because the first adjustment after the aggregate shock eliminates all heterogeneity except for the sign of the adjustment, we only need to keep track of firms until their first adjustment. In other words, steady-state behavior is restored and two numbers are enough to characterize the CIR: $\mathcal{M}^{buy}$, which measures the expected cumulative deviations below to the steady-state mean conditional on a positive investment, and $\mathcal{M}^{sell}$, which measures the expected cumulative deviations above to the steady-state mean conditional on a negative investment. Given these two numbers, we characterize this irreversibility term with a similar strategy used in q, by "amortizing" the change in masses during periods of inaction (formally, the average local drift of the cumulative deviations from steady state in the cross-section). Later, we provide expressions for these two numbers and show how to measure them in the microdata.

Identifying and characterizing the irreversibility term in the CIR is one of the key contributions of our analysis, as it opens the door to studying transitional dynamics in environments with history dependence—that is, in which the first stopping time does not fully absorb the effects of an aggregate shock. These types of problems are labeled "problems with reinjection" by Álvarez and Lippi (2021).

4 On the Aggregate Consequences of Investment Frictions

In this section, we explore in depth how the nature of adjustment frictions matters for the macroeconomy. Specifically, we show that the size of the price wedge relative to the fixed cost matters and that the role of the price wedge hinges on the size of the drift (which consists of productivity growth and capital depreciation) relative to the volatility of the idiosyncratic shocks. We consider three cases that showcase how investment frictions map onto aggregate outcomes and isolate the mechanisms at play.

In what follows, we exploit the decomposition between frictionless and frictional policies in Proposition 1, work in the space of normalized capital-productivity ratios $x \equiv \hat{k} - \hat{k}^{ss}$, and consider second-order approximations to the profit function.

4.1 Zero drift

First, we consider an environment with zero drift ($\nu = 0$). This case is relevant for economies or sectors with low productivity growth and/or a low depreciation rate, in which idiosyncratic shocks are the main drivers of investment. In this driftless and symmetric environment, the price wedge constitutes an important friction because firms expect to purchase and sell capital with equal probability. Proposition 4 characterizes two subcases: only fixed cost and only price wedge.[8]

Proposition 4. *Assume $\nu \to 0$. Construct $\hat{k}^{ss}$ using the price $(p^{buy} + p^{sell})/2 = p(1 - \omega/2)$ such that the value function is symmetric. The effective price wedge equals $\tilde{\omega} = \omega\alpha/\mathcal{U}$ and the effective fixed cost equals $\tilde{\theta} = \left[p(1-\omega/2)\mathcal{U}/(\alpha A^{1/\alpha})\right]^{\frac{\alpha}{1-\alpha}} \theta$; the user cost of capital is $\mathcal{U} = \rho - \sigma^2$; and the discount factor is $r = \rho - \sigma^2/2$. In all symmetric cases $\mathbb{E}[\hat{k}] = \hat{k}^{ss}$, $\mathbb{E}[x] = 0$, and $\mathbb{C}ov[x, a] = 0$.*

[8] The driftless symmetric case with both frictions active is studied in Appendix A.9. The main result is that a marginal increase in one friction, when the other is large, has tiny effects on the macro outcomes.

(i) ***Only fixed cost:*** *If $\omega = 0$, then the inaction thresholds are $\bar{x} = \pm\left(\frac{6\tilde{\theta}\sigma^2}{\alpha(1-\alpha)}\right)^{1/4}$, the reset point is $x^* = 0$, and the macro outcomes are*

$$(40) \qquad \mathbb{V}ar[x] = \frac{\overline{x}^2}{6}; \qquad q = 1 - \frac{\mathcal{U}}{r}\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]; \qquad \frac{CIR(\delta)}{\delta} = \frac{\mathbb{V}ar[x]}{\sigma^2}.$$

(ii) ***Only price wedge:*** *If $\theta = 0$, then the inaction thresholds and the reset points coincide at $\overline{x}^* = \pm\left(\frac{3\tilde{\omega}\sigma^2}{4\alpha(1-\alpha)}\right)^{1/3}$, and the macro outcomes are*

$$(41) \quad \mathbb{V}ar[x] = \frac{\overline{x}^{*2}}{3}; \quad q = 1 - \left(1 + \frac{2}{\alpha}\right)\frac{\mathcal{U}}{r}\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]; \quad \frac{CIR(\delta)}{\delta} = \left(1 + \frac{1}{\sigma^2}\right)\mathbb{V}ar[x].$$

In the two cases above, there is a positive relationship between the corresponding effective investment friction ($\tilde{\theta}$ or $\tilde{\omega}$) and the cross-sectional dispersion $\mathbb{V}ar[x]$. In turn, larger dispersion lowers aggregate productivity $\hat{Y}/\hat{K}$ because of decreasing returns, which reduces q; larger dispersion reduces responsiveness and slows the propagation of aggregate productivity shocks, which increases the CIR. Thus we learn that with a small drift, the cross-sectional dispersion is the main driver behind capital valuation and fluctuations. We also learn that if effective frictions were of the same size, that is, $\tilde{\theta} = \tilde{\omega}$, a price wedge generates a higher $\mathbb{V}ar[x]$, a lower q, and a larger CIR compared with the case with only fixed costs.

4.2 Large drift

Second, we consider a very large drift ($\nu \to \infty$). This case is relevant for economies or sectors in which forces to upsize the capital stock dominate (productivity growth and depreciation) and idiosyncratic shocks play only a minor role for investment. The price wedge is irrelevant here, because firms prefer to downsize by allowing the drift to erode their capital instead of facing the penalty. Proposition 5 establishes this result. Note that instead of taking the drift to infinity, we take an equivalent limit toward zero volatility of idiosyncratic shocks.

Proposition 5. *Let $\nu > 0$ and $\sigma^2 \to 0$ such that $\nu/\sigma^2 \to \infty$. Construct $\hat{k}^{ss}$ using the purchase price p. The effective fixed cost equals $\tilde{\theta} = \left[p\mathcal{U}/(\alpha A^{1/\alpha})\right]^{\frac{\alpha}{1-\alpha}}\theta$ and the effective price wedge is irrelevant. In this case, the user cost is $\mathcal{U} = \rho + \xi^k$ and the discount is $r = \rho - \mu$. The policy is a one-sided inaction region with lower threshold x^- and one reset point x^*. The cross-sectional distribution is uniform over $[x^-, x^*]$ with moments*

$$(42) \qquad \mathbb{E}[x] = \frac{(x^* + \overline{x})}{12}; \qquad \mathbb{V}ar[x] = \frac{(x^* - \overline{x})^2}{12}.$$

The policy solves the non linear system

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} \;=\; -\frac{r\tilde{\theta}}{\sqrt{12}\alpha(1-\alpha)}; \qquad \frac{\mathbb{E}[x]}{\mathbb{V}ar[x]+\mathbb{E}[x]^2} \;=\; -\left(\frac{r}{\nu}+\frac{\alpha+1}{2}\right), \tag{43}$$

and the macro outcomes are

$$q \;=\; 1 \;-\frac{U}{r}(1-\alpha)\left(\mathbb{E}[x]+\frac{\alpha}{2}\mathbb{V}ar[x]\right); \qquad \frac{CIR(\delta)}{\delta} \;=\; 0. \tag{44}$$

The case with a large drift reveals new mechanisms that are absent in symmetric environments. As we have already stated, the price wedge has no effect. Comparing the expression for aggregate q and the CIR with large drift against the driftless cases in Proposition 4, we see that now the average $\mathbb{E}[x]$ matters. Moreover, the frictional average $\mathbb{E}[x]$ and not the frictionless average $\mathbb{E}[\hat{k}]$ is the relevant statistic for the marginal value of capital. The non linear system in (43) that pins down the investment policy implies that larger effective fixed costs $\tilde{\theta}$ increase both the average $\mathbb{E}[x]$ (in absolute value) and the variance $\mathbb{V}ar[x]$ of the normalized capital-productivity ratios x. In fact, the first equation in (43) is an indifference curve that mediates the trade-off between these two moments. The system also implies that the average $\mathbb{E}[x]$ is negative, and thus $\mathbb{E}[\hat{k}] = \hat{k}^{ss} + \mathbb{E}[x] < \hat{k}^{ss}$. As the mean becomes more negative with higher fixed costs, q goes up; but as the variance increases, q goes down. The overall effect depends on the relative elasticities of these moments with respect to $\tilde{\theta}$. Lastly, the CIR equals zero: With an extreme drift, aggregate shocks are immediately absorbed and there are no deviations from steady state (see Corollary 2 in Baley and Blanco, 2021).

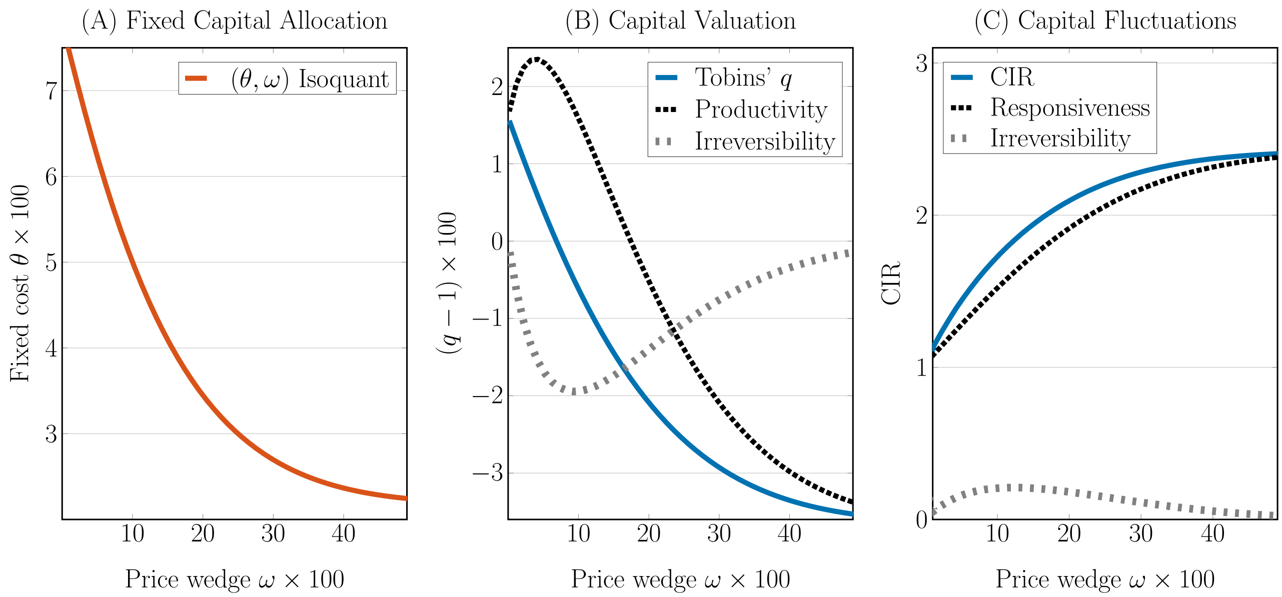
4.3 Intermediate drift

Finally, we consider the empirically relevant case with an intermediate drift (about 11% in our calibration), in which the two investment frictions are relevant.[9] Clearly, varying the relative size of frictions changes many endogenous objects; thus, to deliver a consistent comparison across model configurations we must fix some target. In the spirit of Hsieh and Klenow (2009), we fix the cross-sectional dispersion of marginal products of capital, $\mathbb{V}ar[\hat{k}]$, since it summarizes the pervasiveness of investment frictions. Fixing $\mathbb{V}ar[\hat{k}]$ implicitly fixes the frequency of adjustment and the dispersion of investment rates. Next, we use a calibrated version of the model (to be discussed in the following section) to illustrate the relevance of the price wedge in the aggregate.

For each price wedge $\omega \in [0.1, 0.5]$, we find the fixed cost θ that delivers a constant level of $\mathbb{V}ar[\hat{k}]$. Panel A in Figure III plots one of these isoquants (θ, ω), which happens to be convex. Panels B and C plot q and the CIR against the price wedge ω, computed along the isoquant. In all figures, going from left to right increases the relative importance of the price wedge vis-à-vis the

[9] We refer to Miao (2019) for a characterization of the case with full irreversibility ($\omega = 1$) for any drift $\nu \in \mathbb{R}$.

Figure III – Macroeconomic outcomes for fixed allocation



Notes: Panel A shows the (θ, ω) isoquant that matches the empirical value of $\mathbb{V}ar[\hat{k}] = 0.098$ (see Section 5.3). Panel B plots q and its components, normalizing them as $(q-1) \times 100$. Panel C plots the CIR and its components. Other parameters are from Table I.

fixed cost while delivering the same level of cross-sectional dispersion $\mathbb{V}ar[\hat{k}]$. We observe that as price wedges become dominant, capital valuation decreases (q goes down) and capital fluctuations become more persistent (CIR goes up) relative to environments in which fixed costs dominate.

We further decompose q into its productivity and irreversibility components in (32). The irreversibility term (dashed line) is non monotonic: It goes to zero for low ω (when there is no wedge) and for high ω (when no firm disinvests and the wedge becomes irrelevant). We find that irreversibility has the largest negative impact on q for a wedge of around 10%. To examine the productivity term, note that since $\mathbb{V}ar[\hat{k}]$ remains fixed, this is exclusively determined by capital accumulation reflected in (minus) the average capital-productivity ratio $\mathbb{E}[\hat{k}]$. The productivity term (dotted line) is also non monotonic: At low ω, an increase in the wedge decreases both prices (q) and quantities ($\mathbb{E}[\hat{k}]$) because firms are more cautious about investing; whereas at high ω an increase in the wedge decreases the price (q) but increases quantities ($\mathbb{E}[\hat{k}]$) because firms stop disinvesting.

We also decompose the CIR into its responsiveness and irreversibility components in (38). Because $\mathbb{V}ar[\hat{k}]$ remains fixed, the responsiveness term (dotted) is exclusively determined by the covariance of capital-productivity ratios with their age, $\mathbb{C}ov[\hat{k}, a]$. As shown in Baley and Blanco (2021), this covariance reflects asymmetry in the costs of downsizing vs. upsizing the capital stock. It is zero in driftless and symmetric environments. It is negative when there is drift and fixed costs and counteracts the effect of the variance for the CIR. With price wedges, the covariance turns positive because of the strong downsizing friction. Besides the covariance effect, price wedges have an additional (non monotonic and quantitatively smaller) direct effect on the CIR (dashed line).

Summing up. The previous analysis teach us three lessons. First, the importance of the price wedge for macro outcomes crucially depends on the size of the drift. Second, the effect of frictions on q is ambiguous, as it depends on the mean $\mathbb{E}[x]$ and the variance $\mathbb{V}ar[x]$ of normalized capital-productivity ratios that may move in opposite directions as frictions increase. And third, keeping the capital allocation fixed, an economy or sector in which the price wedge is the dominant friction features lower capital valuation and more persistent capital fluctuations compared with an economy or sector in which fixed costs dominate.

In Section 6, we show that corporate income tax cuts correspond to movements along the (θ, ω) isoquant from left to right, which increases the relative importance of price wedges and changes long-run capital behavior.

5 Measuring macro outcomes with microdata

This section derives mappings from the microdata to parameters, policies, and cross-sectional moments of the invariant distribution $g(\hat{k})$. Together with the relationship established between cross-sectional moments and macro outcomes in Section 3, these mappings connect macro outcomes with microdata.

Given an exogenously determined price wedge ω and panel of observations $\Omega = \{\Delta\hat{k}, \tau\}$, which includes the adjustment size and the duration of inaction, we infer the behavior of non-adjusters in the cross-section and reverse-engineer the reset points $\{\hat{k}^{*-}, \hat{k}^{*+}\}$, the parameters of the productivity process $\{\nu, \sigma^2\}$, and the key cross-sectional moments that enter q and the CIR:

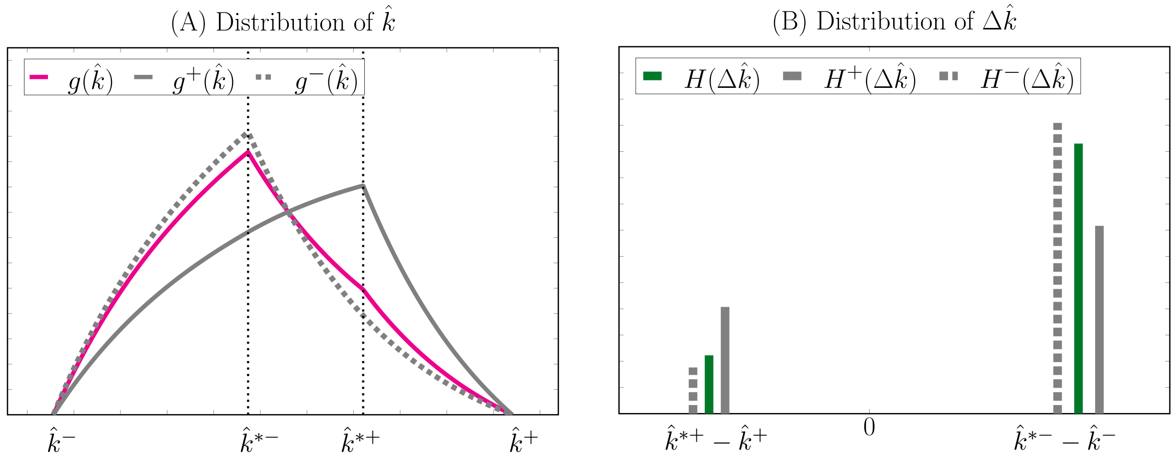
$$\left\{\mathbb{E}[\hat{k}],\ \mathbb{V}ar[\hat{k}],\ \mathbb{C}ov[\hat{k}, a],\ \mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{P}(\hat{k}_s)\phi(\hat{k}_s))\right]\right],\ \mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)]\right]\right\}. \tag{45}$$

To obtain these mappings, we condition adjusters' behavior on the sign of the last adjustment so that their actions remain Markovian. Then, we exploit properties of Markov processes and the fact that the two reset points are constant. Next, we define the conditional densities required to handle the history dependence from irreversibility.

5.1 Conditional distributions

Let $g^-(\hat{k})$ and $g^+(\hat{k})$ denote the stationary density of $\hat{k}$ conditional on the last reset point being $\hat{k}^{*-}$ or $\hat{k}^{*+}$, respectively. They satisfy the same KFE as $g(\hat{k})$, except that they only have one kink at the corresponding reset point. Panel A in Figure IV plots the three densities g, g^-, and

Figure IV – Unconditional and Conditional Distributions of $\hat{k}$ and $\Delta\hat{k}$



Notes: These figures illustrate conditional and unconditional distributions. Panel A plots the unconditional density $g(\hat{k})$ and the densities conditional on the last reset $g^{\pm}(\hat{k})$. Panel B plots the unconditional distribution $H(\Delta\hat{k})$ and the distributions conditional on the last reset $H^{\pm}(\Delta\hat{k})$.

g^{+} (these are proper densities and integrate to 1). We denote expectations computed with these distributions as $\mathbb{E}$, $\mathbb{E}^{-}$, and $\mathbb{E}^{+}$.

Next, we consider the distribution over actions, denoted by $H(\Delta\hat{k},\tau)$, and the distributions of actions *conditional on the last reset point*: $H^{-}(\Delta\hat{k},\tau)$ and $H^{+}(\Delta\hat{k},\tau)$. Panel B of Figure IV plots the marginal distributions of adjustment size, $H(\Delta\hat{k})$, $H^{-}(\Delta\hat{k})$, $H^{+}(\Delta\hat{k})$, where we have integrated out the duration τ; these distributions correspond to probability masses at two points $\Delta\hat{k}=\hat{k}^{*+}-\hat{k}^{+}<0$ and $\Delta\hat{k}=\hat{k}^{*-}-\hat{k}^{-}>0$.

Note that the mass of upward adjustments $H(\hat{k}^{*-}-\hat{k}^{-})$ is larger than the mass of downward adjustments $H(\hat{k}^{*+}-\hat{k}^{+})$. This is because the drift shrinks capital-productivity ratios over time and prompt upward adjustments and because partial irreversibility penalizes downward adjustments. This asymmetry is also observed in the firms' distribution, since g is closer to g^{-}. Second, the conditional masses reflect the autocorrelation in the investment sign; for instance, $H^{-}>H^{+}$ at $\Delta\hat{k}>0$ means that the probability of reseting to $\hat{k}^{*-}$ is larger whenever the last reset point was also $\hat{k}^{*-}$. In other words, positive investments beget future positive investments. We denote with bars the expectations computed with the distributions of adjusters: $\overline{\mathbb{E}}$, $\overline{\mathbb{E}}^{-}$, and $\overline{\mathbb{E}}^{+}$.

From conditional to unconditional distributions. Define the shares of upward $\mathcal{N}^{-}/\mathcal{N}$ and downward $\mathcal{N}^{+}/\mathcal{N}$ adjustments within the *population of adjusters*. By Bayes' law, the unconditional and conditional distribution of adjusters satisfy

$$H(\Delta\hat{k},\tau) \;=\; \frac{\mathcal{N}^{-}}{\mathcal{N}}H^{-}(\Delta\hat{k},\tau) \;+\; \frac{\mathcal{N}^{+}}{\mathcal{N}}H^{+}(\Delta\hat{k},\tau). \tag{46}$$

This relationship is useful for computing moments of adjusters. For example, the average duration of inaction equals the weighted sum of the average conditional durations:

$$\overline{\mathbb{E}}[\tau] \;=\; \overline{\mathbb{E}}\left[\mathbb{E}[\tau|\Delta k]\right] \;=\; \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau] \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau]. \tag{47}$$

However, we need to leverage another approach to recover the unconditional distribution of firms. In that case, the shares must be rescaled by the relative durations of inaction:

$$g(\hat{k}) \;=\; \frac{\mathcal{N}^-}{\mathcal{N}}\frac{\overline{\mathbb{E}}^-[\tau]}{\overline{\mathbb{E}}[\tau]}\, g^-(\hat{k}) \;+\; \frac{\mathcal{N}^+}{\mathcal{N}}\frac{\overline{\mathbb{E}}^+[\tau]}{\overline{\mathbb{E}}[\tau]}\, g^+(\hat{k}) \;=\; \mathcal{N}^-\, \overline{\mathbb{E}}^-[\tau]\, g^-(\hat{k}) \;+\; \mathcal{N}^+\, \overline{\mathbb{E}}^+[\tau]\, g^+(\hat{k}), \tag{48}$$

where we simplify the expression using $\overline{\mathbb{E}}[\tau] = \mathcal{N}^{-1}$; that is, the average duration of inaction equals the inverse of the total frequency of adjusters. This implies that the duration-adjusted frequencies also sum up to one, i.e., $\mathcal{N}^-\, \overline{\mathbb{E}}^-[\tau] + \mathcal{N}^+\, \overline{\mathbb{E}}^+[\tau] = 1$. Why do we need to rescale by duration? The answer is the *fundamental renewal property:* The average behavior in the economy is attributable to firms with longer periods of inaction (which are observed less frequently). Adjusting the shares with their relative duration corrects this observational bias. In environments with irreversibility, the slowly adjusting firms are coincidentally those that make downward adjustments.[10]

To illustrate the power of our "conditional" approach, we use the law of total variance to decompose capital allocation $\mathbb{V}ar[\hat{k}]$ into two terms that condition on the sign of the last adjustment:

$$\underbrace{\mathbb{V}ar[\hat{k}]}_{\text{total}} \;=\; \underbrace{\mathbb{E}\left[\mathbb{V}ar[\hat{k}|\Delta\hat{k}]\right]}_{\text{within}} \;+\; \underbrace{\mathbb{V}ar\left[\mathbb{E}[\hat{k}|\Delta\hat{k}]\right]}_{\text{between}}. \tag{49}$$

The decomposition in (49) is useful for assessing the relative importance of each investment friction in generating capital misallocation. The first term is the average of the variance *within* each conditional distribution g^+ and g^-—that is, the average of $\mathbb{V}ar^-[\hat{k}]$ and $\mathbb{V}ar^+[\hat{k}]$ computed using (57) below conditioning on the sign of $\Delta\hat{k}$ and using the conditional renewal measure as in (47). Both investment frictions add to this dispersion. The second term reflects the distance *between* the conditional means $\mathbb{E}^-[\hat{k}]$ and $\mathbb{E}^+[\hat{k}]$ (computed using (56) below conditioning on the sign of $\Delta\hat{k}$ and using the conditional renewal measure). This term arises exclusively from the price wedge that generates two different means. The larger the price wedge, the further apart are the conditional means and the larger the between variance. Note that this term is zero when only fixed costs are present since there is a unique reset point.

5.2 Data mappings

Next, we use the conditional distributions of adjusters to back out parameters, reset points, and

[10] Appendix A.11 presents an illustrative example of how to use relative frequencies, Bayes' law, and renewal theory to back out unconditional distributions from conditional distributions.

cross-sectional moments. To facilitate exposition, we present the mappings for these objects separately, but in fact they are all recovered simultaneously through a system of equations that can be solved iteratively and substitutes the population moments with their sample counterparts (see Appendix D.4).

Parameters. We begin by recovering the parameters (ν, σ^2) of the stochastic process of capital-productivity ratios in Proposition 6.

Proposition 6. *The parameters of the stochastic process for productivity (ν, σ^2) are recovered from the microdata $\Omega \equiv (\Delta\hat{k}, \tau)$ as follows:*

$$\nu = \frac{\overline{\mathbb{E}}[\Delta\hat{k}]}{\overline{\mathbb{E}}[\tau]}, \tag{50}$$

$$\sigma^2 = \frac{\overline{\mathbb{E}}[(\hat{k}_\tau + \nu\tau)^2] - \overline{\mathbb{E}}[(\hat{k}^*)^2]}{\overline{\mathbb{E}}[\tau]}. \tag{51}$$

Expression (50) recovers the drift (which includes depreciation ξ^k and productivity growth μ) from the average adjustment size times the frequency of adjustment (the inverse of the expected duration of inaction $\mathcal{N} = \overline{\mathbb{E}}[\tau]^{-1}$), while expression (51) recovers the volatility of idiosyncratic shocks from the second moments of the adjustment size, also scaled by frequency.[11]

Reset points. We continue with the reset points. As a preliminary step, we note that the optimal stopping policy τ^* and the optimal reset points $\{\hat{k}^{*-}, \hat{k}^{*+}\}$ satisfy

$$p^{buy} e^{\hat{k}^{*-}} = \mathbb{E}\left[\int_0^{\tau^*} \alpha e^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k})\; e^{-r\tau^*+\hat{k}_{\tau^*}} \middle| \hat{k}_0 = \hat{k}^{*-}\right], \tag{52}$$

$$p^{sell} e^{\hat{k}^{*+}} = \mathbb{E}\left[\int_0^{\tau^*} \alpha e^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k})\; e^{-r\tau^*+\hat{k}_{\tau^*}} \middle| \hat{k}_0 = \hat{k}^{*+}\right]. \tag{53}$$

We obtain this result by applying the envelope theorem for arbitrary choice sets (Milgrom and Segal, 2002) to expression (10) and then using the optimality condition at the reset points. The expressions say that when a firm resets its capital-productivity ratio to either $\hat{k}^{*-}$ or $\hat{k}^{*+}$, it equalizes marginal costs to the marginal benefits. When purchasing capital, the marginal cost is the investment price $p^{buy} e^{\hat{k}^{*-}}$ and the marginal benefit includes the cumulative marginal profits obtained during the expected duration of its inaction period plus the expected value of its undepreciated capital at the next adjustment date. Since we work on log scale, benefits and costs are expressed in percentage points (and thus multiplied by $e^{\hat{k}}$). Proposition 7 presents a mapping from the

[11] We obtained similar mappings from the data to the parameters in Baley and Blanco (2021) for the case without irreversibility. Irreversibility does not change the mapping to the drift, but it changes the mapping to the volatility, because it affects the transition speed across the two reset points.

microdata to the two reset points.

Proposition 7. *Let $\Phi(\nu,\sigma^2) \equiv \log\left(\alpha A/(r+\alpha\nu-\alpha^2\sigma^2/2)\right)$ be a function of structural parameters. For each $\Delta\hat{k}$, construct the stopped capital $\hat{k}_\tau(\Delta\hat{k})$ using (16). Then the reset points are recovered from the microdata $\Omega \equiv (\Delta\hat{k},\tau)$ as follows:*

$$\hat{k}^{*-} = \frac{1}{1-\alpha}\left[\Phi(\nu,\sigma^2) - \log(p^{buy}) + \log\left(\frac{1-\overline{\mathbb{E}}^{-}\left[e^{-\hat{r}\tau+\alpha\left(\hat{k}_\tau-\hat{k}^{*+}\right)}\right]}{1-\overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k})}{p^{buy}}e^{-\hat{r}\tau+\hat{k}_\tau-\hat{k}^{*+}}\right]}\right)\right], \tag{54}$$

$$\hat{k}^{*+} = \frac{1}{1-\alpha}\left[\Phi(\nu,\sigma^2) - \log(p^{sell}) + \log\left(\frac{1-\overline{\mathbb{E}}^{+}\left[e^{-\hat{r}\tau+\alpha\left(\hat{k}_\tau-\hat{k}^{*-}\right)}\right]}{1-\overline{\mathbb{E}}^{+}\left[\frac{p(\Delta\hat{k})}{p^{sell}}e^{-\hat{r}\tau+\hat{k}_\tau-\hat{k}^{*-}}\right]}\right)\right]. \tag{55}$$

The first term $\Phi(\nu,\sigma^2)$ in expressions (54) and (55) reflects the ratio of profitability to the user cost of capital. Through this ratio, both reset states increase with profitability A and idiosyncratic risk σ^2 and decrease with the discount r and the drift ν. The second term shows that reset points decrease with the corresponding investment price: Firms invest more the lower the purchasing price p^{buy} and disinvest less the lower the selling price p^{sell}. Lastly, the third term shows how investment frictions shape the reset points through the marginal profits accrued during periods of inaction (in the numerator) and the resale value (in the denominator).

As a direct measure of irreversibility, consider the difference between the two reset points $(\hat{k}^{*+}-\hat{k}^{*-})$. The term $\Phi(\nu,\sigma^2)$ cancels out in the difference. The second term equals $-(\log(1-\omega))/(1-\alpha) > 0$ and reflects the exogenous price wedge, which is further amplified by the elasticity of output to capital α. The third term reflects history dependence. As long as the optimal policy $(\Delta\hat{k},\tau)$ depends on the last reset point, endogenous irreversibility arises beyond the exogenous price wedge.

Cross-sectional moments. With the parameters and reset points in hand, Proposition 8 recovers steady-state moments of capital-productivity ratios $\hat{k}$.

Proposition 8. *For each inaction spell, find the departing point $\hat{k}^*$ and the ending point $\hat{k}_\tau$ using (15) and (16). Then the unconditional mean and variance of $\hat{k}$ and the covariance between $\hat{k}$ and age a are recovered from the microdata $\Omega \equiv (\Delta\hat{k},\tau)$ as follows:*

$$\mathbb{E}[\hat{k}] = \overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\left(\frac{\hat{k}^*+\hat{k}_\tau}{2}\right)\left(\frac{\hat{k}^*-\hat{k}_\tau}{\overline{\mathbb{E}}[\Delta\hat{k}]}\right)\middle|\Delta\hat{k}\right]\right] + \frac{\sigma^2}{2\nu}, \tag{56}$$

$$\mathbb{V}ar[\hat{k}] = \overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\left((\hat{k}^*-\mathbb{E}[\hat{k}])(\hat{k}_\tau-\mathbb{E}[\hat{k}]) + \frac{(\hat{k}^*-\hat{k}_\tau)^2}{3}\right)\left(\frac{\hat{k}^*-\hat{k}_\tau}{\overline{\mathbb{E}}[\Delta\hat{k}]}\right)\middle|\Delta\hat{k}\right]\right], \tag{57}$$

$$\mathbb{C}ov[\hat{k},a] = \frac{1}{2\nu}\left(\mathbb{V}ar[\hat{k}] - \frac{\overline{\mathbb{E}}[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^2\tau]}{\overline{\mathbb{E}}[\tau]} + \frac{\sigma^2}{2}\frac{\overline{\mathbb{E}}[\tau]}{2}(1+\overline{\mathbb{CV}}^2[\tau])\right). \tag{58}$$

The mapping in (56) recovers the population mean $\mathbb{E}[\hat{k}]$ from the average midpoint between the departing and the ending points of an inaction spell $(\hat{k}^* + \hat{k}_\tau)/2$, where the average is computed under a change of measure induced by the renewal weights $(\hat{k}^* - \hat{k}_\tau)/\overline{\mathbb{E}}[\Delta\hat{k}]$. To recover the population mean, the renewal measure overweighs the midpoints of adjusters with longer periods of inaction, which are more representative in the population.[12] The term $\sigma^2/2\nu$ corrects for the accumulated drift between adjustments. Similarly, the mapping in (57) recovers the population variance $\mathbb{V}ar[\hat{k}]$ from the average distance between the departing point and the mean $(\hat{k}^* - \mathbb{E}[\hat{k}])$, the ending point and the mean $(\hat{k}_\tau - \mathbb{E}[\hat{k}])$, and the departing and ending points $(\hat{k}^* - \hat{k}_\tau)^2$, again computed using the renewal distribution. In these expressions, we compute the inner expectation with H^- or H^+ depending on the sign of the last adjustment and compute the outer expectation with shares of upward $\mathcal{N}^-/\mathcal{N}$ and downward $\mathcal{N}^+/\mathcal{N}$ adjustment in the population. Lastly, expression (58) recovers the covariance $\mathbb{C}ov[\hat{k}, a]$ from the difference in the second moments of the distribution of adjusters relative to non-adjusters.

Irreversibility component of q. Finally, we derive data mappings to back out the irreversibility terms. Proposition 9 recovers the irreversibility term for q in (32), which reflects the way in which expected capital losses affect capital valuation.

Proposition 9. *The irreversibility term of q is negative and we recover it from the microdata as*

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{P}(\hat{k}_s)\phi(\hat{k}_s))\right]\right] = -\frac{\overline{\mathbb{C}ov}\left[\Delta\hat{k}, \mathcal{P}(\hat{k}^*(\Delta\hat{k}))\right]}{\overline{\mathbb{E}}[\tau]} < 0. \tag{59}$$

According to (59), the irreversibility term of q maps to minus the covariance of investment $\Delta\hat{k}$ and capital losses $\mathcal{P}(\Delta\hat{k})$, scaled by average duration. This covariance is positive, since firms purchase capital at the high price and sell capital at the low price. Since the covariance is positive, irreversibility reduces q. Intuitively, firms seek to avoid histories in which, after upsizing, negative productivity shocks will force them to downsize and face the penalty of selling their capital at a discount. Firms also seek to avoid histories in which, after downsizing, positive productivity shocks will force them to upsize and face the penalty of purchasing back capital at a higher price. To minimize the likelihood of these "switching" situations, firms underinvest and underdisinvest, which effectively reduces capital valuation. The strength of this mechanism is measured in the data through this covariance.

Irreversibility component for CIR. Proposition 10 below recovers the irreversibility term for the CIR in (38), which reflects the expected cumulative deviations from the steady-state mean of capital-productivity ratios that arise following the aggregate productivity shock that changes

[12] Without the price wedge, the renewal weights are equal to the relative size of adjustment $\Delta k/\overline{\mathbb{E}}[\Delta\hat{k}]$.

the masses of adjusters across reset points. Before we proceed to the Proposition, we provide the expressions for $\mathcal{M}^{buy}$ and $\mathcal{M}^{sell}$ we used to define the function $\mathcal{M}(\hat{k})$ in (39). These two numbers equal to the cumulative deviations below and above the steady-state mean $\mathbb{E}[\hat{k}]$ conditional on the sign of the last reset. Formally, they are equal to

$$\mathcal{M}^{buy} \equiv (\mathbb{E}^-[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^-[\tau]\,\frac{\mathbb{E}[\mathbb{P}^+]}{\mathbb{P}^{-+}} < 0, \tag{60}$$

$$\mathcal{M}^{sell} \equiv (\mathbb{E}^+[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^+[\tau]\,\frac{\mathbb{E}[\mathbb{P}^-]}{\mathbb{P}^{+-}} > 0. \tag{61}$$

The number $\mathcal{M}^{buy}$ reflects firms' upsizing behavior in steady state (analogously, $\mathcal{M}^{sell}$ reflects firms' downsizing behavior in steady state, mutatis mutandis). Upsizing firms reset their capital-productivity ratio below the unconditional mean and, on average, remain below the mean for the duration of their inaction spell. The average deviation accumulated during one inaction spell is then $(\mathbb{E}^-[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^-[\tau]$. Since the investment sign is serially correlated, upsizing firms remain in an upsizing phase and contribute to negative deviations for several periods; they would only leave this phase after a series of negative shocks causes them to downsize. The ratio $\mathbb{E}[\mathbb{P}^+]/\mathbb{P}^{-+}$ exactly reflects the average time spent in the transient upsizing phase, where $\mathbb{E}[\mathbb{P}^+] \equiv \Pr[\Delta\hat{k}' < 0]$ is the unconditional probability of downsizing and $\mathbb{P}^{-+} \equiv \Pr[\Delta\hat{k}' < 0|\Delta\hat{k} > 0]$ is the probability of downsizing conditional on currently being in an upsizing phase.

Proposition 10. *Then irreversibility term in the CIR is positive and is recovered from the microdata as*

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))\right]\right] = -\frac{\overline{\mathbb{C}ov}[\Delta\hat{k}, \mathcal{M}(\hat{k}^*(\Delta\hat{k}))]}{\overline{\mathbb{E}}[\tau]} > 0. \tag{62}$$

To compute $\mathcal{M}^{buy}$ and $\mathcal{M}^{sell}$, we recover means conditional on the sign of past adjustment as

$$\mathbb{E}^-[\hat{k}] = \frac{(\hat{k}^{*-})^2 - \overline{\mathbb{E}}^-[\hat{k}_\tau^2]}{2\overline{\mathbb{E}}^-[\Delta\hat{k}]} + \frac{\sigma^2}{2}\frac{\overline{\mathbb{E}}^-[\tau]}{\overline{\mathbb{E}}^-[\Delta\hat{k}]}; \qquad \mathbb{E}^+[\hat{k}] = \frac{(\hat{k}^{*+})^2 - \overline{\mathbb{E}}^+[\hat{k}_\tau^2]}{2\overline{\mathbb{E}}^+[\Delta\hat{k}]} + \frac{\sigma^2}{2}\frac{\overline{\mathbb{E}}^+[\tau]}{\overline{\mathbb{E}}^+[\Delta\hat{k}]}; \tag{63}$$

we recover unconditional probabilities as

$$\mathbb{E}[\mathbb{P}^+] = \frac{\overline{\mathbb{E}}[\tau\mathbb{I}(\Delta\hat{k} < 0)]}{\overline{\mathbb{E}}[\tau]}, \qquad \mathbb{E}[\mathbb{P}^-] = \frac{\overline{\mathbb{E}}[\tau\mathbb{I}(\Delta\hat{k} > 0)]}{\overline{\mathbb{E}}[\tau]}; \tag{64}$$

and we recover transition probabilities between reset points as:

$$\mathbb{P}^{+-} = \Pr[\Delta\hat{k}' > 0|\Delta\hat{k} < 0], \qquad \mathbb{P}^{-+} = \Pr[\Delta\hat{k}' < 0|\Delta\hat{k} > 0]. \tag{65}$$

According to (62), the irreversibility term in the CIR equals minus the covariance of investment $\Delta\hat{k}$ with the capital deviations $\mathcal{M}(\Delta\hat{k})$. The covariance is negative: Due to micro-history dependence, firms that make negative investments are expected to remain above the steady-state mean and contribute with positive deviations (similarly, firms that make negative investment are expected to contribute with positive deviations). Because a negative aggregate shock increases the mass of selling firms relative to steady state, and these firms contribute with positive deviations, the CIR goes up.

In summary, expressions (50) to (62) provide inverse mappings from the microdata $\Omega = \{\Delta \hat{k}, \tau\}$ to parameters, reset points, capital allocation $\mathbb{V}ar[\hat{k}]$, capital valuation q, and capital fluctuations CIR. In the next section, we apply these mappings to estimate the aggregate outcomes from the microdata and assess the role of investment frictions in shaping their empirical values.

5.3 Putting the theory to work

We apply the mappings using yearly investment data on manufacturing plants in Chile from the Annual National Manufacturing Survey (*Encuesta Nacional Industrial Anual*) for the period 1980 to 2011. To construct the capital series, we use information on depreciation rates and price deflators from national accounts and Penn World Tables. The sample considers plants that appear in the sample for at least 10 years (more than 60% of the sample) and have more than 10 workers. Appendix D presents all details of the data.

Capital stock and investment rates. We construct the capital stock series using the perpetual inventory method. We include structures, machinery, equipment, and vehicles. Following the theory, a plant's capital stock in year s, k_s, evolves as

$$k_s = (1 - \xi^k) k_{s-1} + I_s/(p(I_s) D_s), \tag{66}$$

where ξ^k is the physical depreciation rate; I_s is the nominal value of investment; $p(I_s)$ is the investment pricing function, which considers different prices for capital purchases and sales; D_s is the gross fixed capital formation deflator; and k_0 is a plant's self-reported nominal capital stock at current prices for the first year in which it is nonnegative. Note that the ratio $I_s/(p(I_s)D_s)$ is the real investment in capital units (the data counterpart to $i_s = \Delta k_s$ in the model).

Calibration of the irreversibility wedge. With irreversibility, we need information on the price wedge in order to compute real investment using the perpetual inventory method with two prices.[13] We pick a price wedge of 15% and set $\omega = 0.15$. This is a conservative number, relative to the estimates in Ramey and Shapiro (2001) and Kermani and Ma (2020). We pick a smaller number to reflect heterogeneity across sectors and types of capital and the possibility of internal

[13] Note that only the price wedge matters for computing investment, not the price level.

transfers of capital through mergers and acquisitions, as well as the fact that empirical estimates are obtained using information from liquidating firms and thus are likely to suffer from selection. Below, we conduct comparative statistics for the irreversibility parameter and discuss how, through the lens of the model, the probabilities of adjustment conditional on the last reset point serve as overidentifying restrictions for the parameter choice.

We construct gross nominal investment i_s with information on purchases, reforms, improvements, and sales of fixed assets, and define the investment rate ι_s as the ratio of real gross investment to the capital stock:[14]

$$\iota_s \equiv \frac{I_s/(p(I_s)D_s)}{k_{s-1}}. \tag{67}$$

For each plant and each inaction spell h, we record the change in the capital-productivity ratio upon action $\Delta\hat{k}_h$ and the spell's duration τ_h. We construct $\Delta\hat{k}_h$ with investment rates from (67):

$$\Delta\hat{k}_h = \begin{cases} \log(1+\iota_h) & \text{if} \quad |\iota_h| > \underline{\iota}, \\ 0 & \text{if} \quad |\iota_h| < \underline{\iota}. \end{cases} \tag{68}$$

The threshold $\underline{\iota} > 0$ reflects the idea that small maintenance investments should be excluded. Following Cooper and Haltiwanger (2006), we set $\underline{\iota} = 0.01$, such that all investment rates below 1% in absolute value are considered to be part of an inaction spell. Then we define an adjustment date T_h from $\Delta\hat{k}_{T_h} \neq 0$ and compute a spell's duration as the difference between two adjacent adjustment dates: $\tau_h = T_h - T_{h-1}$. Finally, we truncate the investment distribution at the 2nd and 98th percentiles to eliminate outliers.[15]

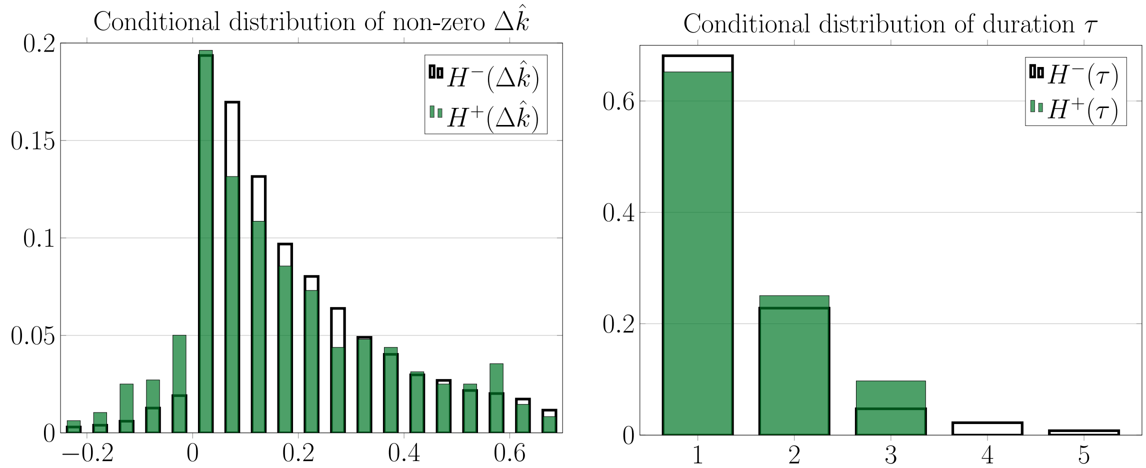
Figure V plots the resulting cross-sectional distribution of non-zero changes in the capital-productivity ratios $\Delta\hat{k}$ and completed inaction spells τ, conditional on a past positive or negative investment. The data show investment patterns consistent with partial irreversibility. In particular, the distribution of investment conditional on a last negative investment $H^+(\Delta k)$ is skewed toward the left of the distribution conditional on a last positive investment H^-, which means that the probability of a negative investment is larger after a negative investment, and vice versa.

Externally calibrated parameters. We externally calibrate several parameters to match average statistics from the Chilean economy between 1980 and 2011. One period equals a year. We set the real interest rate to 6.6% ($\rho = 0.066$) to match the average real interest rate computed by

[14] Note that the investment rate equals $\iota_{T_h} \equiv i_{T_h}/k_{T_h^-} = (k_{T_h} - k_{T_h^-})/k_{T_h^-}$, where $k_{T_h^-} = \lim_{s\uparrow T_h} k_s$. In contrast to the continuous-time model, in which investment is computed as the difference in the capital stock between two consecutive instants, in the data we compute it as the difference between two consecutive years. Potentially, there could be a bias arising from time aggregation as we take a continuous time model to annual data. We leave for further research the assessment of this bias.

[15] Table I in Appendix D presents descriptive statistics on investment rates. In particular, the inaction rate ($|\iota| < 0.01$) equals 40.1%.

Figure V – Empirical Distribution of Observable Actions



Notes: Own calculations using establishment data from Chile. Panel A plots the distribution of *nonzero* changes in capital-productivity ratios and Panel B plots the duration of inaction spells. Solid bars = conditional on departing from $\hat{k}^{*+}$ (last negative investment); white bars = conditional on departing from $\hat{k}^{*-}$ (last positive investment). Sample: Firms with at least 10 years of data, truncation at the 2nd and 98th percentiles of investment rate distribution, and inaction threshold of $\underline{\iota} = 0.01$.

Table I – Parameterization

Technology				Productivity		Normalization	
μ	α	ρ	ω	ν	σ^2	A	p
0.033	0.500	0.066	0.150	0.115	0.055	1.000	6.000

Notes: Baseline parameterization.

the IMF. The productivity growth rate is 3.3% ($\mu = 0.033$) to match the average GDP growth rate. The elasticity of output to capital is set to $\alpha = 0.5$ and conduct comparative statics below. We set the price level $p = 6$ to match an aggregate output-capital ratio of $\hat{Y}/p\hat{K} = 0.23$. Finally, we set $A = 1$ as a normalization.

Estimated parameters. Using the mappings from the microdata to the parameters of the productivity process in (50) and (51), we recover a drift of $\nu = 0.115$ and a volatility of $\sigma^2 = 0.055$. Together with the productivity growth rate, the value for the drift ν implies a physical depreciation rate of $\xi^k = \nu - \mu = 0.082$. Given these values, the adjusted discount is $r = \rho - \mu - \sigma^2/2 = 0.006$ and the user cost is $\mathcal{U} = \rho + \xi^k - \sigma^2 = 0.093$. The parameterization is summarized in Table I.

5.4 Aggregate capital behavior: 1980-2011

Given the parameters and data, we apply all data mappings. Table II reports the investment policy and average macro outcomes in Chile for the period between 1980 and 2011.

We begin by examining the investment policy. From (54) and (55), we recover a positive gap between the two reset points, $\hat{k}^{*+} - \hat{k}^{*-} = 0.568$, which is a tell-tale sign of partial irreversibility. This gap is almost equally explained by the exogenous price wedge, $\log(p^{\text{buy}}/p^{\text{sell}}) = 0.325$, and the endogenous response (computed as a residual) equals 0.243. Using (56) and (57), we estimate a dispersion in marginal products of $\mathbb{V}ar[\hat{k}] = 0.098$.

Table II – Aggregate Capital Behavior

Investment Policy		**Capital Allocation**	
Difference in reset capitals ($\hat{k}^{*+} - \hat{k}^{*-}$)	0.568	Variance	0.098
Exogenous price wedge	0.325		
Endogenous response	0.243		
Capital Valuation		**Capital Fluctuations**	
Tobin's q	1.06	CIR	3.07
Productivity	1.09	Responsiveness	2.29
Irreversibility	-0.03	Irreversibility	0.78

Notes: Objects recovered from theory mappings applied to establishment-level data from Chile. Parameters described in Table I.

We use (32) to recover an average capital valuation of $q = 1.06$. While q is not far from its frictionless value (unity), it would be erroneous to conclude that dynamic frictions are not present; in fact, they are important but affect mainly the stock of capital. The productivity component in (33) is 1.09 and the irreversibility component in (59) is -0.03. As predicted by the theory, irreversibility decreases q.

Lastly, using (38), we recover an average CIR of 3.07, which means that a 1% decrease in aggregate productivity generates a total deviation of aggregate capital above its steady-state value of 3.07%. We further decompose the CIR into its components: responsiveness 2.29 from (57) and irreversibility 0.78 from (62). As predicted by the theory, the CIR's irreversibility term is positive. Indirectly, irreversibility reduces sensitivity to idiosyncratic shocks (increasing $\mathbb{V}ar[\hat{k}]$) and increases the relative cost of downsizing the capital stock (increasing $\mathbb{C}ov[\hat{k}, a]$).

Comparative statics for ω and α. To assess how sensitive are the measured aggregate outcomes to the choice of irreversibility ω and returns to scale α, we conduct two robustness checks.

For the first check, as an overidentifying restriction, we compute conditional probabilities of upsizing and downsizing the capital stock in the data and in the calibrated model for our baseline choice of $(\omega, \alpha) = (0.15, 0.5)$. In the data, the conditional probability of upsizing given a last upward adjustment equals $\mathbb{P}^{--} = 0.96$, whereas the conditional probability of downsizing given a last downward adjustment equals $\mathbb{P}^{++} = 0.13$. In the calibrated model, these untargeted probabilities are close to their empirical counterparts, with values $\mathbb{P}^{--} = 0.98$ and $\mathbb{P}^{++} = 0.22$, respectively.

For the second check, we conduct comparative statics for these two parameters, one at a time, in the ranges $\omega \in \{0.05, 0.15, 0.25\}$ and $\alpha \in \{0.4, 0.5, 0.6\}$. The results, which are analogous to Table II, are reported in Appendix D.6. We note that the estimated productivity process (ν, σ) and the cross-sectional variance of capital to productivity ratios $\mathbb{V}ar[\hat{k}]$ do not vary significantly across parameterizations. Regarding other outcomes, we observe that q and the CIR decrease with both ω and α in the ranges considered. Also, there are interesting interactions between the two parameters, for instance, lower returns to scale α amplify the irreversibility component of the CIR. While variations in these parameters affect the estimates of the macro outcomes, the value of q remains consistently above 1.05 and the CIR consistently above 2.5. Therefore, the general messages about the role of investment frictions on macro outcomes remain valid.

6 The Macroeconomic Effects of Corporate Taxes

This section applies our methodology to study the effects of corporate tax reforms. We introduce a comprehensive tax schedule and characterize the role of corporate taxation in shaping macroeconomic outcomes. We do this in three steps. First, we show that corporate taxes change four parameters: profitability A, the discount factor ρ, the fixed cost θ, and the investment prices $p(\Delta\hat{k})$. Once we redefine these parameters, the investment problem is identical to the one described in Section 2. Second, we analyze the effect of taxation using the notion of *after-tax* effective frictions, which summarize the complex interaction of taxes with investment frictions with an appropriate rescaling of each friction. Third, we assess the effects of corporate income tax cuts using the mappings between macroeconomic outcomes, cross-sectional moments, and microdata.

6.1 A comprehensive tax schedule

Following Summers (1981) and Abel (1982), we introduce a corporate tax system to the firm problem. It includes a corporate income tax, deduction allowances, a personal income tax, and capital gains tax.[16] The firm pays the corporate income tax rate t^c on its cash flow Ay_s net of deductions, where ξ^d denotes the deduction rate. Since the physical and legal depreciation rates differ, we distinguish deductions from the capital stock and denote these with d_s. The state space now includes deductions $V(k, u, d)$.

The after–tax profits per unit of time π_s is given by

$$\pi_s = (1 - t^c)Ay_s + \underbrace{t^c \xi^d d_s}_{\text{deductions}} - \underbrace{t^c p \omega i_s \mathbb{1}_{\{i_s<0\}}}_{\text{capital losses}}, \tag{69}$$

[16] This taxation schedule is also used in the investment models of Poterba and Summers (1983); King and Fullerton (1984); Auerbach (1986); Auerbach and Hines (1986); and Hassett and Hubbard (2002).

where deductions evolve as

$$\log d_s \;=\; \log d_0 - \xi^d s + \sum_{h:T_h \le s} \left(1 + \frac{\theta_{T_h} + pi_{T_h}}{d_{T_h^-}} \right). \tag{70}$$

Let us explain how we model the interaction of taxes with investment frictions. First, we assume that the payments of fixed adjustment costs are capitalized and enter the law of motion of capital deductions in (70). This assumption is sensible to the extent that the fixed costs associated with investment or disinvestment are not fully deducted when they are paid. In contrast, when a firm sells its capital, it loses the future deductions valued at the buying price. For example, if $\omega = 1$ and the firm buys and sells capital immediately, then the capital depreciation allowance does not change. Second, we assume that capital losses are fully deducted from ordinary income in (69) on the date on which they are accrued.[17]

The personal income tax t^p and the capital gains tax t^g alter the firms' discount factor. We assume that equity is purchased by a representative investor with access to a riskless bond with return ρ per unit of time. Let D_s be the dividend per share, P_s the equity price per share, and $E_s = 1$ the number of shares, which we normalize to unity without loss of generality. From the investor's perspective, dividends and bond returns are taxed at the rate t^p, while capital gains that arise from changes in equity prices are taxed at the rate t^g. For any dividend process, no-arbitrage implies equal after-tax returns:

$$(1 - t^p)\rho \, \mathrm{d}s \;=\; (1 - t^g)\frac{\mathbb{E}[\mathrm{d}P_s]}{P_s} \;+\; (1 - t^p)\frac{D_s}{P_s}\, \mathrm{d}s. \tag{71}$$

Condition (71) pins down the time-0 value of the firm, which equals the equity price:

$$V(k_0, u_0, d_0) \;=\; P_0 \;=\; \frac{1 - t^p}{1 - t^g}\, \mathbb{E}_0 \left[\int_0^\infty e^{-\rho \frac{1-t^p}{1-t^g} s} D_s \, \mathrm{d}s \right]. \tag{72}$$

This expression says that the firm maximizes the cum-dividends market value of equity P_0 using the investor's after-tax discount factor $\rho(1 - t^p)/(1 - t^g)$, as in Auerbach (1979). We follow the "tax capitalization" view of the dividend decision and consider dividends as residuals, equal to

[17] In the U.S., the capital gains tax in the corporate sector is computed on observed transactions, as for households. Different from household taxation of capital gains however, firms do not receive a preferential tax rate; that is, firms are not allowed to use capital losses to offset ordinary income taxation. Instead, as with the corporate income tax, losses are carried backward or forward to compensate for gains. We abstract from these dimensions and assume that the corporate income tax is linear and applies to the sum of ordinary and capital income. See Desai and Gentry (2004) for a detailed discussion of capital gains taxes for the corporate sector. We thank Jim Hines for helpful advice on these assumptions.

the cash flow π_s net of investment and capital adjustment costs[18]

$$D_s \, \mathrm{d}s \;=\; \pi_s \, \mathrm{d}s - (\theta_s + (p(i_s) - t^c p \omega \mathbb{1}_{\{i_s<0\}}) i_s) \mathcal{D}(s = T_h), \qquad \mathcal{D}(\cdot) \sim \mathit{Dirac}. \tag{73}$$

Given the tax schedule, Lemma 3 characterizes the firm's problem with corporate taxation. The strategy consists of defining the discounted value of deductions per unit of investment z and using it to rewrite the 3-state problem (k, u, d) as the 1-state problem in terms of the capital-productivity ratio $\hat{k} = \log(k/u)$ already examined in Section 2.3, with four parametric changes and an additive term that reflects deductions d.

Lemma 3. *Define the discounted value of deductions:*

$$z \;\equiv\; \frac{\xi^d}{\rho \frac{1-t^p}{1-t^g} + \xi^d} < 1. \tag{74}$$

The firm's value with taxes can be decomposed as:

$$V(k, u, d) \;=\; \frac{1 - t^p}{1 - t^g} \left[u v(\hat{k}) \;+\; t^c z d \right], \tag{75}$$

where $v(\hat{k})$ solves the investment problem in Lemma 2 with the following four parametric changes:

$$A \;\rightarrow\; (1 - t^c) A, \tag{76}$$

$$\rho \;\rightarrow\; \frac{(1 - t^p)}{(1 - t^g)} \rho, \tag{77}$$

$$\theta \;\rightarrow\; (1 - t^c z) \theta, \tag{78}$$

$$p(\Delta \hat{k}) \;\rightarrow\; \left(1 - t^c z - \omega (1 - t^c) \mathbb{1}_{\{\Delta \hat{k} < 0\}} \right) p. \tag{79}$$

The parametric changes established in Lemma 3 highlight the different channels through which taxes affect the firm value and optimal policy. The corporate income tax t^c directly affects after-tax profitability A in (76). The personal income tax t^p and the capital gains tax t^g scale the discount factor ρ in (77).[19] The discounted value of deductions z affects the firm value through an income effect, as deductions increase additively the firm value in (75), and a substitution effect, as deductions promote investment by reducing the after-tax adjustment costs and after-tax prices in (78) and (79). Additionally, t^p and t^g operate indirectly through z and t^c affects directly the

[18] In the model without corporate taxes, the Modigliani-Miller theorem holds, that is, the firm's value and investment policy—and the implicit dividend policy—are independent of the capital structure. Introducing taxes, in principle, could break this independence (for example, under the trade-off theory of the capital structure; see Hines and Park, 2017). Nevertheless, following the arguments in Miller (1977) and more recently in Abel (2018), we continue to work under the Modigliani-Miller paradigm.

[19] The factor $(1 - t^p)/(1 - t^g)$ also scales A, θ, and $p(\hat{k})$. However, these parameters divide each other in all the expressions that follow, so we can safely ignore this factor.

effective level of partial irreversibility.

Next, we formalize the channels through which taxes affect investment through their interaction with frictions. To simplify the notation, we define the *after-tax* discount $\tilde{r}$ and the *after-tax* user cost of capital $\tilde{\mathcal{U}}$:

$$\tilde{r} \equiv \frac{1-t^p}{1-t^g}\rho - \mu - \frac{\sigma^2}{2}, \tag{80}$$

$$\tilde{\mathcal{U}} \equiv \frac{1-t^p}{1-t^g}\rho + \xi^k - \sigma^2. \tag{81}$$

In particular, the after-tax user cost $\tilde{\mathcal{U}}$ decreases with the personal income tax and increases with the capital gains tax. For the problem to be well defined, we assume $\tilde{r} > 0$ and $\tilde{\mathcal{U}} > 0$.

6.2 After-tax investment frictions

Proposition 11 introduces the notion of after-tax investment frictions, which are analogous to the effective frictions defined in (28) and (29) with the addition of taxes. Then, it signs the relationships with investment frictions.

Proposition 11. *Let* $z \equiv \xi^d/(\rho\frac{1-t^p}{1-t^g} + \xi^d) < 1$. *With corporate taxes, the effective fixed cost* $\tilde{\theta}$ *and effective price wedge* $\tilde{\omega}$ *are given by*

$$\tilde{\theta} = \left(\frac{1-t^c z}{1-t^c}\frac{1}{A}\right)^{\frac{1}{1-\alpha}} \left(\frac{p\tilde{\mathcal{U}}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} \theta, \tag{82}$$

$$\tilde{\omega} = \frac{(1-t^c)\alpha}{\tilde{\mathcal{U}}}\omega. \tag{83}$$

Corporate taxes have the following effects on the after-tax investment frictions:

1. *The effective fixed adjustment cost* $\tilde{\theta}$ *increases with* t^c *and the after-tax user cost* $\tilde{\mathcal{U}}$*, and decreases with the PDV of deductions* z*.*

2. *The effective price wedge* $\tilde{\omega}$ *decreases with the corporate income tax* t^c *and after-tax user cost* $\tilde{\mathcal{U}}$*, and does not change with the PDV of deductions* z*.*

Let us focus on the effects of the corporate income tax, which has opposing effects on after-tax investment frictions. On the one hand, a lower corporate income tax t^c increases profits and therefore *reduces the after-tax fixed cost*. This effect is mediated by the depreciation allowance rate, being lowest when $z = 1$ (in this case t^c is a pure profit tax) and highest when $\xi^d = z = 0$. On the other hand, a lower corporate income tax t^c *increases the after-tax price wedge* because it provides a smaller subsidy to capital losses. As anticipated, a corporate income tax cut generates an inward *displacement* together with a movement *along* the isoquant of after-tax frictions $(\tilde{\theta}, \tilde{\omega})$ (recall Figure III). Next, we apply these insights to study the aggregate effects of corporate taxes.

6.3 A regime shift from high to low taxes

This section explores the macroeconomic effects of a regime change from high to low taxes, focusing on a reduction in the corporate income tax rate and comparing outcomes across steady states.[20] We motivate this exercise with the observation that the top marginal corporate income tax rate experienced a median drop of 17 percentage points across OECD countries, from 42% in 1980 to 25% in 2020.[21] According to the theory, a decline in the corporate income tax rate t^c is equivalent to a reduction in the effective fixed cost $\tilde{\theta}$ and an increase in the effective price wedge $\tilde{\omega}$. Given our calibration, this reform unambiguously reduced misallocation, other things equal, but the consequences for capital valuation and capital fluctuations depend on the magnitude of the various forces we characterized before.

To discipline these forces, we use the parameterization that matches the average Chilean experience summarized in Table I. Additionally, we must take a stand on the size of the fixed cost θ in order to map the changes in the corporate tax onto changes in the after-tax fixed cost $\tilde{\theta}$. Note that taking a stand on the size of the fixed cost was not necessary in order to apply the mappings from microdata to macro outcomes in the previous section. Here, we search for a fixed-cost parameter using the method of moments to minimize the relative distance between two moments in the model and the data: the variance of capital-productivity ratios $\mathbb{V}ar[\hat{k}]$ and the covariance of capital-productivity ratios with the time elapsed since their last adjustment $\mathbb{C}ov[\hat{k}, a]$ (below, we discuss these choices and the implied model fit). We obtain a fixed cost of $\theta = 0.03$.

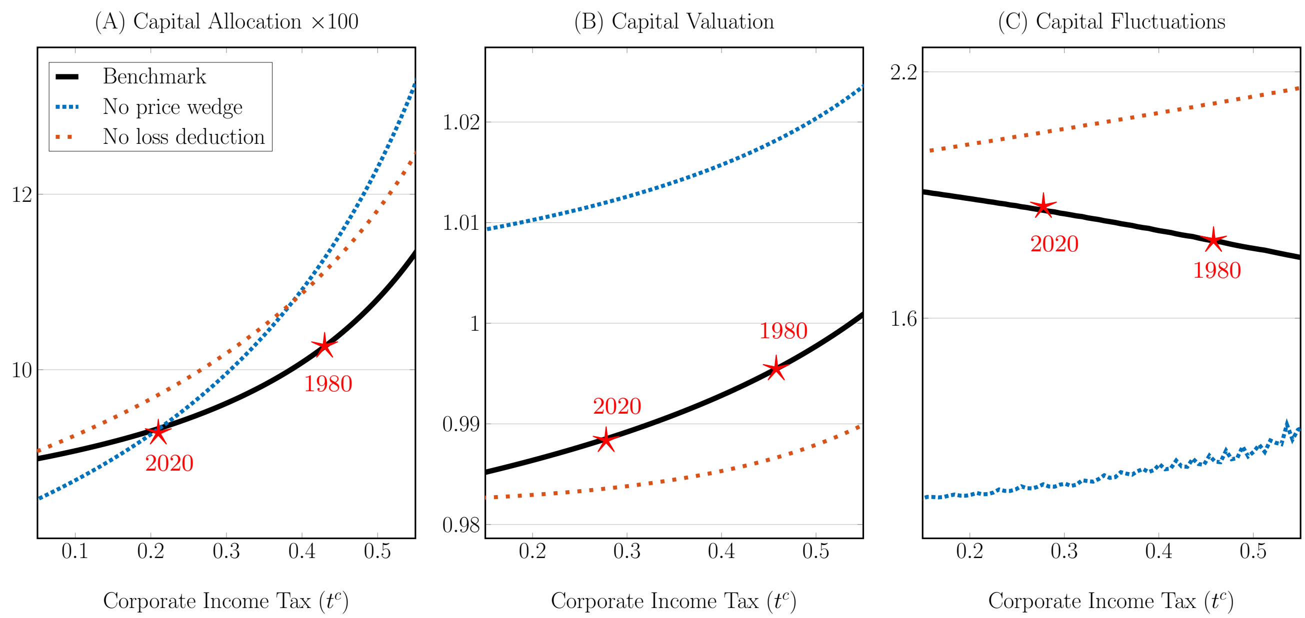
Figure VI illustrates the macroeconomic consequences of a drop in the corporate tax rate. In each panel, we consider three versions of the model. The first is the benchmark model (solid black) that considers a price wedge of $\omega = 0.15$ and allows for capital loss deductions. The second version eliminates the price wedge by setting $\omega = 0$ (by construction, there are no capital losses). The third version maintains the price wedge at the benchmark value but does not allow for capital loss deductions.

Panel A shows the capital allocation $\mathbb{V}ar[\hat{k}]$. We observe that a decline in the tax rate improves the allocation of capital. As predicted by equation (82) and Propositions 4 and 5, lower t^c reduces the effective fixed cost $\tilde{\theta}$ and shrinks the inaction regions, which lowers firms' tolerance for mismatch between their capital and productivity and decreases the dispersion of capital-productivity

[20] We study the transitional dynamics that follow a corporate tax reform in Baley, Blanco and Markiewitz (2022).

[21] In Chile, the evolution of the corporate tax rate is U-shaped. It was 40% in 1980, dropped to 10% in 1984, and then consistently (but infrequently) moved upward until reaching 20% in 2020 (see Appendix D). In this exercise, we identify the high-tax regime with 1980 and the low-tax regime with 2020.

Figure VI – Macro Outcomes and Corporate Income Tax



Notes: Panels A, B, and C plot capital allocation, capital valuation, and capital fluctuations for various levels of the corporate income tax rate t^c and three specifications: Benchmark = baseline model calibrated as in Table I ; No price wedge = baseline model with $\omega = 0$; No loss deduction = Stars correspond to the median values for t^c in the OECD: 1980 = 42%, 2020 = 25%.

ratios. At the same time, it raises the price wedge, which in principle can increase dispersion. In the calibration, the first effect dominates and cross-sectional dispersion falls with the lower tax.

Panel B shows capital valuation q (in black). We discover that q moves in the same direction as the tax rate. When t^c decreases, firms invest more and the average capital-productivity ratio $\mathbb{E}[\hat{k}]$ goes up. Abundant capital is less valuable and q goes down. At the same time, the allocation of capital improves, $\mathbb{V}ar[\hat{k}]$ falls (see Panel A), and q goes up. However, the price wedge becomes more important and the implied subsidy for capital losses falls. This force also pushes q down.

Finally, Panel C shows capital fluctuations measured by the CIR (in black). We find that the CIR increases with the tax cut, which means that aggregate productivity shocks propagate more slowly when taxes are low. While there is lower dispersion, the larger after-tax price wedge generates more history dependence and persistent deviations from steady state.

Comparing the results across the various model configurations—with and without price wedge and with and without capital loss deductions—highlights the importance of price wedges and their tax treatment for the macroeconomy. We observe important differences in the size and sign of the elasticities of aggregate outcomes with respect to corporate taxes. For instance, ignoring the price wedge reverts the sign of the relationship between the CIR and the corporate tax rate.

In summary, a drop in the corporate income tax rate reduces capital misallocation, reduces capital valuation, and slows the propagation of aggregate productivity shocks. While we focus here on the aggregate outcomes, our results can be also applied when considering cross-sectional responses to corporate tax changes. In particular, cross-sectional differences in the relative importance of fixed costs θ and price wedges ω—across firms, industries, sectors, or types of capital—may cause heterogeneous responses to an identical change in t^c across the board. Alternatively, cross-sectional differences in depreciation allowances z, as documented by House and Shapiro (2008) and Zwick and Mahon (2017), should bring heterogeneous responses to t^c, controlling for fixed costs and price wedges. These observations offer a complementary identification strategy that exploits ex ante heterogeneity instead of heterogeneity in the treatment. We leave testing this prediction using cross-sectoral data for future work.

A remark on the calibration of fixed costs. Let us compare the values for the macro outcomes reported in Table II—recovered directly from the microdata mappings—with the corresponding values in Figure VI—, which are obtained by simulating the calibrated model. We see that all values in the data are consistently larger than those produced by the model. The reason for this discrepancy is that our model with a symmetric fixed cost is extremely parsimonious and cannot reproduce the large variance of capital-productivity ratios $\mathbb{V}ar[\hat{k}]$ and the large covariance of capital-productivity ratios and their age $\mathbb{C}ov[\hat{k}, a]$ recovered from the data. The simulated method of moments strikes a balance between these two moments in the data, but falls below their empirical values.

In previous work (Baley and Blanco, 2021), we demonstrated that the symmetric fixed cost model is unable to replicate the empirical values of these two moments and showed how it should be augmented in order to match them. Introducing a time-dependent component in adjustment, such as random opportunities for free adjustment, increases the variance $\mathbb{V}ar[\hat{k}]$. Introducing asymmetric fixed costs that depend on the adjustment sign increases the covariance $\mathbb{C}ov[\hat{k}, a]$ (the price wedge already pushes the covariance up, but it is not quantitatively enough). Augmenting the model in these two directions is straightforward and necessary to conduct a fully fledged quantitative analysis. Nevertheless, we have opted to keep the model as simple as possible to facilitate exposition and highlight the key mechanisms at work in the cleanest way.

7 Final thoughts

We propose a new laboratory to study the macroeconomic implications of partial irreversibility, its interaction with fixed adjustment costs, and corporate taxes. Our results—and particularly the tax application—highlight the need to disentangle the role of price wedges vis-à-vis fixed adjustment to correctly assess the aggregate effects of tax policy. Moreover, the analysis puts forward a new channel for policy intervention: Corporate tax policy can change the effective size of fixed costs and price wedges—technological constraints or market prices typically outside the control of a policymaker—and structurally change the long-run behavior of aggregate capital and the macroeconomy more broadly.

A key innovation in our analysis is to characterize sufficient statistics for models with lumpy adjustment with more than one reset point. In this paper, the price wedge generates two reset points for the capital to productivity ratio. Two reset points also arise in other environments in which the marginal benefit (or cost) of taking an action is different from the marginal benefit (or cost) of reversing that action, such as durables, housing, inventories, and labor adjustment. More generally, our methodology is flexible enough to accommodate a finite number of reset points, and the mappings from microdata to macro outcomes can be applied as long as there is enough data to discipline the transitions across reset points. For example, a potential application of sufficient statistics with multiple reset points are models with fixed costs and financial frictions, in which firm would adjust its capital to a point that depends on their availability of funds; in this case, data on leverage could in principle be useful to discipline the transition probabilities. We leave for future research these explorations.

We conclude by discussing two limitations in our analysis of transitional dynamics following aggregate shocks. The first limitation is that we consider small shocks to the distribution of firms and focus on first-order perturbations, and thus we ignore potential nonlinearities and the response to large shocks. The second limitation is that we consider transitional dynamics in small open economies in which prices are fixed and do not respond to aggregate shocks. Contemporaneous work makes progress in these directions within the context of price-setting models. Alexandrov (2021) characterizes the CIR for non-marginal shocks, and Álvarez, Lippi and Souganidis (2022) use mean-field games to characterize the role of strategic complementarities. We leave for future research developing these areas in the context of investment.

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The Macroeconomics of Partial Irreversibility

Isaac Baley and Andrés Blanco

Online Appendix

Contents

A Proofs

A.1 Auxiliary Theorems

Auxiliary Theorem 1. *Let X be a (sub)martingale on the filtered space $(\Omega, \mathbb{P}, \mathcal{F})$ and let τ be a stopping time. If $(\{X_t\}_t, \tau)$ is a well-defined stopping process, then*

$$\mathbb{E}[X_\tau](\geq) = \mathbb{E}[X_0]. \tag{A.1}$$

This is the Optional Sampling Theorem (OST), see Theorem 4.4 in Stokey (2009) for the proof.

Auxiliary Theorem 2. *Let g be a real valued function of a Brownian motion x_t, F the ergodic distribution of x, and τ a stopping time. Assume that $\int g(x)\,\mathrm{d}F(x) = \lim_{T\to\infty} T^{-1}\int_0^T g(x_t)\,\mathrm{d}t$ for all initial x_0 and a constant reset state $x_\tau = x^*$. The following relationship holds:*

$$\underbrace{\mathbb{E}\left[\int_0^\tau g(x_t)\,\mathrm{d}t \middle| x_0 = x^*\right]}_{\textit{occupancy measure}} = \underbrace{\int g(x)\,\mathrm{d}F(x)}_{\textit{steady-state mass}} \quad \underbrace{\mathbb{E}\left[\tau \middle| x_0 = x^*\right]}_{\textit{proportionality constant}}. \tag{A.2}$$

Proof. This result establishes the equivalence between the steady-state distribution and the occupancy measure. The occupancy measure (LHS) is the average time an agent's state spends at a given value. It is proportional to the stationary mass of agents at that particular state (RHS), with a proportionality constant equal to the expected time between adjustments. For example, if $g(x) = x^m$, then $\mathbb{E}\left[\int_0^\tau x_t^m\,\mathrm{d}t | x_0 = x^*\right] = \mathbb{E}[x^m]\mathbb{E}[\tau | x_0 = x^*]$. For notation clarity, we use $\overline{\mathbb{E}}[\cdot] = \mathbb{E}[\cdot | x_0 = x^*]$. See Appendix B.2 for the proof and Stokey (2009) for more details.

A.2 Proof of Lemmas 1, 2, and 3

The proof has 3 steps. First, we derive the sufficient conditions that characterize the optimal value function $V(k,u,d)$ and the policy $(k^-, k^{*-}, k^{*+}, k^+)$. Second, to reduce the dimensionality of the state space, we guess that $V(k,u,d)$ can be separated as $V(k,u,d) = \frac{1-t^p}{1-t^c}\left[uv(\hat{k}) + t^c zd\right]$, where $v(\hat{k})$ is a function of the log capital-to-productivity ratio $\hat{k} \equiv \log(k/u)$ and z is the discounted value of deductions. The corresponding policy is $(\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+)$. Third, we verify the guess by showing that the sufficient conditions for V are equivalent to those satisfied by v. Lastly, we express the optimal policy as a system of equations in fundamental parameters.

To simplify the notation, we denote the state evaluated at the lower border of inaction as $S^- \equiv (k^-, u, d)$ and evaluated after a positive adjustment as $S^{*-} \equiv \left(k^{*-}, u, d + \theta u + p^{buy}\Delta k\right)$, where $\Delta k = k^{*-} - k^-$. Analogously, we denote the state evaluated at the upper border of inaction as $S^- \equiv (k^-, u, d)$ and evaluated after a negative adjustment as $S^{*+} \equiv \left(k^{*+}, u, d + \theta u + p^{sell}\Delta k\right)$, where $\Delta k = k^{*+} - k^+$.

Step 1: Characterize the value function $V(k,u,d)$. Recall the recursive problem in (5). Substituting output (1), after-tax dividends (69), and adjustment costs $\theta_s = \theta u_s$ we obtain

$$\begin{aligned} V(k_0,u_0,d_0) &= \max_{\{T_h,\, i_{T_h}\}_{h=1}^\infty} \left(\frac{1-t^p}{1-t^g}\right)\mathbb{E}\left[\int_0^\infty e^{-\rho\frac{1-t^p}{1-t^g}s}[(1-t^c)Au_s^{1-\alpha}k_s^\alpha + t^c\xi^d d_s]\,\mathrm{d}s \right. \\ &\quad \left. - \sum_{h=1}^\infty e^{-\rho\frac{1-t^p}{1-t^g}T_h}\left(\theta u_{T_h} + \left(p\left(i_{T_h}\right) - t^c\omega\mathbb{1}_{\{i_{T_h}<0\}}\right) i_{T_h}\right)\right]. \end{aligned} \tag{A.3}$$

Since $\left(\frac{1-t^p}{1-t^g}\right)$ is multiplicative, we can ignore this term from the optimal investment policy hereon. Using the principle of optimality, we write the problem recursively as a stopping-time problem with initial conditions $(k_0, u_0, d_0) = (k, u, d)$:

$$\begin{aligned} V(k,u,d) &= \max_{\tau,\,\Delta k_\tau} \mathbb{E}\left[\int_0^\tau e^{-\rho\frac{1-t^p}{1-t^g}s}[(1-t^c)Au_s^{1-\alpha}k_s^\alpha + t^c\xi^d d_s]\,\mathrm{d}s \right. \\ &\quad \left. + e^{-\rho\frac{1-t^p}{1-t^g}\tau}\left(-\theta u_\tau - \left(p(\Delta k_\tau) - t^c\omega\mathbb{1}_{\{\Delta k_\tau<0\}}\right)\Delta k_\tau + V\left(k_\tau + \Delta k_\tau, u_\tau, d_\tau + u_\tau\theta + \tilde{p}(\Delta k_\tau)\Delta k_\tau\right)\right)\right]. \end{aligned} \tag{A.4}$$

The optimal policy $(k^-, k^{*-}, k^{*+}, k^+)$ satisfies the system of sufficient conditions (A.5) to (A.15).

1. In the interior of the inaction region, i.e., $k \in (k^-, k^+)$, $V(S)$ solves the HJB equation:

$$\rho \frac{1-t^p}{1-t^g} V(S) = u(1-t^c)A\left(\frac{k}{u}\right)^\alpha + t^c \xi^d d - \xi^k k \frac{\partial V(S)}{\partial k} - \xi^d d \frac{\partial V(S)}{\partial d} + \left(\mu + \frac{\sigma^2}{2}\right) u \frac{\partial V(S)}{\partial u} + \frac{(\sigma u)^2}{2} \frac{\partial^2 V(S)}{\partial u^2}. \tag{A.5}$$

2. The value matching conditions for all (u, d), that equalize the value of adjusting to the value of inaction at the borders of the inaction region $\{k^-, k^+\}$:

$$V(S^{*-}) - p\Delta k - \theta u = V(S^-), \tag{A.6}$$
$$V(S^{*+}) - p(1-\omega(1-t^c))\Delta k - \theta u = V(S^+). \tag{A.7}$$

3. The two optimality conditions for the reset capitals $\{k^{*-}, k^{*+}\}$:

$$\frac{\partial V(S^{*-})}{\partial k} = \left[1 - \frac{\partial V(S^{*-})}{\partial d}\right] p, \tag{A.8}$$
$$\frac{\partial V(S^{*+})}{\partial k} = \left[1 - \frac{\partial V(S^{*+})}{\partial d}\right] p(1-\omega(1-t^c)). \tag{A.9}$$

4. The six smooth pasting conditions:

$$\frac{\partial V(S^-)}{\partial k} = \left[1 - \frac{\partial V(S^-)}{\partial d}\right] p, \tag{A.10}$$
$$\frac{\partial V(S^+)}{\partial k} = \left[1 - \frac{\partial V(S^+)}{\partial d}\right] p(1-\omega(1-t^c)), \tag{A.11}$$
$$\frac{\partial V(S^{*-})}{\partial u} = \theta + \frac{\partial V(S^-)}{\partial u}, \tag{A.12}$$
$$\frac{\partial V(S^{*-})}{\partial u} = \theta + \frac{\partial V(S^-)}{\partial u}, \tag{A.13}$$
$$\frac{\partial V(S^{*+})}{\partial d} = \frac{\partial V(S^+)}{\partial d}. \tag{A.14}$$
$$\frac{\partial V(S^{*+})}{\partial d} = \frac{\partial V(S^+)}{\partial d}. \tag{A.15}$$

For additional details about the sufficiency of HJB equations, value matching, optimality and smooth pasting conditions, see Oksendal (2007), Baley and Blanco (2019), and Baley and Blanco (2021).

Step 2: Guess of separable value function. Define the new parameters

$$z \equiv \frac{\xi^d}{\rho\frac{1-t^p}{1-t^g} + \xi^d}; \qquad \tilde{r} \equiv \rho\frac{1-t^p}{1-t^g} - \mu - \sigma^2/2; \qquad \nu \equiv \mu + \xi^k, \tag{A.16}$$

and the fixed cost and price of investment

$$\tilde{A} = (1-t^c)A, \tag{A.17}$$
$$\tilde{\theta} = (1-t^c z)\theta, \tag{A.18}$$
$$\tilde{p}^{buy} = (1-t^c z)\, p, \tag{A.19}$$
$$\tilde{p}^{sell} = (1-t^c z - \omega(1-t^c))\, p, \tag{A.20}$$
$$\tilde{p}\,(i_s) \equiv \tilde{p}^{\text{buy}} \mathbb{1}_{\{i_s>0\}} + \tilde{p}^{\text{sell}} \mathbb{1}_{\{i_s<0\}}. \tag{A.21}$$

We guess a separable value function:

$$V(S) = \frac{1-t^p}{1-t^g}\left[uv(\hat{k}) + t^c zd\right], \tag{A.22}$$

where the function $v(\hat{k})$ is defined as

$$v(\hat{k}) \equiv \max_{\tau,\Delta\hat{k}} \mathbb{E}\left[\int_0^\tau \tilde{A}e^{-\tilde{r}s+\alpha\hat{k}_s}\,\mathrm{d}s + e^{-\tilde{r}\tau}\left(-\tilde{\theta} - \tilde{p}(\Delta\hat{k})(e^{\hat{k}_\tau+\Delta\hat{k}} - e^{\hat{k}_\tau}) + v(\hat{k}_\tau + \Delta\hat{k})\right)\right]. \tag{A.23}$$

The value function $v(\hat{k})$ and the optimal policy $\{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ satisfy the following three conditions:

(i) In the interior of the inaction region, $v(\hat{k})$ solves the HJB equation:

$$\tilde{r}v(\hat{k}) = \tilde{A}e^{\alpha\hat{k}} - \nu v'(\hat{k}) + \frac{\sigma^2}{2}v''(\hat{k}). \tag{A.24}$$

(ii) At the borders of the inaction region, $v(\hat{k})$ satisfies the value-matching conditions:

$$v(\hat{k}^-) = v(\hat{k}^{*-}) - \tilde{\theta} + \tilde{p}^{buy}(e^{\hat{k}^-} - e^{\hat{k}^{*-}}), \tag{A.25}$$

$$v(\hat{k}^+) = v(\hat{k}^{*+}) - \tilde{\theta} + \tilde{p}^{sell}(e^{\hat{k}^+} - e^{\hat{k}^{*+}}). \tag{A.26}$$

(iii) At the borders of the inaction region and the two reset states, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$v'(\hat{k}) = \tilde{p}^{buy}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \tag{A.27}$$

$$v'(\hat{k}) = \tilde{p}^{sell}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \tag{A.28}$$

Step 3: Verify the guess. We verify that the guess in (A.22) and (A.23) is correct by showing the equivalence between the sufficient conditions of V, (A.5) to (A.15), and the sufficient conditions of v, (A.24) to (A.28). Given our guess, the following relationships between the derivatives of V and the derivatives of v hold:

$$\frac{\partial V(S)}{\partial u} = v(\hat{k}) - v'(\hat{k}), \tag{A.29}$$

$$\frac{\partial^2 V(S)}{\partial u^2} = -\frac{v'(\hat{k})}{u} + \frac{v''(\hat{k})}{u}, \tag{A.30}$$

$$\frac{\partial V(S)}{\partial k} = \frac{u}{k}v'(\hat{k}), \tag{A.31}$$

$$\frac{\partial V(S)}{\partial d} = t^c z. \tag{A.32}$$

Verify HJB. We start with the HJB by substituting the guess into (A.5)

$$\rho\frac{1-t^p}{1-t^g}\left[uv(\hat{k}) + t^c zd\right] = u(1-t^c)Ae^{\alpha\hat{k}} + t^c\xi^d d - \xi^k k\frac{u}{k}v'(\hat{k}) - t^c z\xi^d d + \left(\mu + \frac{\sigma^2}{2}\right)u(v(\hat{k}) - v'(\hat{k})) + \frac{\sigma^2 u^2}{2}\left(\frac{v''(\hat{k})}{u} - \frac{v'(\hat{k})}{u}\right).$$

Using the definition of z to cancel the term $t^c\xi^d d$ on both sides, and rearranging we obtain:

$$\left(\rho\frac{1-t^p}{1-t^g} - \mu - \frac{\sigma^2}{2}\right)uv(\hat{k}) = u\tilde{A}e^{\alpha\hat{k}} - (\mu + \xi^k)uv'(\hat{k}) + \frac{\sigma^2}{2}uv''(\hat{k}).$$

Using the new parameters $\nu \equiv \mu + \xi^k$ and $\tilde{r} \equiv \rho\frac{1-t^p}{1-t^g} - \mu - \sigma^2/2$, and cancelling u, we obtain the HJB in (A.24):

$$\tilde{r}v(\hat{k}) \;=\; \tilde{A}e^{\alpha\hat{k}} - \nu v'(\hat{k}) + \frac{\sigma^2}{2}v''(\hat{k}).$$

Verify value matching. We verify the value matching conditions by substituting the guess into (A.6) and (A.7).

$$\begin{aligned} uv(\hat{k}^{*-}) \;+\; t^c z(d+\theta u + p\Delta k) \;-\; p\Delta k \;-\; \theta u \;&=\; uv(\hat{k}^-) \;+\; t^z zd, \\ uv(\hat{k}^{*+}) \;+\; t^c z(d+\theta u + p\Delta k) \;-\; p(1-\omega(1-t^c))\Delta k \;-\; \theta u \;&=\; uv(\hat{k}^-) \;+\; t^z zd. \end{aligned}$$

Writing positive investment as $\Delta k = u(e^{\Delta\hat{k}-\hat{k}^-} - e^{\hat{k}^-})$, negative investment as $\Delta k = u(e^{\Delta\hat{k}-\hat{k}^+} - e^{\hat{k}^+})$, cancelling $t^d d$, dividing both sides by u, we obtain (A.25) and (A.26):[22]

$$\begin{aligned} v(\hat{k}^{*-}) - (1-t^c z)p(e^{\Delta\hat{k}-\hat{k}^-} - e^{\hat{k}^-}) - \underbrace{\theta(1-t^c z)}_{=\tilde{\theta}} \;&=\; v(\hat{k}^-), \\ v(\hat{k}^{*+}) - (1-t^c z - \omega(1-t^c))p(e^{\Delta\hat{k}-\hat{k}^+} - e^{\hat{k}^+}) - \underbrace{\theta(1-t^c z)}_{=\tilde{\theta}} \;&=\; v(\hat{k}^-). \end{aligned}$$

Verify optimality and smooth pasting. We verify the optimality conditions and the smooth pasting conditions for capital by substituting the guess into (A.8), (A.9), (A.10) and (A.11)

$$\begin{aligned} \frac{u}{k}v'(\hat{k}^{*-}) \;=\; (1-t^c z)p \;&\Longleftrightarrow\; v'(\hat{k}^{*-}) \;=\; e^{\hat{k}^{*-}}(1-t^c z)p, \\ \frac{u}{k}v'(\hat{k}^{*-}) \;=\; (1-t^c z-\omega(1-t^c))p \;&\Longleftrightarrow\; v'(\hat{k}^{*-}) \;=\; e^{\hat{k}^{*+}}(1-t^c z-\omega(1-t^c))p, \\ \frac{u}{k}v'(\hat{k}^{-}) \;=\; (1-t^c z)p \;&\Longleftrightarrow\; v'(\hat{k}^{-}) \;=\; e^{\hat{k}^{-}}(1-t^c z)p, \\ \frac{u}{k}v'(\hat{k}^{+}) \;=\; (1-t^c z-\omega(1-t^c))p \;&\Longleftrightarrow\; v'(\hat{k}^{-}) \;=\; e^{\hat{k}^{+}}(1-t^c z-\omega(1-t^c))p. \end{aligned}$$

which are equal to the expressions in (A.27) and (A.28). To verify the smooth-pasting for idiosyncratic productivity, we substitute the guess into (A.12) and (A.13) and then substitute (A.27) and (A.28) to rewrite $v'(\cdot)$ in terms of prices, which yields the conditions in (A.25) and (A.26):

$$\begin{aligned} v(\hat{k}^{*-}) \;&=\; \tilde{\theta} + v(\hat{k}^-) \;+\; (1-t^c z)p(e^{\hat{k}^{*-}-\hat{k}^-}) \\ v(\hat{k}^{*+}) \;&=\; \tilde{\theta} + v(\hat{k}^+) \;+\; (1-t^c z-\omega(1-t^c))p(e^{\hat{k}^{*+}-\hat{k}^+}) \end{aligned}$$

Finally, the smooth pasting conditions for deductions (A.14) and (A.15) are trivially satisfied given the linearity of the guess. We conclude that the guess is correct.

A.3 Generalized hazard model

See Appendix B for further details.

A.4 Cross-sectional distributions with non-stochastic fixed cost of adjustments

Let $G(\hat{k})$ be the distribution of firms over their log capital-productivity ratio and let $g(\hat{k})$ be its continuous marginal density. Also, let $\mathcal{N}^-$, $\mathcal{N}^+$, and $\mathcal{N} = \mathcal{N}^- + \mathcal{N}^+$ be the frequencies of positive, negative, and non-zero adjustments

[22] To express investment in terms of $\hat{k}$, we start from the expression $\Delta\hat{k} \;=\; \log(1+\Delta k/k_\tau)$, which yields $\Delta k = e^{\Delta\hat{k}}k_\tau - k_\tau$; multiply and divide by u_τ and substitute the definition of $\hat{k}$ to get: $\Delta k = u_\tau e^{\Delta\hat{k}+\hat{k}_\tau} - e^{\hat{k}_\tau}$. Then we use $\hat{k}_\tau = \hat{k}^+$ or $\hat{k}_\tau = \hat{k}^-$ accordingly.

in the *total population*, which are equal to the mass of firms that adjust to $\hat{k}^{*-}$, to $\hat{k}^{*+}$, or to either point.[23] The density and frequencies solve the following system, which includes: a Kolmogorov forward equation that describes the evolution of capital-productivity ratios inside the inaction region (excluding the two reset points)

$$\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k}) = 0, \quad \text{for all} \quad \hat{k} \in (\hat{k}^-, \hat{k}^+) \setminus \{\hat{k}^{*-}, \hat{k}^{*+}\}; \tag{A.33}$$

three border conditions

$$g(\hat{k}) = 0, \quad \text{for } \hat{k} \in \{\hat{k}^{*-}, \hat{k}^{*+}\}, \tag{A.34}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\, \mathrm{d}\hat{k} = 1; \tag{A.35}$$

two resetting conditions

$$\underbrace{\frac{\sigma^2}{2} \lim_{\hat{k}\downarrow\hat{k}^-} g'(\hat{k})}_{\mathcal{N}^-} = \frac{\sigma^2}{2} \left[\lim_{\hat{k}\uparrow\hat{k}^{*-}} g'(\hat{k}) - \lim_{\hat{k}\downarrow\hat{k}^{*-}} g'(\hat{k}) \right], \tag{A.36}$$

$$\underbrace{-\frac{\sigma^2}{2} \lim_{\hat{k}\uparrow\hat{k}^+} g'(\hat{k})}_{\mathcal{N}^+} = \frac{\sigma^2}{2} \left[\lim_{\hat{k}\uparrow\hat{k}^{*+}} g'(\hat{k}) - \lim_{\hat{k}\downarrow\hat{k}^{*+}} g'(\hat{k}) \right], \tag{A.37}$$

(A.38)

and two continuity conditions at the reset points (not reported). Condition (A.34) sets the mass of firms at the inaction thresholds equal to zero. Condition (A.35) ensures that g is a density. Conditions (A.36) and (A.37) relate the masses of upward and downward adjustments to the discontinuities in the derivative of g at the reset points. In a small period of time ds, the mass $\mathcal{N}^-$ that "exits" the inaction region by hitting the lower threshold—equal to $\frac{\sigma^2}{2} \lim_{\hat{k}\downarrow\hat{k}^-} g'(\hat{k})$—must coincide with the mass of firms that "enters" at the reset point $\hat{k}^{*-}$—equal to the jump in g'. This argument is analogous for $\mathcal{N}^+$; in fact, it is straightforward to verify that conditions (A.33) to (A.36) jointly imply condition (A.37), and thus it is redundant.

A.5 Cross-sectional distributions with random fixed cost of adjustments

Let $g(\hat{k})$ be the cross-sectional distribution of $\hat{k}$. It satisfies a Kolmogorov forward equation with its boundary conditions.

Without partial irreversibility.

$$\Lambda(\hat{k})g(\hat{k}) = \nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k}), \quad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^*\} \tag{A.39}$$

$$g(\hat{k}^{\pm}) = 0 \tag{A.40}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\, \mathrm{d}\hat{k} = 1, \tag{A.41}$$

$$g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^*\}), \mathbb{C}^2(\{\hat{k}^*\}) \tag{A.42}$$

[23] To avoid any confusion with our notation, we emphasize that the sign in the exponent of an object refers to the last reset point, not to the sign of the adjustment.

With partial irreversibility.

$$\text{(A.43)} \qquad \Lambda(\hat{k})g(\hat{k}) = \nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k}), \qquad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^{*-}, \hat{k}^{*+}\}$$

$$\text{(A.44)} \qquad g(\hat{k}^{\pm}) = 0$$

$$\text{(A.45)} \qquad \int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k} = 1,$$

$$\text{(A.46)} \qquad g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^{*-}, \hat{k}^{*+}\}), \mathbb{C}^2(\{\hat{k}^{*-}, \hat{k}^{*+}\})$$

A.6 Proof of Proposition 2

Proposition 3 characterizes the aggregate Tobin's q as a function of the aggregate output-to-capital ratio $\hat{Y}/\hat{K}$ and the expected capital gains. Let us depart from the definition of aggregate Tobin's q in (31)

$$\text{(A.47)} \qquad q \equiv \frac{1}{p} \int_{\hat{k}^-}^{\hat{k}^+} \phi(\hat{k}) \frac{v'(\hat{k})}{e^{\hat{k}}} \,\mathrm{d}G(\hat{k}) = \frac{\mathbb{E}[v'(\hat{k})]}{p\hat{K}}.$$

We characterize the numerator,

$$\text{(A.48)} \qquad \mathbb{E}[v'(\hat{k})] = \int_{\hat{k}^-}^{\hat{k}^+} v'(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k},$$

by combining the HJB equation for $v'(\hat{k})$ and the KFE for $g(\hat{k})$ into a single "master equation".

A.6.1 Without partial irreversibility

The value $v'(k)$ satisfies the following conditions:

$$\text{(A.49)} \qquad r v'(\hat{k}) = \alpha A e^{\alpha \hat{k}} - \nu v''(\hat{k}) + \frac{\sigma^2}{2} v'''(\hat{k}) + \Lambda(\hat{k}) \left[p e^{\hat{k}} - v'(\hat{k}) \right], \forall \hat{k} \in (\hat{k}^-, \hat{k}^+)$$

$$\text{(A.50)} \qquad v'(\hat{k}) = p e^{\hat{k}}, \quad \text{for} \quad \hat{k} \in \{\hat{k}^-, \hat{k}^*, \hat{k}^+\},$$

$$\text{(A.51)} \qquad v'(\hat{k}) \in \mathbb{C}, \mathbb{C}^1$$

and $g(\hat{k})$ satisfies a KFE and boundary conditions in (A.39) to (A.42). From equation (A.39), we solve for the adjustment hazard $\Lambda(\hat{k}) = (\nu g'(\hat{k}) + (\sigma^2/2) g''(\hat{k})/g(\hat{k})$; then substitute the hazard into (A.49), multiply both sides by $g(\hat{k})$, and join common terms, to obtain the Master equation that is valid for all $\hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^*\}$:

$$\text{(A.52)} \qquad \begin{aligned} r v'(\hat{k}) g(\hat{k}) &= \alpha A e^{\alpha \hat{k}} g(\hat{k}) - \nu v''(\hat{k}) g(\hat{k}) + \frac{\sigma^2}{2} v'''(\hat{k}) g(\hat{k}) + \left[\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k}) \right] \left[p e^{\hat{k}} - v'(\hat{k}) \right] \\ &= \alpha A e^{\alpha \hat{k}} g(\hat{k}) - \nu \left(v'(\hat{k}) g(\hat{k}) \right)' + \frac{\sigma^2}{2} \left(v''(\hat{k}) g(\hat{k}) - v'(\hat{k}) g'(\hat{k}) \right)' + p e^{\hat{k}} \left[\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k}) \right]. \end{aligned}$$

Next, we integrate both sides in the interval $[\hat{k}^-, \hat{k}^+]$.

$$\text{(A.53)} \qquad \begin{aligned} \underbrace{r \int_{\hat{k}^-}^{\hat{k}^+} v'(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k}}_{qp\hat{K}} &= \alpha A \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} e^{\alpha \hat{k}} g(\hat{k})\,\mathrm{d}\hat{k}}_{\hat{Y}} - \nu \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} \left(v'(\hat{k}) g(\hat{k}) \right)' \mathrm{d}\hat{k}}_{T_1} \\ &+ \frac{\sigma^2}{2} \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} \left(v''(\hat{k}) g(\hat{k}) - v'(\hat{k}) g'(\hat{k}) \right)' \mathrm{d}\hat{k}}_{T_2} + p \left[\nu \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}} g'(\hat{k})\,\mathrm{d}\hat{k}}_{T_3} + \frac{\sigma^2}{2} \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}} g''(\hat{k})\,\mathrm{d}\hat{k}}_{T_4} \right]. \end{aligned}$$

In the LHS, we recognize aggregate q. In the first term of the RHS, we recognize the average output-to-productivity ratio in (33). The remaining terms, labelled T_1, T_2, T_3 and T_4, are computed following a common strategy: split the domain into two regions, $[\hat{k}^-, \hat{k}^+] = [\hat{k}^-, \hat{k}^*] \cup [\hat{k}^*, \hat{k}^{*+}]$; integrate by parts if needed; use the border conditions $g(\hat{k}^\pm) = 0$ in (A.40); and use continuity conditions for g in (A.42) and for v' in (A.51).

$$T_1 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} \left(v'(\hat{k})g(\hat{k})\right)' \mathrm{d}\hat{k} \;=\; \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*}}_{v'(\hat{k}^*)g(\hat{k}^*)} \;+\; \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{-v'(\hat{k}^*)g(\hat{k}^*)} \;=\; 0. \tag{A.54}$$

$$T_3 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}} g'(\hat{k})\,\mathrm{d}\hat{k} \;=\; \underbrace{e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*}}_{e^{\hat{k}^*}g(\hat{k}^*)} \;+\; \underbrace{e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{-e^{\hat{k}^*}g(\hat{k}^*)} - \int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}} g(\hat{k})\,\mathrm{d}\hat{k} = -\hat{K}. \tag{A.55}$$

$$T_4 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}} g''(\hat{k})\,\mathrm{d}\hat{k} \;=\; e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} \;+\; e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} \;-\; T_3. \tag{A.56}$$

$$\begin{aligned} T_2 \;&=\; \int_{\hat{k}^-}^{\hat{k}^+} \left(v''(\hat{k})g(\hat{k}) - v'(\hat{k})g'(\hat{k})\right)' \mathrm{d}\hat{k} \;=\; \underbrace{v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*}}_{v''(\hat{k}^*)g(\hat{k}^*)} \;+\; \underbrace{v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{-v''(\hat{k}^*)g(\hat{k}^*)} \\ &\quad -\; \left[v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} \;+\; v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] \\ &=\; -p\left[e^{\hat{k}^*}g'(\hat{k}^*) \;-e^{\hat{k}^-}g'(\hat{k}^-) \;+\; e^{\hat{k}^+}g'(\hat{k}^+) \;-e^{\hat{k}^*}g'(\hat{k}^*)\right] \;=\; -p(T_3 \;+\; T_4). \end{aligned} \tag{A.57}$$

Substituting equations (A.54) to (A.56) into (A.53):

$$rqp\hat{K} \;=\; \alpha A\hat{Y} \;-\; \frac{\sigma^2}{2}p(T_3 + T_4) \;+\; p\left[\nu T_3 \;+\; \frac{\sigma^2}{2}T_4\right] \;=\; \alpha A\hat{Y} \;+\; p\hat{K}\left(\frac{\sigma^2}{2} - \nu\right).$$

Dividing both sides by $p\hat{K}$ we obtain the result:

$$q \;=\; \frac{1}{r}\left[\frac{\alpha A\hat{Y}}{p\hat{K}} \;+\; \frac{\sigma^2}{2} - \nu\right]. \tag{A.58}$$

A.6.2 With partial irreversibility

The proof follows the same steps as without partial irreversibility, but using the auxiliary pricing function $\mathcal{P}(\hat{k})$.

Auxiliary pricing function $\mathcal{P}(\hat{k})$. We propose the following auxiliary price-deviation function $\mathcal{P}(\hat{k}) \in \mathbb{C}^2$ that satisfies all the requirements and is a member of the polynomial family:

$$\mathcal{P}(\hat{k}) \;\equiv\; \begin{cases} 0 & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ -\omega \sum_{i=0}^{5} \mathcal{P}_i \left(\frac{\hat{k}-\hat{k}^{*-}}{\hat{k}^{*+}-\hat{k}^{*-}}\right)^i & \text{if } \hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+}) \\ -\omega & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+] \end{cases} \tag{A.59}$$

with coefficients $\mathcal{P}_0 = \frac{p^{\text{buy}}-p}{p^{\text{sell}}-p^{\text{buy}}}, \mathcal{P}_1 = \mathcal{P}_2 = 0, \mathcal{P}_3 = 10, \mathcal{P}_4 = -15$, and $\mathcal{P}_5 = 6$ or equivalently:

$$\mathcal{P}(\hat{k}) \;=\; -\omega\left[10\left(\frac{\hat{k} - \hat{k}^{*-}}{\hat{k}^{*+} - \hat{k}^{*-}}\right)^3 - 15\left(\frac{\hat{k} - \hat{k}^{*-}}{\hat{k}^{*+} - \hat{k}^{*-}}\right)^4 + 6\left(\frac{\hat{k} - \hat{k}^{*-}}{\hat{k}^{*+} - \hat{k}^{*-}}\right)^5\right] \quad \mathit{for}\ \hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+}) \tag{A.60}$$

Characterization of q with irreversibility. Inside the inaction region, $\hat{k} \in (\hat{k}^-, \hat{k}^+)$, the value $v'(k)$ satisfies the following conditions:

$$(A.61)\qquad rv'(\hat{k}) = \alpha A e^{\alpha\hat{k}} - \nu v''(\hat{k}) + \frac{\sigma^2}{2}v'''(\hat{k}) + \begin{cases} \Lambda(\hat{k})\left[p^{\text{buy}}e^{\hat{k}} - v'(\hat{k})\right] & \text{if } \hat{k}\in[\hat{k}^-,\hat{k}^{*-}) \\ 0 & \text{if } \hat{k}\in[\hat{k}^{*-},\hat{k}^{*+}] \\ \Lambda(\hat{k})\left[p^{\text{sell}}e^{\hat{k}} - v'(\hat{k})\right] & \text{if } \hat{k}\in(\hat{k}^{*+},\hat{k}^+] \end{cases}$$

$$(A.62)\qquad v'(\hat{k}) = p^{\text{buy}}e^{\hat{k}}, \quad \text{for } \hat{k}\in\{\hat{k}^-,\hat{k}^{*-}\},$$

$$(A.63)\qquad v'(\hat{k}) = p^{\text{sell}}e^{\hat{k}}, \quad \text{for } \hat{k}\in\{\hat{k}^{*+},\hat{k}^+\},$$

$$(A.64)\qquad v'(\hat{k}) \in \mathbb{C},\mathbb{C}^1.$$

Since $\Lambda(\hat{k})=0$ between the two reset points, $[\hat{k}^{*-},\hat{k}^{*+}]$, the HBJ can be written compactly as

$$(A.65)\qquad rv'(\hat{k}) = \alpha A e^{\alpha\hat{k}} - \nu v''(\hat{k}) + \frac{\sigma^2}{2}v'''(\hat{k}) + \Lambda(\hat{k})\left[p(\mathcal{P}(\hat{k})+1)e^{\hat{k}} - v'(\hat{k})\right], \ \forall\hat{k}\in(\hat{k}^-,\hat{k}^+)$$

$$(A.66)\qquad v'(\hat{k}) = p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}, \quad \text{for } \hat{k}\in\{\hat{k}^-,\hat{k}^{*-},\hat{k}^{*+},\hat{k}^+\}.$$

where $\mathcal{P}(\hat{k})$ is defined above. In turn, the $g(\hat{k})$ satisfies the KFE and boundary conditions in (A.43) to (A.46). From equation (A.43), we solve for the adjustment hazard $\Lambda(\hat{k})$, substitute it into (A.65), and obtain the Master Equation that is valid for all $\hat{k}\in(\hat{k}^-,\hat{k}^+)/\{\hat{k}^{*-},\hat{k}^{*+}\}$. Integrating:

$$(A.67)\qquad rqp\hat{K} = \alpha A\hat{Y} - \nu\underbrace{\int_{\hat{k}^-}^{\hat{k}^+}\left(v'(\hat{k})g(\hat{k})\right)'\mathrm{d}\hat{k}}_{T_1} + \frac{\sigma^2}{2}\underbrace{\int_{\hat{k}^-}^{\hat{k}^+}\left(v''(\hat{k})g(\hat{k}) - v'(\hat{k})g'(\hat{k})\right)'\mathrm{d}\hat{k}}_{T_2}$$
$$+ \nu\underbrace{\int_{\hat{k}^-}^{\hat{k}^+}p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k}}_{T_3} + \frac{\sigma^2}{2}\underbrace{\int_{\hat{k}^-}^{\hat{k}^+}p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g''(\hat{k})\,\mathrm{d}\hat{k}}_{T_4}$$

In this case, to compute the terms T_1,T_2,T_3,T_4, we must split the domain into three regions, $[\hat{k}^-,\hat{k}^+]=[\hat{k}^-,\hat{k}^{*-})\cup[\hat{k}^{*-},\hat{k}^{*+}]\cup(\hat{k}^*,\hat{k}^{*+}]$; integrate by parts when needed; use the border conditions $g(\hat{k}^\pm)=0$ in (A.44); and use continuity conditions for g in (A.46) and for v' in (A.64).

(A.68)

$$T_1 = \int_{\hat{k}^-}^{\hat{k}^+}\left(v'(\hat{k})g(\hat{k})\right)'\mathrm{d}\hat{k} = \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}}}_{v'(\hat{k}^{*-})g(\hat{k}^{*-})} + \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}}}_{v'(\hat{k}^{*+})g(\hat{k}^{*+})-v'(\hat{k}^{*-})g(\hat{k}^{*-})} + \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{-v'(\hat{k}^{*+})g(\hat{k}^{*+})} = 0.$$

(A.69)

$$\begin{aligned} T_2 &= \int_{\hat{k}^-}^{\hat{k}^+}\left(v''(\hat{k})g(\hat{k}) - v'(\hat{k})g'(\hat{k})\right)'\mathrm{d}\hat{k} \\ &= \underbrace{\left[v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right]}_{=0} - \left[v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] \\ &= -\left[p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] \end{aligned}$$

(A.70)

$$T_3 = \int_{\hat{k}^-}^{\hat{k}^+} p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k} = \underbrace{p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=\,0}$$

$$-\int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left[p(\mathcal{P}(\hat{k})+1)+p\mathcal{P}'(\hat{k})\right]g(\hat{k})\,\mathrm{d}\hat{k} = -\mathbb{E}\left[e^{\hat{k}}\left(p(\mathcal{P}(\hat{k})+1)+p\mathcal{P}'(\hat{k})\right)\right].$$

(A.71)

$$T_4 = \int_{\hat{k}^-}^{\hat{k}^+} p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g''(\hat{k})\,\mathrm{d}\hat{k} = \underbrace{p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + p(\mathcal{P}(\hat{k})+1)e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{T_2}$$

(A.72)

$$\begin{aligned}
&-\Big[\int_{\hat{k}^{*+}}^{\hat{k}^+}\left(p(\mathcal{P}(\hat{k})+1)+p\mathcal{P}'(\hat{k})\right)e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}}\left(p(\mathcal{P}(\hat{k})+1)+p\mathcal{P}'(\hat{k})\right)e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k} + \\
&\int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left(p(\mathcal{P}(\hat{k})+1)+p\mathcal{P}'(\hat{k})\right)g'(\hat{k})\,\mathrm{d}\hat{k}\Big] \\
&= -T_2 - T_3 - p\int_{\hat{k}^-}^{\hat{k}^+}\mathcal{P}'(\hat{k})e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k} \\
&= -T_2 - T_3 - p\left[\underbrace{\mathcal{P}'(\hat{k})e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \mathcal{P}'(\hat{k})e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \mathcal{P}'(\hat{k})e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=\,0} - \int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}}\left[\mathcal{P}'(\hat{k})+\mathcal{P}''(\hat{k})g(\hat{k})\right]\mathrm{d}\hat{k}\right] \\
&= -T_2 - T_3 + p\,\mathbb{E}\left[e^{\hat{k}}\left(\mathcal{P}'(\hat{k})+\mathcal{P}''(\hat{k})\right)\right]
\end{aligned}$$

Substituting (A.68) to (A.71) into (A.67)

$$\begin{aligned}
\text{(A.73)}\qquad rq\hat{K} &= \frac{\alpha A\hat{Y} + \nu T_3 + \frac{\sigma^2}{2}(T_2 + T_4)}{p} \\
&= \frac{\alpha A\hat{Y} + \nu T_3 - \frac{\sigma^2}{2}\left(T_3 - p\,\mathbb{E}\left[e^{\hat{k}}\left(\mathcal{P}'(\hat{k})+\mathcal{P}''(\hat{k})\right)\right]\right)}{p} \\
&= \alpha A\hat{Y} + \left(\frac{\sigma^2}{2}-\nu\right)\hat{K} + \left(\frac{\sigma^2}{2}-\nu\right)\mathbb{E}\left[e^{\hat{k}}\left(\mathcal{P}(\hat{k})+\mathcal{P}'(\hat{k})\right)\right] + \frac{\sigma^2}{2}\mathbb{E}\left[e^{\hat{k}}\left(\mathcal{P}'(\hat{k})+\mathcal{P}''(\hat{k})\right)\right]
\end{aligned}$$

Dividing both sides by $\hat{K} = \mathbb{E}[e^{\hat{k}}]$ and defining the weights $\phi(\hat{k}) = e^{\hat{k}}/\hat{K}$ we obtain the result:

$$\text{(A.74)}\quad q = \frac{1}{r}\left[\frac{\alpha A\hat{Y}}{p\hat{K}} + \left(\frac{\sigma^2}{2}-\nu\right) + \left(\frac{\sigma^2}{2}-\nu\right)\mathbb{E}\left[\phi(\hat{k})\left(\mathcal{P}(\hat{k})+\mathcal{P}'(\hat{k})\right)\right] + \frac{\sigma^2}{2}\mathbb{E}\left[\phi(\hat{k})\left(\mathcal{P}'(\hat{k})+\mathcal{P}''(\hat{k})\right)\right]\right]$$

If we apply Ito's lemma to $\phi(\hat{k})\tilde{\mathcal{P}}(\hat{k})$ and using the Markov property of $\{\hat{k}_s\}$, we have

$$\begin{aligned}
\text{(A.75)}\qquad \mathbb{E}_s[\mathrm{d}e^{\hat{k}_s}\mathcal{P}(\hat{k}_s)|\hat{k}_s = \hat{k}] &= \left[-\nu e^{\hat{k}_s}\left(\mathcal{P}(\hat{k})+\mathcal{P}'(\hat{k})\right) + \frac{\sigma^2}{2}e^{\hat{k}}\left[(\mathcal{P}(\hat{k})+2\tilde{\mathcal{P}}'(\hat{k})+\mathcal{P}''(\hat{k})\right]\right]\mathrm{d}s \\
&= \left(\frac{\sigma^2}{2}-\nu\right)\left[\phi(\hat{k})\left(\mathcal{P}(\hat{k})+\mathcal{P}'(\hat{k})\right)\right] + \frac{\sigma^2}{2}\left[\phi(\hat{k})\left(\mathcal{P}'(\hat{k})+\mathcal{P}''(\hat{k})\right)\right]\mathrm{d}s
\end{aligned}$$

Thus, taking the mean over $\hat{k}$ using $g(\hat{k})$

$$q \;=\; \frac{1}{r}\left(\frac{\alpha A\hat{Y}}{p\hat{K}} \;+\; \left(\frac{\sigma^2}{2}-\nu\right) \;+\; \mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{P}(\hat{k}_s)\phi(\hat{k}_s))\right]\right]\right). \tag{A.76}$$

A.6.3 Aggregate productivity

We approximate the aggregate productivity term. Using a second-order Taylor expansion, we obtain:

$$\frac{\hat{Y}}{\hat{K}} \;=\; \frac{\mathbb{E}[e^{\alpha\hat{k}}]}{\mathbb{E}[e^{\hat{k}}]} \;=\; \exp\left\{-(1-\alpha)\left(\mathbb{E}[\hat{k}]+\frac{\alpha}{2}\mathbb{V}ar[\hat{k}]\right)\right\} \;+\; o(2). \tag{A.77}$$

A.6.4 Irreversibility

Now, we show (59).

Lemma A.1. *The following relation holds for*

$$\mathbb{E}\left[\frac{\mathbb{E}[\mathrm{d}\phi(\hat{k})\mathcal{P}(\hat{k})]}{\mathrm{d}s}\right] = \frac{\overline{\mathbb{E}}\left[\mathbb{E}\left[\phi(\hat{k}_\tau)\mathcal{P}(\hat{k}_\tau)(\hat{k}_\tau)|\Delta\hat{k}\right]\right]-\overline{\mathbb{E}}\left[\mathbb{E}\left[\phi(\hat{k}^*(\Delta\hat{k}))\mathcal{P}(\hat{k}^*)|\Delta\hat{k}\right]\right]}{\overline{\mathbb{E}}[\tau]} \;\approx\; -\frac{\overline{\mathbb{C}ov}\left[\Delta\hat{k},\mathcal{P}(\hat{k}^*)\right]}{\overline{\mathbb{E}}[\tau]} < 0 \tag{A.78}$$

Proof. Define $Y_s = \phi(\hat{k}_s)\mathcal{P}(\hat{k}_s)$. We apply Ito's Lemma to Y_t to obtain:

$$\mathrm{d}Y_s = \left[-\nu\phi(\hat{k}_s)\left(\mathcal{P}(\hat{k}_s)+\mathcal{P}'(\hat{k}_s)\right)+\frac{\sigma^2}{2}\phi(\hat{k}_s)\left(\mathcal{P}(\hat{k}_s)+2\mathcal{P}'(\hat{k}_s)+\mathcal{P}''(\hat{k}_s)\right)\right]\mathrm{d}s+\phi(\hat{k}_s)\left(\mathcal{P}(\hat{k}_s)+\mathcal{P}'(\hat{k}_s)\right)dB_s. \tag{A.79}$$

Applying the OST, we have

$$\mathbb{E}^{\pm}\left[\phi(\hat{k}_\tau)\mathcal{P}(\hat{k}_\tau)\right]-\phi(\hat{k}^{*\pm})\mathcal{P}(\hat{k}^{*\pm}) = -\nu\mathbb{E}^{\hat{k}^{*\pm}}\left[\int_0^\tau\phi(\hat{k}_s)\left(\mathcal{P}(\hat{k}_t)+\mathcal{P}'(\hat{k}_t)\right)\mathrm{d}s\right]+\frac{\sigma^2}{2}\mathbb{E}^{\hat{k}^{*\pm}}\left[\int_0^\tau\phi(\hat{k}_s)\left(\mathcal{P}(\hat{k}_s)+2\mathcal{P}'(\hat{k}_s)+\mathcal{P}''(\hat{k}_s)\right)\right.$$

$$\cdots+\underbrace{\mathbb{E}^{\hat{k}_0}\left[\int_0^\tau\phi(\hat{k}_t)\left(\mathcal{P}(\hat{k}_t)+\mathcal{P}'(\hat{k}_t)\right)dW_s\right]}_{=0} \tag{A.80}$$

Using the occupancy measure, $\hat{k}_\tau = \hat{k}^*(\Delta\hat{k})-\Delta\hat{k}$, and taking expectations again to average across positive and negative adjustments, we get:

$$-\nu\mathbb{E}\left[\phi(\hat{k})\left(\mathcal{P}(\hat{k})+\mathcal{P}'(\hat{k})\right)\right]+\frac{\sigma^2}{2}\left[\phi(\hat{k})\left(\mathcal{P}(\hat{k})+2\mathcal{P}'(\hat{k})+\mathcal{P}''(\hat{k})\right)\right] \tag{A.81}$$

$$= \frac{\overline{\mathbb{E}}\left[\mathbb{E}\left[\phi(\hat{k}_\tau)\mathcal{P}(\hat{k}_\tau)(\hat{k}_\tau)|\Delta\hat{k}\right]\right]-\overline{\mathbb{E}}\left[\mathbb{E}\left[\phi(\hat{k}^*(\Delta\hat{k}))\mathcal{P}(\hat{k}^*)|\Delta\hat{k}\right]\right]}{\overline{\mathbb{E}}[\tau]} \tag{A.82}$$

Note that $\mathcal{P}(\hat{k}_\tau) = \mathcal{P}(\hat{k}^*)$ since it only depends on the sign of investment in the outer inaction region, thus

$$-\nu\mathbb{E}\left[\phi(\hat{k})\left(\mathcal{P}(\hat{k})+\mathcal{P}'(\hat{k})\right)\right]+\frac{\sigma^2}{2}\left[\phi(\hat{k})\left(\mathcal{P}(\hat{k})+2\mathcal{P}'(\hat{k})+\mathcal{P}''(\hat{k})\right)\right] \tag{A.83}$$

$$= \frac{\overline{\mathbb{E}}\left[\mathbb{E}\left[(\phi(\hat{k}_\tau)-\phi(\hat{k}^*(\Delta\hat{k})))\mathcal{P}(\hat{k}^*)|\Delta\hat{k}\right]\right]}{\overline{\mathbb{E}}[\tau]} \tag{A.84}$$

Finally, using the following approximations

$$\mathbb{E}[e^{\hat{k}}] = e^{\mathbb{E}[\hat{k}]}\mathbb{E}[e^{\hat{k}-\mathbb{E}[e^{\hat{k}}]}] \approx e^{\mathbb{E}[\hat{k}]}\mathbb{E}[1+\hat{k}-\mathbb{E}[e^{\hat{k}}]))] = e^{\mathbb{E}[\hat{k}]} \tag{A.85}$$

$$\phi(\hat{k}_\tau) - \phi(\hat{k}^*(\Delta\hat{k}) \approx e^{\hat{k}_\tau - \mathbb{E}[\hat{k}]} - e^{\hat{k}^*(\Delta\hat{k})-\mathbb{E}[\hat{k}]} \tag{A.86}$$

$$\approx \hat{k}_\tau - \hat{k}^*(\Delta\hat{k}) = -\Delta\hat{k} \tag{A.87}$$

With this approximation

$$\mathbb{E}\left[\frac{\mathbb{E}[\mathrm{d}\phi(\hat{k})\mathcal{P}(\hat{k})]}{\mathrm{d}s}\right] \approx -\frac{\overline{\mathbb{E}}[\Delta\hat{k}\mathcal{P}(\hat{k}^*)]}{\overline{\mathbb{E}}[\tau]} = -\frac{\overline{\mathbb{E}}[\Delta\hat{k}]\overbrace{\overline{\mathbb{E}}[\mathcal{P}(\hat{k}^*)]}^{=0} + \overline{\mathbb{C}ov}\left[\Delta\hat{k}\mathcal{P}(\hat{k}^*)\right]}{\overline{\mathbb{E}}[\tau]}. \tag{A.88}$$

Since $\mathcal{P}(\hat{k}^*) > 0$ when $\Delta\hat{k} > 0$ and $\mathcal{P}(\hat{k}^*) < 0$ when $\Delta\hat{k} < 0$, the covariance term in negative. □

A.7 Proof of Proposition 4

Proposition 4 characterizes the CIR as a function of cross-sectional steady-state moments. Let $g(\hat{k})$ be the capital-productivity steady-state distribution and $g_t(\hat{k})$ the distribution t-periods after an aggregate productivity shock of size $\delta > 0$, with $g_0(\hat{k}) = g(\hat{k} - \delta)$. Let $f(\hat{k})$ be a continuous function of $\hat{k}$. Define the cumulative impulse response of the function f as follows:

$$\mathrm{CIR}(f, \delta) \equiv \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} f(\hat{k}) \left[g_s(\hat{k}) - g(\hat{k})\right] \mathrm{d}\hat{k}\,\mathrm{d}s. \tag{A.89}$$

Note that in the main text we take $f(\hat{k}) = \hat{k}$.

Strategy. The proof has four steps. Step 1 we express the CIR as the integral of a value function $m(\hat{k})$ and $g'(\hat{k})$. Step 2 characterizes the terminal value of the value function. Step 3 constructs the master equation. Step 4 characterizes the CIR as a function of steady-state moments.

First order approximation. Start from the definition of the CIR, (1) operates over the integral; (2) uses conditional expectation, where $g_s(\hat{k}|\hat{k}_0)d\hat{k}$ is the probability of the state $\hat{k}$ at time s with initial condition $\hat{k}_0$; (3) uses the definition of the initial condition; (4) and (5) apply Fubini's theorem and the definition of limit of an integral; (7) applies a Taylor approximation over δ.

$$\begin{aligned}
\mathrm{CIR}(f,\delta) &= \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} f(\hat{k}) \left(g_s(\hat{k}) - g(\hat{k}) \right) \mathrm{d}\hat{k}\, \mathrm{d}s \\
&=^{(1)} \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) g_s(\hat{k})\, \mathrm{d}\hat{k}\, \mathrm{d}s \\
&=^{(2)} \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) g_s(\hat{k}|\hat{k}_0) g_0(\hat{k}_0)\, \mathrm{d}\hat{k}_0 \right] \mathrm{d}\hat{k}\, \mathrm{d}s \\
&=^{(3)} \int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0 - \delta)\, \mathrm{d}\hat{k}_0 \right] \mathrm{d}\hat{k}\, \mathrm{d}s \\
&=^{(4)} \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_0^\infty \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) g_s(\hat{k}|\hat{k}_0)\, \mathrm{d}\hat{k}\, \mathrm{d}s \right] g(\hat{k}_0 - \delta)\, \mathrm{d}\hat{k}_0 \\
&=^{(5)} \int_{\hat{k}^-}^{\hat{k}^+} \left[\lim_{\mathcal{T}\to\infty} \underbrace{\int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) g_s(\hat{k}|\hat{k}_0)\, \mathrm{d}\hat{k}\, \mathrm{d}s}_{\equiv\, m_{\mathcal{T}}(\hat{k})} \right] g(\hat{k}_0 - \delta)\, \mathrm{d}\hat{k}_0 \\
&=^{(6)} \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g(\hat{k} - \delta)\, \mathrm{d}\hat{k} \\
&=^{(7)} -\delta \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g'(\hat{k})\, \mathrm{d}\hat{k} + o(\delta^2)
\end{aligned}$$

where we define

$$m_{\mathcal{T}}(\hat{k}) \equiv \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) g_s(\hat{k}|\hat{k}_0)\, \mathrm{d}\hat{k}\, \mathrm{d}s \tag{A.90}$$

A.7.1 Without partial irreversibility

Without partial irreversibility there is one investment price p and one reset state $\hat{k}^*$.

Step 1: Up to first order, the CIR equals:

$$\mathrm{CIR}(f,\delta) \;=\; -\delta \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g'(\hat{k})\, \mathrm{d}\hat{k} + o(\delta^2), \tag{A.91}$$

where the function $m(\hat{k}) \in \mathcal{C}, \mathcal{C}^1$ satisfies the following HJB and border conditions:

$$0 \;=\; f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \Lambda(\hat{k})(m(\hat{k}^*) - m(\hat{k})), \tag{A.92}$$

$$0 \;=\; m(\hat{k}^*) - m(\hat{k}^\pm), \tag{A.93}$$

$$0 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k})\, \mathrm{d}\hat{k}. \tag{A.94}$$

1. Show that $\lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) = m(\hat{k})$ for all $\hat{k}$.
   See Baley and Blanco (2021).

2. We show that $\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\, \mathrm{d}\hat{k} = 0$.
   Substitute the definition of $m_{\mathcal{T}}(\hat{k})$ in the integral. In the following equalities, (1) and (2) use Fubini's theorem and Bayes' theorem; (3) uses the fact that $g(\hat{k})$ is the steady-state distribution; (4) solves the first

and second integrals.

$$\text{(A.95)} \qquad \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s \right] g(\hat{k}_0)\,\mathrm{d}\hat{k}_0$$
$$=^{(1)} \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}\hat{k}_0\,\mathrm{d}s$$
$$=^{(2)} \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) \left[\int_{\hat{k}^-}^{\hat{k}^+} g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0)\,\mathrm{d}\hat{k}_0 \right] \mathrm{d}\hat{k}\,\mathrm{d}s$$
$$=^{(3)} \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})] \right) g(\hat{k})\,\mathrm{d}\hat{k}\,\mathrm{d}s =^{(4)} 0.$$

3. Show that $\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = 0$.

   Write $m(\hat{k}) = \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k})$ inside the integral, pull the limit outside the integral, and use the previous result in (A.95) to get:

$$\text{(A.96)} \qquad \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} \;=\; \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} \;=\; \lim_{\mathcal{T}\to\infty} \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = 0$$

4. HJB and border conditions for $m_{\mathcal{T}}(\hat{k})$.

   The function $m_{\mathcal{T}}(\hat{k})$ satisfies the following HBJ with border conditions:

$$0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\mathcal{T}} - \nu \frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}} + \frac{\sigma^2}{2} \frac{\mathrm{d}^2 m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}^2} + \Lambda(\hat{k})(m_{\mathcal{T}}(\hat{k}^*) - m_{\mathcal{T}}(\hat{k}))$$
$$0 = m_{\mathcal{T}}(\hat{k}^*) - m_{\mathcal{T}}(\hat{k}^{\pm})$$
$$0 = \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k}$$

5. HJB and border conditions for $m(\hat{k})$.

   Taking the limit to $\mathcal{T} \to \infty$, and using point-wise convergence of $m_{\mathcal{T}}(\hat{k})$, we have the result.

Step 2: We characterize the terminal value $m(\hat{k}^*) = -\mathbb{C}ov[a, f(\hat{k})]$.
Proof of Step 2. Observe that $m(\hat{k})$ satisfies the following recursive representation

$$\text{(A.97)} \qquad m(\hat{k}) \;=\; \mathbb{E}\left[\int_0^{\tau} (f(\hat{k}_s) - \mathbb{E}[f(\hat{k})])\,\mathrm{d}s + m(\hat{k}^*) \middle| \hat{k}_0 = \hat{k} \right].$$

Define the following auxiliary function

$$\text{(A.98)} \qquad z(\hat{k}|\varphi) \;=\; \mathbb{E}\left[\int_0^{\tau} e^{\varphi s}(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})])\,\mathrm{d}s + e^{\varphi\tau} m(\hat{k}^*) \middle| \hat{k}_0 = \hat{k} \right].$$

and note that $z(\hat{k}|0) = m(\hat{k})$. The auxiliary function $z(\hat{k}|\varphi)$ satisfies the following HBJ and border condition

$$\text{(A.99)} \qquad -\varphi z(\hat{k}|\varphi) + \Lambda(\hat{k}) \left(z(\hat{k}|\varphi) - m(\hat{k}^*) \right) = f(\hat{k})) - \mathbb{E}[f(\hat{k})] - \nu \frac{\partial z(\hat{k}|\varphi)}{\partial \hat{k}} + \frac{\sigma^2}{2} \frac{\partial^2 z(\hat{k}|\varphi)}{\partial \hat{k}^2},$$

$$\text{(A.100)} \qquad z(\hat{k}^{\pm}, \varphi) \;=\; m(\hat{k}^*).$$

Since $z(\hat{k}|0) = m(\hat{k})$, (A.96) implies that $\int_{\hat{k}^-}^{\hat{k}^+} z(\hat{k}|0) g(\hat{k})\,\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k})\,\mathrm{d}\hat{k} = 0$. Taking the derivative with respect to φ in (A.99), we have that

$$(\Lambda(\hat{k})-\varphi)\frac{\partial z(\hat{k}|\varphi)}{\partial\varphi}-z(\hat{k}|\varphi) \;=\; -\nu\frac{\partial^2 z(\hat{k},\varphi)}{\partial\hat{k}\partial\varphi}+\frac{\sigma^2}{2}\frac{\partial^3 z(\hat{k}|\varphi)}{\partial\hat{k}^2\partial\varphi}, \tag{A.101}$$

$$\frac{\partial z(\hat{k}^{\pm}|\varphi)}{\partial\varphi} \;=\; 0. \tag{A.102}$$

Using the Schwarz's theorem to exchange partial derivatives, evaluating at $\varphi=0$, and using $z(\hat{k}|0)=m(\hat{k})$

$$\Lambda(\hat{k})\frac{\partial z(\hat{k}|0)}{\partial\varphi}-m(\hat{k}) = -\nu\frac{\partial}{\partial\hat{k}}\left(\frac{\partial z(\hat{k}|0)}{\partial\varphi}\right)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial\hat{k}^2}\left(\frac{\partial z(\hat{k}|0)}{\partial\varphi}\right), \tag{A.103}$$

$$\frac{\partial z(\hat{k}^{\pm}|0)}{\partial\varphi} \;=\; 0. \tag{A.104}$$

(A.103) and (A.103) are the HBJ and border conditions for $\frac{\partial z(\hat{k}|0)}{\partial\varphi}=\mathbb{E}\left[\int_0^\tau m(\hat{k}_s)\,\mathrm{d}s\Big|k_0=\hat{k}\right]$. Evaluating $\frac{\partial z(\hat{k}|0)}{\partial\varphi}$ at $\hat{k}^*$, and using the occupancy measure, we write the previous equation as:

$$\frac{\partial z(\hat{k}^*|0)}{\partial\varphi} \;=\; \mathbb{E}\left[\int_0^\tau m(\hat{k}_s)\,\mathrm{d}s\Big|k_0=\hat{k}^*\right] \;=\; \mathbb{E}[\tau]\mathbb{E}[m(\hat{k})] \;=\; 0 \tag{A.105}$$

At the same time, we have that

$$\frac{\partial z(\hat{k}^*|0)}{\partial\varphi} \;=\; \mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s)-\mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s+\tau m(\hat{k}^*)\Big|\,\hat{k}_0=\hat{k}^*\right], \tag{A.106}$$

Together (A.105) and (A.106) imply:

$$0=\mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s)-\mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s\Big|\,\hat{k}_0=\hat{k}\right]+\mathbb{E}\left[\tau|\,\hat{k}_0=\hat{k}\right]m(\hat{k}^*). \tag{A.107}$$

Solving for $m^{\hat{k}^*}$:

$$m(\hat{k}^*)=-\frac{\mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s)-\mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s\Big|\,\hat{k}_0=\hat{k}\right]}{\mathbb{E}\left[\tau|\,\hat{k}_0=\hat{k}\right]}. \tag{A.108}$$

Thus, we find the terminal value:

$$m(\hat{k}^*)=-\mathbb{E}[a(f(\hat{k})-\mathbb{E}[f(\hat{k})])] \;=\; -\mathbb{C}ov[a,f(\hat{k})]. \tag{A.109}$$

Step 3: In this step we show

$$-\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k}=\mathbb{C}ov\left[f(\hat{k}),\frac{\hat{k}+\nu a}{\sigma^2}\right] \tag{A.110}$$

Substituting $\Lambda(k)=\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}$ and using equation (A.155)

$$\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}m(\hat{k})=f(\hat{k})-\mathbb{E}[f(\hat{k})]-\nu m'(\hat{k})+\frac{\sigma^2}{2}m''(\hat{k})+\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}m(\hat{k}^*). \tag{A.111}$$

Multiplying by $g(\hat{k})\hat{k}$ and taking the integral between $\hat{k}^-$ and $\hat{k}^+$

$$0 = \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu T_1 + \frac{\sigma^2}{2}T_2 + m(\hat{k}^*)T_3 \tag{A.112}$$

$$T_1 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right] \mathrm{d}\,\hat{k} \tag{A.113}$$

$$T_2 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right] \mathrm{d}\,\hat{k} \tag{A.114}$$

$$T_3 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left(\nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k})\right) \mathrm{d}\,\hat{k}. \tag{A.115}$$

T_1 is equal to

$$\begin{aligned} T_1 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right] \mathrm{d}\,\hat{k} \\ &=^{(1)} \int_{\hat{k}^-}^{\hat{k}^*} \hat{k}\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right] \mathrm{d}\,\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} k\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right] \mathrm{d}\,\hat{k} \\ &=^{(2)} \int_{\hat{k}^-}^{\hat{k}^*} \hat{k}\frac{d\left(m(\hat{k})g(\hat{k})\right)}{\mathrm{d}\,\hat{k}} \mathrm{d}\,\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} k\frac{d\left(m(\hat{k})g(\hat{k})\right)}{\mathrm{d}\,\hat{k}} \mathrm{d}\,\hat{k} \\ &=^{(3)} \underbrace{\hat{k}m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}m(\hat{k})g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{=0} - \left[\int_{\hat{k}^-}^{\hat{k}^*} m(\hat{k})g(\hat{k})\,\mathrm{d}\,\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\,\hat{k}\right] \\ &=^{(4)} -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\,\hat{k} \\ &= 0 \end{aligned} \tag{A.116}$$

In step (1), we divide the integral across the discontinuity points; in (2), we use $m'(\hat{k})g(\hat{k})+m(\hat{k})g'(\hat{k}) = \frac{d\left(m(\hat{k})g(\hat{k})\right)}{\mathrm{d}\,\hat{k}}$; in (3), we use integration by parts, the border conditions, and continuity of $m(\hat{k})$ and $g(\hat{k})$ around $\hat{k}^*$.

T_2 satisfies

$$\text{(A.117)}\quad \begin{aligned}
T_2 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right] \mathrm{d}\,\hat{k} \\
&=^{(1)} \int_{\hat{k}^-}^{\hat{k}^*} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right] \mathrm{d}\,\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right] \mathrm{d}\,\hat{k} \\
&=^{(2)} \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Big|_{\hat{k}^*}^{\hat{k}^+} \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^*} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right] \mathrm{d}\,\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right] \mathrm{d}\,\hat{k}\right] \\
&=^{(3)} \underbrace{\hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0} - m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] \\
&\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^*} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right] \mathrm{d}\,\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right] \mathrm{d}\,\hat{k}\right] \\
&=^{(4)} -m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] - \int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\,\hat{k} + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} \\
&=^{(5)} -m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] - \left[\underbrace{m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k}\right] + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} \\
&= -m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k}.
\end{aligned}$$

In step (1), we divide the integral across the discontinuity points; in (2), we use the equality $m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k}) = \frac{\mathrm{d}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\,\hat{k}}$ and integration by parts. In step (3), we use continuity of $m'(\hat{k})$ and $g(\hat{k})$ around $\hat{k}^*$ and the border condition $g(\hat{k}^+) = g(\hat{k}^-) = 0$ for $\hat{k}m'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+} = 0$. We use the border conditions $m(\hat{k}^+) = m(\hat{k}^-)m(\hat{k}^*)$ for $\hat{k}m(\hat{k})g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} = m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right]$. Step (5) uses integration by part to the term to the term $\int_{\hat{k}^-}^{\hat{k}^+} m'(\hat{k})g(\hat{k})\,\mathrm{d}\,\hat{k}$.

Following similar steps as before, T_3 is equal to

$$\text{(A.118)}\quad \begin{aligned}
T_3 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left(\nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k})\right) \mathrm{d}\,\hat{k} \\
&= \nu\left[\int_{\hat{k}^-}^{\hat{k}^*} \hat{k}g'(\hat{k})\,\mathrm{d}\,\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}g'(\hat{k})\,\mathrm{d}\,\hat{k}\right] + \frac{\sigma^2}{2}\left[\int_{\hat{k}^-}^{\hat{k}^*} \hat{k}g''(\hat{k})\,\mathrm{d}\,\hat{k} + \int_{\hat{k}^*}^{\hat{k}^+} \hat{k}g''(\hat{k})\,\mathrm{d}\,\hat{k}\right] \\
&= \nu\left[\underbrace{\hat{k}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{=0} - \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\,\hat{k}}_{=1}\right] + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} - \int_{\hat{k}^-}^{\hat{k}^+} g'(\hat{k})\,\mathrm{d}\,\hat{k}\right] \\
&= -\nu + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} - \underbrace{g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0}\right] \\
&= -\nu + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right]
\end{aligned}$$

Using step 2, and the expressions for T_1, T_2, and T_3 in equations (A.117) to (A.119)

$$\begin{aligned}
0 &= \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu T_1 + \frac{\sigma^2}{2}T_2 + m(\hat{k}^*)T_3 \\
&= \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu 0 + \frac{\sigma^2}{2}\left[-m(\hat{k}^*)\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k}\right] \\
&\cdots + m(\hat{k}^*)\left[-\nu + \frac{\sigma^2}{2}\left[\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} + \hat{k}g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right]\right] \\
&= \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} + \nu\mathbb{E}\left[a\left(f(\hat{k}) - \mathbb{E}\left[f(\hat{k})\right]\right)\right] \\
&= \mathbb{C}ov\left[f(\hat{k}), \hat{k} + \nu a\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k}. \iff -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k}) = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k} + \nu a}{\sigma^2}\right]
\end{aligned} \tag{A.119}$$

(A.120)

Using step 1 and the previous equation

$$\frac{\mathrm{d}CIR(f,\delta)}{\mathrm{d}\delta}\bigg|_{\delta=0} = -\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k} + \nu a}{\sigma^2}\right] \tag{A.121}$$

A.7.2 With partial irreversibility

Step 1: Up to first order, the CIR equals:

$$\mathrm{CIR}(f,\delta) = -\delta\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2) \tag{A.122}$$

where the function $m(\hat{k}) \in \mathcal{C}, \mathcal{C}^1$ satisfies the following HJB and border conditions:

$$0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2}m''(\hat{k}) + \Lambda(\hat{k})(\mathcal{M}(\hat{k}^*) - m(\hat{k})), \tag{A.123}$$

$$0 = \mathcal{M}(\hat{k}^*) - m(\hat{k}^{\pm}), \tag{A.124}$$

$$0 = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}. \tag{A.125}$$

with where $\tilde{\mathcal{M}}(\hat{k}) \in \mathbb{C}^2$ is defined as

$$\tilde{\mathcal{M}}(\hat{k}) = \begin{cases} m(\hat{k}^{*-}) & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ m(\hat{k}^{*+}) & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+] \end{cases} \tag{A.126}$$

Proof of Step 1. Following the same steps as in the case without irreversibility, we obtain:

$$\mathrm{CIR}(f,\delta) = -\delta\int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2). \tag{A.127}$$

1. Show that $\lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) = m(\hat{k})$ for all $\hat{k}$.

   Let $\{T_i\}_{i=0}^{N(\mathcal{T})}$ be the adjustment dates between 0 and $\mathcal{T}$ for all $i = 1, 2, \ldots, N(\mathcal{T}) - 1$ and $T_0 = 0$ and $T_{N(\mathcal{T})} = \mathcal{T}$. Then, we can write $m_{\mathcal{T}}(\hat{k})$ as

$$m_{\mathcal{T}}(\hat{k}) = \mathbb{E}\left[\sum_{i=1}^{N(\mathcal{T})}\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})]\right)\Bigg|\,\hat{k}_0 = \hat{k}\right] \tag{A.128}$$

   Taking the limit $\mathcal{T} \to \infty$, we obtain the following equalities: (1) divides the sum; (2) uses the indicator function to write the finite sum; (3) uses the fact that

$$\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\mathbb{I}(N(\mathcal{T})\geq i+1)\middle|\hat{k}_0=\hat{k}\right]=1,\forall i;$$

(4) uses the fact that $\mathbb{E}\left[\sum_{i=1}^{\infty}\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right]$ is an independent of $\mathcal{T}$.

(A.129)

$$
\begin{aligned}
&lim_{\mathcal{T}\to\infty}m_{\mathcal{T}}(\hat{k})\\
&=\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\sum_{i=1}^{N(\mathcal{T})}\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right]\\
&=^{(1)}\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\sum_{i=1}^{N(\mathcal{T})-1}\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s+\lim_{\mathcal{T}\to\infty}\int_{N(\mathcal{T})-1}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right]\\
&=^{(2)}\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\sum_{i=1}^{\infty}\mathbb{I}(N(\mathcal{T})\geq i+1)\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s+\lim_{\mathcal{T}\to\infty}\int_{N(\mathcal{T})-1}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right]\\
&=^{(3)}\mathbb{E}\left[\sum_{i=1}^{\infty}\int_{T_{i-1}}^{T_i}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s+\lim_{\mathcal{T}\to\infty}\int_{N(\mathcal{T})-1}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right]\\
&=^{(4)}\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\int_{\tau_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right]+\text{terms independent of }\mathcal{T}
\end{aligned}
$$

2. Show that $\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\int_{N(\mathcal{T})-1}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right]$ is independent of $\hat{k}$ and $\mathcal{T}$. Let us define

(A.130)
$$\mathbb{V}(\hat{k}^{*\pm},\mathcal{T})=\mathbb{E}\left[\int_{\tau_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_{\tau^-_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}}=\hat{k}^{*\pm}\right].$$

Let $P_N^+(\hat{k})=\mathbb{E}\left[\mathbb{E}[\hat{k}_{\tau_N^-}\geq\hat{k}^{*+}\middle|\hat{k}_0=\hat{k}\right]$ and $P_N^-(\hat{k})=\mathbb{E}\left[\mathbb{E}[\hat{k}_{\tau_N^-}\leq\hat{k}^{*-}\middle|\hat{k}_0=\hat{k}\right]$. Then, we using conditional expectation, we can write

$$
\begin{aligned}
&=\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\int_{\tau_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\hat{k}_0=\hat{k}\right],\\
&=^{(1)}\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\mathbb{V}(\hat{k}^{*+},\mathcal{T})\lim_{\mathcal{T}\to\infty}P^+_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}(\hat{k}_0)+\lim_{\mathcal{T}\to\infty}\mathbb{V}(\hat{k}^{*-},\mathcal{T})\lim_{\mathcal{T}\to\infty}P^-_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}(\hat{k}_0)\middle|\hat{k}_0=\hat{k}\right],\\
&=^{(2)}\lim_{\mathcal{T}\to\infty}\mathbb{V}(\hat{k}^{*+},\mathcal{T})P^{+,\infty}+\lim_{\mathcal{T}\to\infty}\mathbb{V}(\hat{k}^{*-},\mathcal{T})P^{-,\infty},\\
&=^{(3)}\mathbb{V}^{\infty}(\hat{k}^{*+})P^{+,\infty}+\mathbb{V}^{\infty}(\hat{k}^{*-})P^{-,\infty}.
\end{aligned}
$$

Here, in step (1) we use law of iterated expectation. Step (2) comes from convergence of discrete Markov chains (see chapter 11 of Stokey (1989)). To show this claim, define $P_N(\hat{k})=\left[P_N^-(\hat{k});P_N^+(\hat{k})\right]\in\mathbb{R}^{2\times1}$, then

(A.131)
$$P_N(\hat{k})=P^TP_{N-1}(\hat{k}),$$

where $P=[P_1^-(\hat{k}^{*-}),1-P_1^+(\hat{k}^{*-});1-P_1^-(\hat{k}^{*+});P_1^+(\hat{k}^{*+})]\in\mathbb{R}^{2\times2}$ is a 2×2 transition probability where the rows are the transition probability. Under the assumption that $P_1^-(\hat{k}^{*-}),P_1^+(\hat{k}^{*+})\in(0,1)$, we that that

(A.132)
$$\lim_{N\to\infty}P_N(\hat{k})=\lim_{N\to\infty}P^{N-1}P_1(\hat{k})=[P^{-\infty};P^{+\infty}].$$

where the last equality comes from Theorem 11.1 in Stokey (1989). So, $\lim_{\mathcal{T}\to\infty}P^+_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}(\hat{k}_0)$ is independent of $\mathcal{T}$ and $\hat{k}_0$. See Baley and Blanco (2021) and Alexandrov (2021) for the convergence of $\lim_{\mathcal{T}\to\infty} \mathbb{V}(\hat{k}^{*\pm}, \mathcal{T}) = \mathbb{V}^{\infty}(\hat{k}^{*\pm})$.

3. Show that $\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\, \mathrm{d}\hat{k} = 0$.

   Using the property that $\Lambda(\hat{k}) = 0$ for all $\hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+})$, we can write in a simple form the HJB and border conditions satisfied by $m_{\mathcal{T}}(\hat{k})$:

$$
\begin{aligned}
&\text{(A.133)} \qquad 0 = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\mathcal{T}} - \nu \frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{d\hat{k}} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}^2} + \Lambda(\hat{k})(\tilde{\mathcal{M}}_{\mathcal{T}}(\hat{k}) - m_{\mathcal{T}}(\hat{k}))\\
&\text{(A.134)} \qquad 0 = \tilde{\mathcal{M}}_{\mathcal{T}}(\hat{k}^{*\pm}) - m_{\mathcal{T}}(\hat{k}^{\pm}),\\
&\text{(A.135)} \qquad 0 = \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k}) g(\hat{k})\, \mathrm{d}\hat{k}.
\end{aligned}
$$

   where $\tilde{\mathcal{M}}_{\mathcal{T}}(\hat{k}) \in \mathbb{C}^2$ is defined as

$$
\text{(A.136)} \qquad \tilde{\mathcal{M}}_{\mathcal{T}}(\hat{k}) = \begin{cases} m_{\mathcal{T}}(\hat{k}^{*-}) & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ m_{\mathcal{T}}(\hat{k}^{*+}) & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+] \end{cases}
$$

   Taking the limit to $\mathcal{T} \to \infty$ and using point-wise convergence of $m_{\mathcal{T}}(\hat{k})$, we have the result.

Step 2: Define the following objects

$$
\text{(A.137)} \qquad \mathbb{E}^{\pm}[f(\hat{k})] \equiv \frac{\mathbb{E}\left[\int_0^{\tau} f(\hat{k}_s)\, \mathrm{d}s \,\middle|\, \hat{k}_0 = \hat{k}^{*\pm}\right]}{\mathbb{E}\left[\tau \,\middle|\, \hat{k}_0 = \hat{k}^{*\pm}\right]},
$$

$$
\text{(A.138)} \qquad P^{\pm +} \equiv \mathbb{E}\left[\mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}} \,\middle|\, \hat{k}_0 = \hat{k}^{*\pm}\right].
$$

where $\mathbb{E}^{\pm}[f(\hat{k})]$ is the mean $f(k)$ conditional of a positive or negative last investment and $P^{\pm+}$ is the probability of a negative investment after a positive or negative investment. Then

$$
\begin{aligned}
m(\hat{k}^{*-}) &= -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^-[\tau]}{1 - P^{--}},\\
\text{(A.139)} \qquad m(\hat{k}^{*+}) &= -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^+[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^+[\tau]}{1 - P^{++}}.
\end{aligned}
$$

Proof of Step 2. Observe that $\mathcal{M}(f, \hat{k})$ satisfies the following recursive representation

$$
\text{(A.140)} \qquad m(\hat{k}) = \mathbb{E}\left[\int_0^{\tau} \left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right) \mathrm{d}s + m(\hat{k}) \,\middle|\, \hat{k}_0 = \hat{k}\right].
$$

Define an auxiliary function $z(\hat{k}|\varphi)$ as follows:

$$
\text{(A.141)} \qquad z(\hat{k}|\varphi) \equiv \mathbb{E}\left[\int_0^{\tau} e^{\varphi s}\left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})]\right) \mathrm{d}s + e^{\varphi \tau} m(\hat{k}) \,\middle|\, \hat{k}_0 = \hat{k}\right].
$$

and note the relationship: $z(\hat{k}|0) = m(\hat{k})$, $z(\cdot|\varphi) \in \mathcal{C}, \mathcal{C}^1$ for all φ, and

$$
\begin{aligned}
&\text{(A.142)} \qquad -\varphi z(\hat{k}|\varphi) + \Lambda(\hat{k})\left(z(\hat{k}|\varphi) - v^{f*}(\hat{k})\right) = f(\hat{k})) - \mathbb{E}[f(\hat{k})] - \nu z'(\hat{k}|\varphi) + \frac{\sigma^2}{2} z''(\hat{k}|\varphi),\\
&\text{(A.143)} \qquad z(\hat{k}^{\pm}|\varphi) = m(\hat{k}^*).
\end{aligned}
$$

Since $z(\hat{k}|0) = m(\hat{k})$, then we have $\int_{\hat{k}^-}^{\hat{k}^+} z(\hat{k}|0) g(\hat{k})\, \mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k})\, \mathrm{d}\hat{k} = 0$. Taking the derivative with respect to φ in (A.142), we have that

$$(\Lambda(\hat{k}) - \varphi)\frac{\partial z(\hat{k}|\varphi)}{\partial \varphi} - z(\hat{k}|\varphi) = -\nu \frac{\partial^2 z(\hat{k}|\varphi)}{\partial \hat{k}\partial \varphi} + \frac{\sigma^2}{2}\frac{\partial^3 z(\hat{k}|\varphi)}{\partial \hat{k}^2 \partial \varphi}, \tag{A.144}$$

$$\frac{\partial z(\hat{k}^{\pm}, \varphi)}{\partial \varphi} = 0. \tag{A.145}$$

Using the Schwarz's theorem to exchange partial derivatives and evaluating at $\varphi = 0$:

$$\Lambda(\hat{k}) \left.\frac{\partial z(\hat{k}, \varphi)}{\partial \varphi}\right|_{\varphi=0} - m(\hat{k}) = -\nu \frac{\partial \frac{\partial \left.\frac{\partial z(\hat{k},\varphi)}{\partial \varphi}\right|_{\varphi=0}}{\partial \varphi}}{\partial \hat{k}} + \frac{\sigma^2}{2}\frac{\partial^2 \left.\frac{\partial z(\hat{k},\varphi)}{\partial \varphi}\right|_{\varphi=0}}{\partial \hat{k}^2}, \tag{A.146}$$

$$\left.\frac{\partial z(\hat{k}, \varphi)}{\partial \varphi}\right|_{\varphi=0} = 0. \tag{A.147}$$

From the previous equation, using the occupancy measure and the renewal distribution, we have that

$$\begin{aligned}
\frac{\overline{\mathbb{E}}\left[\frac{\partial z(\hat{k}^*(\Delta\hat{k})|0)}{\partial \varphi}\right]}{\overline{\mathbb{E}}[\tau]} &= \frac{\overline{\mathbb{E}}\left[\mathbb{E}\left[\int_0^\tau m(\hat{k}_s)ds | k_0 = \hat{k}^*(\Delta\hat{k})\right]\right]}{\overline{\mathbb{E}}[\tau]}, \\
&= \mathbb{E}[m(\hat{k})], \\
&= 0.
\end{aligned} \tag{A.148}$$

Therefore, $\overline{\mathbb{E}}\left[\frac{\partial z(\hat{k}^*(\Delta\hat{k})|0)}{\partial \varphi}\right] = 0$. Using this result, the renewal distribution, the OST, and the $(\mathcal{R}^+, \mathcal{R}^-) = \left(\frac{\mathcal{N}^+}{\mathcal{N}}, \frac{\mathcal{N}^-}{\mathcal{N}}\right)$, we have

$$\begin{aligned}
0 &= \frac{\overline{\mathbb{E}}\left[\frac{\partial z(\hat{k}^*(\Delta\hat{k})|0)}{\partial \varphi}\right]}{\overline{\mathbb{E}}[\tau]}, \\
&= \frac{\overline{\mathbb{E}}\left[\mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s + \tau m(\hat{k}^*)\middle|\, \hat{k}_0 = \hat{k}^*(\Delta\hat{k})\right]\right]}{\overline{\mathbb{E}}[\tau]}, \\
&= \mathbb{E}\left[a\left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right)\right] + \frac{\overline{\mathbb{E}}\left[\mathbb{E}[\tau v^{f*}(\hat{k}_\tau)|\hat{k}_0 = \hat{k}^*(\Delta\hat{k})]\right]}{\overline{\mathbb{E}}[\tau]}, \\
&= \mathbb{C}ov\left[a, f(\hat{k})\right] + \frac{\mathcal{R}^-\mathbb{E}^-[\tau v^{f*}(\hat{k}_\tau)] + \mathcal{R}^+\mathbb{E}^+[\tau v^{f*}(\hat{k}_\tau)]}{\mathbb{E}[\tau]}, \\
&= \mathbb{C}ov\left[a, f(\hat{k})\right] + \frac{\mathcal{R}^-\mathbb{E}^-[\tau\left(m(\hat{k}^{*+})\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+}) + m(\hat{k}^{*-})\left(1 - \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})\right)\right)]}{\mathbb{E}[\tau]} \dots \\
&+ \frac{\mathcal{R}^+\mathbb{E}^+[\tau\left(m(\hat{k}^{*+})\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+}) + m(\hat{k}^{*-})\left(1 - \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})\right)\right)]}{\mathbb{E}[\tau]} \\
&= \mathbb{C}ov\left[a, f(\hat{k})\right] + m(\hat{k}^{*-}) + (m(\hat{k}^{*+}) - m(\hat{k}^{*-}))\frac{\overline{\mathbb{E}}[\tau\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\overline{\mathbb{E}}[\tau]}
\end{aligned} \tag{A.149}$$

To characterize $m(\hat{k}^{*+}) - m(\hat{k}^{*-})$, observe that

$$m(\hat{k}^{*-}) = \left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\overline{\mathbb{E}}^-[\tau] + (1 - P^{--})(\hat{k}^{*+}) + P^{--}m(\hat{k}^{*-}) \tag{A.150}$$

where $\mathbb{E}^{-}[f(\hat{k})]$ is the expected $\hat{k}$ conditional of a positive investment. Thus,

$$-\left(m(\hat{k}^{*+})-m(\hat{k}^{*-})\right)=\frac{\left(\mathbb{E}^{-}[f(\hat{k})]-\mathbb{E}[f(\hat{k})]\right)\overline{\mathbb{E}}^{-}[\tau]}{1-P^{--}} \tag{A.151}$$

From (A.149) and (A.151), we have that

$$m(\hat{k}^{*-})=-\mathbb{C}ov\left[f(\hat{k}),a\right]+\frac{\overline{\mathbb{E}}[\tau\mathbb{I}(\hat{k}_\tau\geq\hat{k}^{*+})]}{\overline{\mathbb{E}}[\tau]}\frac{\left(\mathbb{E}^{-}[f(\hat{k})]-\mathbb{E}[f(\hat{k})]\right)\overline{\mathbb{E}}^{-}[\tau]}{1-P^{--}}. \tag{A.152}$$

With similar steps as before, it is easy to show that

$$m(\hat{k}^{*+})=-\mathbb{C}ov\left[f(\hat{k}),a\right]+\frac{\mathbb{E}[\tau\mathbb{I}(\hat{k}_\tau\leq\hat{k}^{*-})]}{\mathbb{E}[\tau]}\frac{\left(\mathbb{E}^{+}[f(\hat{k})]-\mathbb{E}[f(\hat{k})]\right)\mathbb{E}^{+}[\tau]}{1-P^{++}}. \tag{A.153}$$

Step 3: In this step, we show

$$\frac{\text{CIR}(f,\delta)}{\delta}=\mathbb{C}ov\left[f(\hat{k}),\frac{\hat{k}+\nu a}{\sigma^2}\right]+\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)]\right] \tag{A.154}$$

Proof of Step 3. From steps 1 and 2

$$0 = f(\hat{k})-\mathbb{E}[f(\hat{k})]-\nu m'(\hat{k})+\frac{\sigma^2}{2}m''(\hat{k})+\Lambda(\hat{k})(\mathcal{M}(\hat{k})+\mathbb{C}ov\left[f(\hat{k}),a\right]-m(\hat{k})), \tag{A.155}$$

$$0 = \mathcal{M}(\hat{k}^*)+\mathbb{C}ov\left[f(\hat{k}),a\right]-m(\hat{k}^{\pm}), \tag{A.156}$$

$$0 = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}. \tag{A.157}$$

Using (A.43) to get $\Lambda(k)=\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}$ and using equation (A.155)

$$\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}m(\hat{k})=f(\hat{k})-\mathbb{E}[f(\hat{k})]-\nu m'(\hat{k})+\frac{\sigma^2}{2}m''(\hat{k})+\frac{\nu g'(\hat{k})+\frac{\sigma^2}{2}g''(\hat{k})}{g(\hat{k})}(\mathcal{M}(\hat{k})+\mathbb{C}ov\left[f(\hat{k}),a\right]) \tag{A.158}$$

Multiplying by $g(\hat{k})\hat{k}$ and taking the integral between $\hat{k}^-$ and $\hat{k}^+$

$$0=\mathbb{E}\left[f(\hat{k})\hat{k}\right]-\mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right]-\nu T_1+\frac{\sigma^2}{2}T_2+\nu T_3+\frac{\sigma^2}{2}T_4 \tag{A.159}$$

$$T_1=\int_{\hat{k}^-}^{\hat{k}^+}\hat{k}\left[m'(\hat{k})g(\hat{k})+m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k} \tag{A.160}$$

$$T_2=\int_{\hat{k}^-}^{\hat{k}^+}\hat{k}\left[m''(\hat{k})g(\hat{k})-m(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} \tag{A.161}$$

$$T_3=\int_{\hat{k}^-}^{\hat{k}^+}(\mathcal{M}(\hat{k})+\mathbb{C}ov\left[f(\hat{k}),a\right])\hat{k}g'(\hat{k})\,\mathrm{d}\hat{k} \tag{A.162}$$

$$T_4=\int_{\hat{k}^-}^{\hat{k}^+}(\mathcal{M}(\hat{k})+\mathbb{C}ov\left[f(\hat{k}),a\right])\hat{k}g''(\hat{k})\,\mathrm{d}\hat{k}. \tag{A.163}$$

Next, we compute each of the terms T_j, for $j\in\{1,2,3,4\}$. Using the fact that $\left[m'(\hat{k})g(\hat{k})+m(\hat{k})g'(\hat{k})\right]=$

$\left(m(\hat{k})g(\hat{k})\right)'$ and $\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0$

(A.164)
$$\begin{aligned} T_1 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m'(\hat{k})g(\hat{k}) + m(\hat{k})g'(\hat{k})\right]\mathrm{d}\hat{k} \\ &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left(m(\hat{k})g(\hat{k})\right)'\mathrm{d}\hat{k} \\ &= \underbrace{\hat{k}m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \hat{k}m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0 \end{aligned}$$

Using similar steps as before

(A.165)
$$\begin{aligned} T_2 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[m''(\hat{k})g(\hat{k}) - m(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} \\ &= \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\frac{\mathrm{d}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\frac{d\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\right]}{\mathrm{d}\hat{k}}d\hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\frac{\mathrm{d}\left[m'(\hat{k})g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}\mathrm{d}\hat{k} \\ &= \hat{k}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Bigg|_{\hat{k}^-}^{\hat{k}^{*-}} + \hat{k}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+} \\ &\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^{*-}}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\,\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\,\hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\,\hat{k}\right] \\ &= \underbrace{\hat{k}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\Bigg|_{\hat{k}^-}^{\hat{k}^+} + \hat{k}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^-}\right] \\ &\cdots - \left[\int_{\hat{k}^-}^{\hat{k}^+}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\,\mathrm{d}\,\hat{k} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})d\hat{k}\right] \\ &= -\left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}\right] - \int_{\hat{k}^-}^{\hat{k}^+}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\,\mathrm{d}\,\hat{k} + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})\frac{dg(\hat{k})}{d\hat{k}}\,\mathrm{d}\,\hat{k} \\ &= -\left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\ &- \left[\underbrace{m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\ &= -\left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k}. \\ &= -\left[\left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right)\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right)\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right)\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] \\ &+ 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} \end{aligned}$$

The term T_3 is equal to

$$T_3 = \int_{\hat{k}^-}^{\hat{k}^+} \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k} g'(\hat{k})\,\mathrm{d}\hat{k}$$

(A.166)

$$=^{(1)} \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) g'(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) g'(\hat{k})\,\mathrm{d}\hat{k}$$
$$+ \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) g'(\hat{k})\,\mathrm{d}\hat{k}$$

(A.167)

$$=^{(2)} \underbrace{\left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k} g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k} g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k} g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0}$$
$$- \int_{\hat{k}^-}^{\hat{k}^+} \left[\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right] + \hat{k}\mathcal{M}'(\hat{k})\right] g(\hat{k})\,\mathrm{d}\hat{k}$$

(A.168)

$$=^{(3)} -\mathbb{C}ov\left[f(\hat{k}), a\right] - \mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right]$$

Step (1) divides the integration domain in the discontinuity points. Step (2) uses continuity of $\mathcal{M}(\hat{k})$ and $g(\hat{k})$, together with the boundaries conditions of $g(\hat{k}^{\pm}) = 0$.

Finally, T_4 is equal to

(A.169)

$$T_4 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \frac{d^2 g(\hat{k})}{\mathrm{d}\hat{k}^2}\,\mathrm{d}\hat{k}$$
$$= \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \frac{d^2 g(\hat{k})}{\mathrm{d}\hat{k}^2}\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \frac{d^2 g(\hat{k})}{\mathrm{d}\hat{k}^2}\,\mathrm{d}\hat{k}$$
$$+ \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \frac{d^2 g(\hat{k})}{\mathrm{d}\hat{k}^2}\,\mathrm{d}\hat{k}$$
$$= \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}}$$
$$+ \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k} g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+} - \int_{\hat{k}^-}^{\hat{k}^+} \left[\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right] + \hat{k}\mathcal{M}'(\hat{k})\right] \frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\,\mathrm{d}\hat{k}$$
$$= \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}$$
$$\cdots - \underbrace{\left[\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right] + \hat{k}\mathcal{M}'(\hat{k})\right] \hat{k} g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^+}}_{=0} + \int_{\hat{k}^-}^{\hat{k}^+} \left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right] g(\hat{k})\,\mathrm{d}\hat{k}$$
$$= \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^-}^{\hat{k}^{*-}} + \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \left(\mathcal{M}(\hat{k}) + \mathbb{C}ov\left[f(\hat{k}), a\right]\right) \hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}$$
$$+ \mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right]$$

From equations (A.163) to (A.170)

$$0 = \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu T_1 + \frac{\sigma^2}{2}T_2 + \nu T_3 + \frac{\sigma^2}{2}T_4$$

$$= \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} - \nu\mathbb{E}\left[v^{f**}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{d^2v^{f**}(\hat{k})}{\mathrm{d}\hat{k}^2}\right]$$

(A.170)

$$= \mathbb{C}ov\left[f(\hat{k}),\hat{k}\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} + \nu\mathbb{C}ov\left[f(\hat{k}),a\right] - \nu\mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k})\right].$$

Therefore, we obtain:

(A.171) $$\frac{\mathrm{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}),\frac{\hat{k}+\nu a}{\sigma^2}\right] - \frac{\nu}{\sigma^2}\mathbb{E}\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + \frac{1}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + o(\delta)$$

Finally, if we apply Ito's lemma to $\hat{k}\mathcal{M}(\hat{k})$, we have that

(A.172) $$\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))|\hat{k}_s = \hat{k}] = \left[-\nu\left[\mathcal{M}(\hat{k}) + \hat{k}\mathcal{M}'(\hat{k})\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}) + \hat{k}\mathcal{M}''(\hat{k}_s)\right]\right]\mathrm{d}s$$

(A.173) $$\frac{\mathrm{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}),\frac{\hat{k}+\nu a}{\sigma^2}\right] + \frac{1}{\sigma^2}\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)]\right]$$

A.7.3 Irreversibility

Now, we show (62).

Lemma B.2. *The following relation holds for*

(A.174) $$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)\right]\right] = -\frac{\overline{\mathbb{C}ov}[\Delta\hat{k},\mathcal{M}(\Delta\hat{k})]}{\overline{\mathbb{E}}[\tau]} > 0.$$

Proof. If we apply Ito's lemma to $\hat{k}_s\mathcal{M}(\hat{k}_s)$, we have that

(A.175) $$\mathbb{E}_s[\mathrm{d}(\hat{k}_s\mathcal{M}(\hat{k}_s))] = \left[-\nu\left[\mathcal{M}(\hat{k}_s) + \hat{k}_s\mathcal{M}'(\hat{k}_s)\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\mathcal{M}'(\hat{k}_s) + \hat{k}_s\mathcal{M}''(\hat{k}_s)\right]\right]\mathrm{d}s$$

Taking the integral between 0 and τ, and applying the occupancy measure

(A.176) $$\frac{\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\hat{k}_\tau\mathcal{M}(\hat{k}_\tau)|\Delta\hat{k}\right]\right] - \overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\hat{k}^*\mathcal{M}(\hat{k}^*)|\Delta\hat{k}\right]\right]}{\overline{\mathbb{E}}[\tau]} = \mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)\right]\right]$$

Operating, we have that

(A.177) $$-\frac{\overline{\mathbb{E}}\left[\Delta\hat{k}\mathcal{M}(\Delta\hat{k})\right]}{\overline{\mathbb{E}}[\tau]} = \mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)\right]\right]$$

□

A.8 Proof of Proposition 1

This proposition splits the investment problem into static and dynamic components. Then substitutes the parametric changes introduced by taxation and redefines the investment frictions in terms of after-tax profits.

A.8.1 Flow profits with user cost of capital

Depart from the sufficient optimality conditions (HJB, value matching, optimality and smooth pasting) established in Lemma 2:

$$r v(\hat{k}) = A e^{\alpha \hat{k}} - \nu v'(\hat{k}) + \frac{\sigma^2}{2} v''(\hat{k}), \quad \hat{k} \in (\hat{k}^-, \hat{k}^+), \tag{A.178}$$

$$v(\hat{k}^-) = v(\hat{k}^{*-}) - \theta + p^{buy}(e^{\hat{k}^-} - e^{\hat{k}^{*-}}), \tag{A.179}$$

$$v(\hat{k}^+) = v(\hat{k}^{*+}) - \theta + p^{sell}(e^{\hat{k}^+} - e^{\hat{k}^{*+}}), \tag{A.180}$$

$$v'(\hat{k}) = p^{buy} e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \tag{A.181}$$

$$v'(\hat{k}) = p^{sell} e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \tag{A.182}$$

Define the auxiliary function:

$$\mathcal{W}(\hat{k}) \equiv \frac{v(\hat{k}) - p e^{\hat{k}}}{A} \implies v(\hat{k}) = A \mathcal{W}(\hat{k}) + p e^{\hat{k}}. \tag{A.183}$$

Substitute the new expression for $v(\hat{k})$ in (A.183) into both sides of (A.178):

$$r A \mathcal{W}(\hat{k}) + r p e^{\hat{k}} = A e^{\alpha \hat{k}} - \nu A \mathcal{W}'(\hat{k}) - \nu p e^{\hat{k}} + \frac{\sigma^2}{2} A \mathcal{W}''(\hat{k}) + \frac{\sigma^2}{2} p e^{\hat{k}}. \tag{A.184}$$

Divide by A, join terms with $pe^{\hat{k}}$, and rearrange, to obtain:

$$r \mathcal{W}(\hat{k}) = \underbrace{e^{\alpha \hat{k}} - \left(r + \nu - \frac{\sigma^2}{2}\right) \frac{p}{A} e^{\hat{k}}}_{\Pi(\hat{k})} - \nu \mathcal{W}'(\hat{k}) + \frac{\sigma^2}{2} \mathcal{W}''(\hat{k}), \tag{A.185}$$

where flow profits including the user cost are defined as:

$$\Pi(\hat{k}) \equiv e^{\alpha \hat{k}} - \left(r + \nu - \frac{\sigma^2}{2}\right) \frac{p}{A} e^{\hat{k}}. \tag{A.186}$$

Substitute the auxiliary function into the value matching conditions (A.179) to (A.180) and simplify:

$$\begin{aligned} \mathcal{W}(\hat{k}^-) &= \mathcal{W}(\hat{k}^{*-}) - \frac{\theta}{A} + \frac{(p^{buy} - p)}{A}(e^{\hat{k}^-} - e^{\hat{k}^{*-}}) \\ \mathcal{W}(\hat{k}^+) &= \mathcal{W}(\hat{k}^{*+}) - \frac{\theta}{A} + \frac{(p^{buy} - p)}{A}(e^{\hat{k}^+} - e^{\hat{k}^{*-}}) \end{aligned} \tag{A.187}$$

Substitute the auxiliary function into the smooth pasting and optimality conditions (A.181) to (A.182) and simplify:

$$\begin{aligned} \mathcal{W}'(\hat{k}) &= \frac{(p^{buy} - p)}{A} e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \\ \mathcal{W}'(\hat{k}) &= \frac{(p^{sell} - p)}{A} e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \end{aligned} \tag{A.188}$$

Summarizing, the optimality conditions satisfied by $\mathcal{W}(\hat{k})$ are: (A.185), (A.186), (A.187), and (A.188).

A.8.2 Static component

Define the static policy $\hat{k}^{ss}$ as the capital-productivity ratio that maximizes $\Pi(\hat{k})$ in (A.186):

$$\text{(A.189)} \qquad \hat{k}^{ss} = \frac{1}{1-\alpha}\log\left(\frac{\alpha A}{p(r+\nu-\sigma^2/2)}\right) = \frac{1}{1-\alpha}\log\left(\frac{\alpha A}{p(\rho+\xi^k-\sigma^2)}\right),$$

where in the second equality we have substituted $r = \rho - \mu - \sigma^2/2$. Next, do an infinite-order Taylor approximation of flow profits around the static policy $\hat{k}^{ss}$, denoting the n-th derivative with $\Pi^n(\cdot)$:

$$\text{(A.190)} \quad \Pi(\hat{k}) =^{(1)} \sum_{n=0}^{\infty}\frac{\Pi^n(\hat{k}^{ss})}{n!}(\hat{k}-\hat{k}^{ss})^n =^{(2)} e^{\alpha\hat{k}^{ss}}\sum_{n=0}^{\infty}\frac{(\alpha^n-\alpha)}{n!}(\hat{k}-\hat{k}^{ss})^n =^{(3)} e^{\alpha\hat{k}^{ss}}\left(e^{\alpha(\hat{k}-\hat{k}^{ss})}-\alpha e^{(\hat{k}-\hat{k}^{ss})}\right).$$

where equality (2) substitutes the derivatives of flow profits evaluated at the static policy

$$\text{(A.191)} \qquad \Pi^n(\hat{k}^{ss}) = \alpha^n e^{\alpha\hat{k}} - \left(r+\nu-\frac{\sigma^2}{2}\right)\frac{p}{A}e^{\hat{k}}\Big|_{\hat{k}=\hat{k}^{ss}} = e^{\alpha\hat{k}^{ss}}(\alpha^n-\alpha);$$

and equality (3) applies the definition of the exponential function $f(z) = e^z = \sum_{n=0}^{\infty} z^n/n!$ evaluated at the points $z = \alpha(\hat{k}-\hat{k}^{ss})$ and $z = (\hat{k}-\hat{k}^{ss})$. Thus we obtain the following expression for flow profits:

$$\text{(A.192)} \qquad \Pi(\hat{k}) = e^{\alpha\hat{k}^{ss}}\left(e^{\alpha(\hat{k}-\hat{k}^{ss})}-\alpha e^{(\hat{k}-\hat{k}^{ss})}\right).$$

A.8.3 Dynamic component

Consider the normalized capital-productivity ratios $x = \hat{k}-\hat{k}^{ss}$. Its dynamics are given by $\mathrm{d}x_s = \mathrm{d}\hat{k}_s = -\nu\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$. Using the normalized ratios x, the auxiliary function $\mathcal{W}(\hat{k})$ in (A.183), and the static policy $\hat{k}^{ss}$ in (A.189), define the function $\mathcal{V}(x)$, the flow profits $\Pi(x)$, and the dynamic policy $\mathcal{X}$ as follows:

$$\text{(A.193)} \qquad \mathcal{V}(x) \equiv \frac{\mathcal{W}(x+\hat{k}_{ss})}{e^{\alpha\hat{k}^{ss}}},$$

$$\text{(A.194)} \qquad \pi(x) \equiv \frac{\Pi(x+\hat{k}_{ss})}{e^{\alpha\hat{k}^{ss}}} = e^{\alpha x}-\alpha e^{x},$$

$$\text{(A.195)} \qquad \mathcal{X} \equiv (x^-, x^{*-}, x^{*+}, x^+) = (\hat{k}^- - \hat{k}^{ss}, \hat{k}^{*-}-\hat{k}^{ss}, \hat{k}^{*+}-\hat{k}^{ss}, \hat{k}^+ - \hat{k}^{ss}).$$

Rewrite the optimality conditions (A.185), (A.186), (A.187), and (A.188) using $\mathcal{V}(x)$, $\pi(x)$, and $\mathcal{X}$ as follows:

$$\text{(A.196)} \qquad \begin{aligned} r\mathcal{V}(x) &= \pi(x) - \nu\mathcal{V}'(x) + \frac{\sigma^2}{2}\mathcal{V}''(x), \quad x\in(x^-,x^+),\\ \mathcal{V}(x^-) &= \mathcal{V}(x^{*-}) - \frac{\theta}{Ae^{\alpha\hat{k}^{ss}}} + \frac{p^{buy}-p}{Ae^{(\alpha-1)\hat{k}^{ss}}}(e^{x^-}-e^{x^{*-}}),\\ \mathcal{V}(x^+) &= \mathcal{V}(x^{*+}) - \frac{\theta}{Ae^{\alpha\hat{k}^{ss}}} + \frac{p^{sell}-p}{Ae^{(\alpha-1)\hat{k}^{ss}}}(e^{x^+}-e^{x^{*+}}),\\ \mathcal{V}'(x) &= \frac{p^{buy}-p}{Ae^{(\alpha-1)\hat{k}^{ss}}}e^x, \quad x\in\left\{x^-,x^{*-}\right\},\\ \mathcal{V}'(x) &= \frac{p^{sell}-p}{Ae^{(\alpha-1)\hat{k}^{ss}}}e^x, \quad x\in\left\{x^+,x^{*+}\right\}. \end{aligned}$$

Remark: We construct the user cost of capital with p, i.e., the price of positive investment. Nevertheless, it is easy to check that the proofs above hold for any p.

A.8.4 Adding corporate taxes

Substitute the parametric changes that incorporate corporate taxation in the problem:

$$\text{(A.197)}\quad A \to (1-t^c)A, \qquad \rho \to \left(\frac{1-t^p}{1-t^g}\right)\rho, \qquad p(\Delta\hat{k}) \to \left(1 - t^c z - \omega(1-t^c)\mathbb{1}_{\{\Delta\hat{k}<0\}}\right)p, \qquad \theta \to (1-t^c z)\theta,$$

into the static policy:

$$\text{(A.198)}\quad \hat{k}^{ss} = \frac{1}{1-\alpha}\log\left(\frac{1-t^c}{1-t^c z}\frac{\alpha A}{p\tilde{\mathcal{U}}}\right)$$

where $\tilde{\mathcal{U}} \equiv \frac{1-t^p}{1-t^g}\rho + \xi^k - \sigma^2$ is the after-tax user cost of capital. Then, define the effective fixed cost $\tilde{\theta}$, which is scaled by after-tax profits; the effective price wedge $(\tilde{p}^{buy}, \tilde{p}^{sell})$, which is scaled by after-tax output-capital ratio; and the effective discount factor $\tilde{r}$ as follows:

$$\text{(A.199)}\quad \tilde{\theta} \equiv \frac{1-t^c z}{1-t^c}\frac{\theta}{Ae^{\alpha\hat{k}^{ss}}},$$

$$\text{(A.200)}\quad (\tilde{p}^{buy}, \tilde{p}^{sell}) \equiv \frac{1-t^c z}{1-t^c}\frac{(0, -\omega(1-t^c)p)}{Ae^{(\alpha-1)\hat{k}^{ss}}},$$

$$\text{(A.201)}\quad \tilde{r} \equiv \left(\frac{1-t^p}{1-t^g}\right)\rho - \mu - \frac{\sigma^2}{2}.$$

Using these definitions, convert the system in (A.196) simplifies to the final expressions:

$$\text{(A.202)}\quad \begin{aligned} \tilde{r}\mathcal{V}(x) &= e^{\alpha x} - \alpha e^x - \nu\mathcal{V}'(x) + \frac{\sigma^2}{2}\mathcal{V}''(x), \quad x \in (x^-, x^+), \\ \mathcal{V}(x^-) &= h(x^{*-}) - \tilde{\theta} + \tilde{p}^{buy}(e^{x^-} - e^{x^{*-}}), \\ \mathcal{V}(x^+) &= h(x^{*+}) - \tilde{\theta} + \tilde{p}^{sell}(e^{x^+} - e^{x^{*+}}), \\ \mathcal{V}'(x) &= \tilde{p}^{buy}e^x, \quad x \in \{x^-, x^{*-}\}, \\ \mathcal{V}'(x) &= \tilde{p}^{sell}e^x, \quad x \in \{x^+, x^{*+}\}. \end{aligned}$$

The previous system solves the stopping problem:

$$\text{(A.203)}\quad \mathcal{V}(x) = \max_{\tau,\Delta x}\mathbb{E}\left[\int_0^\tau e^{-\tilde{r}\tau}\left(e^{\alpha x_s} - \alpha e^{x_s}\right)\mathrm{d}s + e^{\tilde{r}\tau}\left(-\tilde{\theta} + \tilde{p}(\Delta x)(e^{x_\tau+\Delta x} - e^{x_\tau}) + \mathcal{V}(x_\tau + \Delta x)\right)\Big| x_0 = x\right]$$

$$\text{(A.204)}\quad \tilde{p}(\Delta x) = \tilde{p}^{\text{buy}}\mathbb{1}_{\{\Delta x>0\}} + \tilde{p}^{\text{sell}}\mathbb{1}_{\{\Delta x<0\}}.$$

A.9 Proof of Proposition 4

This proposition computes the optimal investment policy and the macro outcomes for driftless symmetric models. Assume $\nu \to 0$ (symmetry stochastic process) and following the remark in A.8.3, we can use $p(1-\frac{\omega}{2})$ to construct the static component and $\hat{k}^{ss}$. Under this normalization, the positive and negative capital gains are given by $\frac{p^{buy}-p(1-\frac{\omega}{2})}{Ae^{(\alpha-1)\hat{k}^{ss}}} = \frac{\omega\alpha}{2U} =: \tilde{p}$ and $\frac{p^{sell}-p(1-\frac{\omega}{2})}{Ae^{(\alpha-1)\hat{k}^{ss}}} = -\frac{\omega\alpha}{2U} = -\tilde{p}$. Similarly, for the fixed cost $\tilde{\theta} = \frac{\theta}{Ae^{\alpha\hat{k}^{ss}}} = \theta\left[\frac{p(1-\omega/2)U}{\alpha A^{1/\alpha}}\right]$. Finally, under this normalization, since optimal dynamic policies $\mathcal{X}$ are symmetric in all the following cases, then $\mathbb{E}[x] = 0$ and $\mathbb{C}ov[x,a] = 0$. We characterize the case with both frictions active. It nests the solution for only fixed costs (setting $x^* = 0$) and only price wedge (setting $x^* = \overline{x}$).

A.9.1 Sufficient optimality conditions

The solution is characterized by a symmetric inaction region with borders $\pm\overline{x}$ and two reset points $\pm x^*$ that satisfy the following system of equations:

$$\tilde{r}\mathcal{V}(x) = e^{\alpha x} - \alpha e^x + \frac{\sigma^2}{2}\mathcal{V}''(x), \quad \forall x \in (-\overline{x}, \overline{x}), \tag{A.205}$$

$$\mathcal{V}(-\overline{x}) = \mathcal{V}(-x^*) - \tilde{p}(e^{-x^*} - e^{-\overline{x}}) - \tilde{\theta}, \tag{A.206}$$

$$\mathcal{V}(\overline{x}) = \mathcal{V}(x^*) - \tilde{p}(e^{\overline{x}} - e^{x^*}) - \tilde{\theta}, \tag{A.207}$$

$$\mathcal{V}'(x) = \tilde{p}e^x, \quad x \in \{-\overline{x}, -x^*\}, \tag{A.208}$$

$$\mathcal{V}'(x) = -\tilde{p}e^x, \quad x \in \{\overline{x}, x^*\}. \tag{A.209}$$

A.9.2 Optimal policy

Assume $\tilde{p}$ and $\tilde{\theta}$ are small. A second order Taylor approximation of the flow profits $\pi(x) = e^{\alpha x} - \alpha e^x$ in (A.194) around the (frictionless) point $x = 0$ yields:

$$\pi(x) = \pi(0) + \pi'(0)x + \frac{\pi''(0)}{2}x^2 = 1 - \alpha + \frac{\alpha^2 - \alpha}{2}x^2 = (1 - \alpha) - \frac{\alpha(1-\alpha)}{2}x^2. \tag{A.210}$$

For the characterization, we ignore the constant term. Next, a first order approximation of the exponential function $e^x = 1 + x$ yields $e^{\overline{x}} - e^{x^*} = \overline{x} - x^*$ and $e^{-x^*} - e^{-\overline{x}} = \overline{x} - x^*$. Under these approximations, we write the system of equations as:

$$\tilde{r}\mathcal{V}(x) = -\frac{\alpha(1-\alpha)}{2}x^2 + \frac{\sigma^2}{2}\mathcal{V}''(x), \quad \forall x \in (-\overline{x}, \overline{x}), \tag{A.211}$$

$$\mathcal{V}(-\overline{x}) = \mathcal{V}(-x^*) - \tilde{p}(\overline{x} - x^*) - \tilde{\theta}, \tag{A.212}$$

$$\mathcal{V}(\overline{x}) = \mathcal{V}(x^*) - \tilde{p}(\overline{x} - x^*) - \tilde{\theta}, \tag{A.213}$$

$$\mathcal{V}'(x) = \tilde{p}(1 + x), \quad x \in \{-\overline{x}, -x^*\}, \tag{A.214}$$

$$\mathcal{V}'(x) = -\tilde{p}(1 + x), \quad x \in \{\overline{x}, x^*\}. \tag{A.215}$$

Since the problem is symmetric, we only work with the positive domain and use equations (A.211), (A.213), and (A.215). Next, we derive two conditions that pin down the optimal policy by approximating the value function and its derivatives. We denote the n^{th} derivative with $\mathcal{V}^n(x)$ for $n > 2$.

To derive the first condition, we approximate the value function with a 4^{th} order Taylor expansion around $x = 0$, noting that odd derivatives evaluated at zero are equal to zero by symmetry:

$$\mathcal{V}(x) = \mathcal{V}(0) + \frac{\mathcal{V}''(0)}{2!}x^2 + \frac{\mathcal{V}^4(0)}{4!}x^4. \tag{A.216}$$

The second derivative of (A.211) evaluated at zero equals:

$$\tilde{r}\mathcal{V}''(0) = -\alpha(1 - \alpha) + \frac{\sigma^2}{2}\mathcal{V}^4(0). \tag{A.217}$$

Taking the limit $\tilde{r} \to 0$ in the previous expression, we find $\mathcal{V}^4(0)$:

$$\mathcal{V}^4(0) = \frac{2\alpha(1-\alpha)}{\sigma^2}. \tag{A.218}$$

Substitute the approximation to $\mathcal{V}(x)$ into the value matching condition in (A.212) and simplify:

$$\begin{aligned}
\mathcal{V}(\overline{x}) &= \mathcal{V}(x^*) \; - \; \tilde{p}(\overline{x}-x^*) - \tilde{\theta} \\
\mathcal{V}(0) + \frac{\mathcal{V}''(0)}{2}\overline{x}^2 + \frac{\mathcal{V}^4(0)}{24}\overline{x}^4 &= \mathcal{V}(0) + \frac{\mathcal{V}''(0)}{2}x^{*2} + \frac{\mathcal{V}^4(0)}{24}x^{*4} \; - \; \tilde{p}(\overline{x}-x^*) - \tilde{\theta} \\
\frac{\mathcal{V}''(0)}{2}(\overline{x}^2 - x^{*2}) + \frac{\alpha(1-\alpha)}{12\sigma^2}(\overline{x}^4 - x^{*4}) &= \; - \; \tilde{p}(\overline{x}-x^*) - \tilde{\theta} \\
\frac{\mathcal{V}''(0)}{2}(\overline{x}^2 - x^{*2}) + \frac{\alpha(1-\alpha)}{12\sigma^2}(\overline{x}^2 + x^{*2})(\overline{x}^2 - x^{*2}) &= \; - \; \tilde{p}(\overline{x}-x^*) - \tilde{\theta} \\
\mathcal{V}''(0) + \frac{\alpha(1-\alpha)}{6\sigma^2}(\overline{x}^2 + x^{*2}) &= \; - \; 2\left(\frac{\tilde{p}(\overline{x}-x^*) + \tilde{\theta}}{(\overline{x}^2 - x^{*2})}\right)
\end{aligned}$$

To derive a second condition, we approximate the first derivative of the value function with a 3^rd order Taylor approximation, noting that odd derivatives evaluated at zero are zero by symmetry:

(A.219)
$$\mathcal{V}'(x) = \frac{\mathcal{V}''(0)}{1!}x + \frac{\mathcal{V}^4(0)}{3!}x^3$$

Evaluating at $\overline{x}$ and x^* and substituting the expression for the fourth derivative:

(A.220)
$$-\tilde{p}(1+\overline{x}) = \mathcal{V}''(0)\overline{x} + \frac{\alpha(1-\alpha)}{3\sigma^2}\overline{x}^3; \qquad -\tilde{p}(1+x^*) = \mathcal{V}''(0)x^* + \frac{\alpha(1-\alpha)}{3\sigma^2}x^{*3}.$$

Taking the difference between the two previous conditions and subtracting $\tilde{\theta}$

(A.221)
$$-\tilde{p}(\overline{x}-x^*) = \mathcal{V}''(0)(\overline{x}-x^*) + \frac{\alpha(1-\alpha)}{3\sigma^2}(\overline{x}^3 - x^{*3})$$

and solving for $\mathcal{V}''(0)$:

(A.222)
$$-\frac{\mathcal{V}''(0)}{2} = \frac{\tilde{p}}{2} + \frac{\alpha(1-\alpha)}{6\sigma^2}(\overline{x}^2 + x^{*2} + \overline{x}x^*)$$

Back into optimality:

(A.223)
$$\overline{x}x^*(\overline{x}+x^*) = \frac{3\sigma^2\tilde{p}}{\alpha(1-\alpha)}$$

(A.224)
$$\overline{x}^4 - x^{*4} = \overline{x}x^*(\overline{x}+x^*)(\overline{x}-x^*)\left(1+\overline{x}+x^*\right) + \frac{6\sigma^2\tilde{\theta}}{\alpha(1-\alpha)}$$

The optimal policy nests the two extreme cases:

1. With only a fixed cost, $\tilde{p}=0$ and $x^*=0$. The first condition is irrelevant and the second condition pins down the optimal policy:

   (A.225)
   $$\overline{x}^4 = \frac{6\sigma^2\tilde{\theta}}{\alpha(1-\alpha)} \quad \Longrightarrow \quad \overline{x} = \left(\frac{6\sigma^2\tilde{\theta}}{\alpha(1-\alpha)}\right)^{1/4}.$$

2. With only a price wedge, $\tilde{\theta}=0$ and $\overline{x}^* = x^* = \overline{x}$. The second condition is irrelevant and the first condition pins down the optimal policy:

   (A.226)
   $$2\overline{x}^{*3} = \frac{3\sigma^2\tilde{p}}{\alpha(1-\alpha)} \quad \Longrightarrow \quad \overline{x}^* = \left(\frac{3\sigma^2\tilde{p}}{2\alpha(1-\alpha)}\right)^{1/3}.$$

A.9.3 Cross-sectional and renewal distributions and adjustment probabilities

Cross-sectional distribution. The stationary density $g(x)$ solves the KFE with border, continuity, and reinjection (exit mass equals entry mass) conditions:

$$0 = \frac{\sigma^2}{2} g''(x), \tag{A.227}$$

$$g(\overline{x}) = g(-\overline{x}) = 0, \tag{A.228}$$

$$\int_{-\overline{x}}^{\overline{x}} g(x)\,\mathrm{d}x = 1, \tag{A.229}$$

$$\lim_{x\downarrow -x^*} g(x) = \lim_{x\uparrow -x^*} g(x), \qquad \lim_{x\downarrow x^*} g(x) = \lim_{x\uparrow x^*} g(x), \tag{A.230}$$

$$\lim_{x\downarrow -\overline{x}} g'(x) = \lim_{x\uparrow -x^*} g'(x) - \lim_{x\downarrow -x^*} g'(x), \qquad \lim_{x\uparrow \overline{x}} g'(x) = \lim_{x\downarrow x^*} g'(x) - \lim_{x\uparrow x^*} g'(x). \tag{A.231}$$

Solving for $g(x)$, we obtain a linear function:

$$g''(x) = 0, \quad g'(x) = A, \quad g(x) = Ax + B. \tag{A.232}$$

We split the state-space into three segments $[-\overline{x}, -x^*] \cup [-x^*, x^*] \cup [x^*, \overline{x}]$ and consider three different functions $g_k(x) = A_k x + B_k$ for $j = 1, 2, 3$, one for each segment. Evaluating at the border conditions, we obtain relationships for (A_1, B_1) and (A_3, B_3):

$$\left.\begin{aligned} -A_1\overline{x} + B_1 = 0 \\ A_3\overline{x} + B_3 = 0 \end{aligned}\right\} \implies \overline{x} = B_1/A_1 = -B_3/A_3. \tag{A.233}$$

Evaluating at the reinjection conditions, we obtain A_2:

$$\left.\begin{aligned} A_1 = A_1 - A_2 \\ A_3 = A_3 - A_2 \end{aligned}\right\} \implies A_2 = 0. \tag{A.234}$$

Evaluating at the continuity conditions, using $A_2 = 0$ we obtain for (A_1, B_1) and (A_3, B_3):

$$\left.\begin{aligned} B_2 &= -A_1 x^* + B_1 \\ B_2 &= \;\; A_3 x^* + B_3 \end{aligned}\right\} \implies x^* = \frac{B_1 - B_2}{A_1} = \frac{B_2 - B_3}{A_3}. \tag{A.235}$$

Finally, we use the fact that the density integrates to one:

$$\begin{aligned} 1 &= \int_{-\overline{x}}^{-x^*} (A_1 x + B_1)\,\mathrm{d}x + \int_{-x^*}^{x^*} B_2\,\mathrm{d}x + \int_{x^*}^{\overline{x}} (A_3 x + B_3)\,\mathrm{d}x \\ &= \left(A_1 \frac{x^2}{2} + B_1 x\right)\Big|_{-\overline{x}}^{-x^*} + B_2 x\Big|_{-x^*}^{x^*} + \left(A_3 \frac{x^2}{2} + B_3 x\right)\Big|_{x^*}^{\overline{x}} \\ &= A_1 \left(\frac{x^{*2} - \overline{x}^2}{2}\right) + B_1(\overline{x} - x^*) + 2B_2 x^* + A_3 \left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + B_3(\overline{x} - x^*) \\ &= (A_3 - A_1)\left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + 2B_2 x^* + (B_1 + B_3)(\overline{x} - x^*). \end{aligned} \tag{A.236}$$

Substituting $B_1 = \overline{x}A_1$ and $B_3 = -\overline{x}A_3$ from (A.233) into the previous expression:

$$\begin{aligned} 1 &= (A_3 - A_1)\left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + 2B_2 x^* - \overline{x}(A_3 - A_1)(\overline{x} - x^*) \\ &= (A_3 - A_1)(\overline{x} - x^*)\left[\frac{\overline{x} + x^*}{2} - \overline{x}\right] + 2B_2 x^* \\ &= (A_3 - A_1)\frac{(\overline{x} - x^*)^2}{2} + 2B_2 x^*. \end{aligned} \tag{A.237}$$

Therefore, the cross-sectional density is equal to:

$$g(x) = \frac{1}{\overline{x}^2 - x^{*2}} \begin{cases} \overline{x} + x & \text{for} \quad x \in [-\overline{x}, -x^*] \\ \overline{x} - x^* & \text{for} \quad x \in [-x^*, x^*] \\ \overline{x} - x & \text{for} \quad x \in [x^*, \overline{x}]. \end{cases} \tag{A.238}$$

Renewal probabilities and relative shares. The renewal probabilities (the mass of adjusters from each reset point) are equal to:

$$\mathcal{N}^- = \frac{\sigma^2}{2} \lim_{x \downarrow -\overline{x}} g'(x) = \frac{\sigma^2}{2} A_1 = \frac{\sigma^2}{2} \frac{1}{(\overline{x}^2 - x^{*2})} \tag{A.239}$$

$$\mathcal{N}^+ = -\frac{\sigma^2}{2} \lim_{x \uparrow \overline{x}} g'(x) = -\frac{\sigma^2}{2} A_3 = \frac{\sigma^2}{2} \frac{1}{(\overline{x}^2 - x^{*2})}. \tag{A.240}$$

The shares of total, upward, and downward adjustment are:

$$\mathcal{N} = \mathcal{N}^- + \mathcal{N}^+ = \frac{\sigma^2}{(\overline{x}^2 - x^{*2})} \tag{A.241}$$

$$\frac{\mathcal{N}^-}{\mathcal{N}} = \frac{1}{2}; \qquad \frac{\mathcal{N}^+}{\mathcal{N}} = \frac{1}{2}. \tag{A.242}$$

Probability of negative adjustment. Let $\mathbb{P}^+(x) \equiv \Pr[\Delta x < 0 | x]$ denote the probability of doing a negative adjustment (after hitting the upper bound) conditional on the state x. It solves the HJB with border conditions:

$$0 = \mathbb{P}^{+\prime\prime}(x); \quad \mathbb{P}^+(\overline{x}) = 1; \quad \mathbb{P}^+(-\overline{x}) = 0 \tag{A.243}$$

Solving for $\mathbb{P}^+(x) = Ax + B$ and evaluating at the border conditions:

$$\left.\begin{array}{r} A\overline{x} + B = 1 \\ -A\overline{x} + B = 0 \end{array}\right\} \implies \left.\begin{array}{r} A = 1/2\overline{x} \\ B = 1/2 \end{array}\right\} \implies \mathbb{P}^+(x) = \frac{\overline{x} + x}{2\overline{x}} = \frac{1}{2} + \frac{x}{2\overline{x}}. \tag{A.244}$$

The unconditional probability of a negative adjustment is:

$$\mathbb{E}[\mathbb{P}^+] = \frac{1}{2} + \frac{1}{2\overline{x}} \mathbb{E}[x] = \frac{1}{2}. \tag{A.245}$$

The probability of a negative adjustment conditional on the last adjusting being positive (a switch in adjustment sign) equals:

$$\mathbb{P}^+(-x^*) \equiv \Pr[\Delta x < 0 | -x^*] = \frac{\overline{x} - x^*}{2\overline{x}}. \tag{A.246}$$

Probability of positive adjustment. Let $\mathbb{P}^-(x) \equiv \Pr[\Delta x > 0 | x]$ denote the probability of doing a positive adjustment (after hitting the lower bound) conditional on the state x. It solves the HJB with border conditions:

$$0 = \mathbb{P}^{-\prime\prime}(x); \quad \mathbb{P}^-(-\overline{x}) = 1; \quad \mathbb{P}^-(\overline{x}) = 0. \tag{A.247}$$

Solving for $\mathbb{P}^-(x) = Ax + B$ and evaluating at the border conditions:

$$\left.\begin{array}{r} -A\overline{x} + B = 1 \\ A\overline{x} + B = 0 \end{array}\right\} \quad \left.\begin{array}{r} A = -1/2\overline{x} \\ B = 1/2 \end{array}\right\} \quad \mathbb{P}^-(x) = \frac{\overline{x} - x}{2\overline{x}} = \frac{1}{2} - \frac{x}{2\overline{x}}. \tag{A.248}$$

The unconditional probability of a positive adjustment is:

$$\mathbb{E}[\mathbb{P}^-] = \frac{1}{2} - \frac{1}{2\overline{x}} \mathbb{E}[x] = \frac{1}{2}. \tag{A.249}$$

The probability of a positive adjustment conditional on the last adjusting being negative (a switch in adjustment sign) equals:

$$\mathbb{P}^-(x^*) \equiv \Pr[\Delta x > 0 | x^*] \;=\; \frac{\overline{x} - x^*}{2\overline{x}}. \tag{A.250}$$

A.9.4 Expected duration of inaction

Let $T(x) \equiv \mathbb{E}[\tau|x]$. It solves the HJB with border conditions:

$$0 \;=\; 1 + \frac{\sigma^2}{2} T''(x), \qquad T(\overline{x}) \;=\; T(-\overline{x}) \;=\; 0. \tag{A.251}$$

Solving for $T(x)$:

$$T''(x) \;=\; -\frac{2}{\sigma^2}, \quad T'(x) \;=\; -\frac{2}{\sigma^2}x + A, \quad T(x) \;=\; -\frac{x^2}{\sigma^2} + Ax + B. \tag{A.252}$$

Evaluating at the border conditions, we obtain values for A and B:

$$\left.\begin{array}{r} -\dfrac{\overline{x}^2}{\sigma^2} + A\overline{x} + B = 0 \\ -\dfrac{\overline{x}^2}{\sigma^2} - A\overline{x} + B = 0 \end{array}\right\} \implies \left.\begin{array}{r} 2A\overline{x} = 0 \\ -\dfrac{2\overline{x}^2}{\sigma^2} + 2B = 0 \end{array}\right\} \implies \left.\begin{array}{r} A = 0 \\ B = \dfrac{\overline{x}^2}{\sigma^2} \end{array}\right\} \implies T(x) = \frac{\overline{x}^2 - x^2}{\sigma^2}. \tag{A.253}$$

The expected duration of inaction given the current state $\mathbb{E}[\tau|x]$, the expected duration of a complete inaction spell conditional on the last reset point $(\overline{\mathbb{E}}^+[\tau], \overline{\mathbb{E}}^-[\tau])$, and the unconditional expected duration of inaction $\overline{\mathbb{E}}[\tau]$ are given by:

$$\mathbb{E}[\tau|x] \;=\; \frac{\overline{x}^2 - x^2}{\sigma^2}, \tag{A.254}$$

$$\overline{\mathbb{E}}^+[\tau] \;=\; \overline{\mathbb{E}}^-[\tau] = \frac{\overline{x}^2 - x^{*2}}{\sigma^2}, \tag{A.255}$$

$$\overline{\mathbb{E}}[\tau] \;=\; \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau] + \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau] = \frac{\overline{x}^2 - x^{*2}}{\sigma^2}, \tag{A.256}$$

where the shares of upward and downward adjustment are identical: $\mathcal{N}^+/\mathcal{N} = \mathcal{N}^-/\mathcal{N} = 1/2$.

A.9.5 Cross-sectional means

Let $m(x) \equiv \mathbb{E}\left[\int_0^\tau x_s \,\mathrm{d}s | x_0 = x\right]$. It solves the HJB with border conditions:

$$0 = x + \frac{\sigma^2}{2} m''(x), \quad m(\overline{x}) = m(-\overline{x}) = 0. \tag{A.257}$$

Solving for $m(x)$:

$$m''(x) = -\frac{2}{\sigma^2}x, \qquad m'(x) = -\frac{x^2}{\sigma^2} + A, \qquad m(x) = -\frac{x^3}{3\sigma^2} + Ax + B. \tag{A.258}$$

Evaluating at the border conditions, we obtain values for A and B:

$$\left.\begin{array}{r} -\dfrac{\overline{x}^3}{3\sigma^2} + A\overline{x} + B = 0 \\ \dfrac{\overline{x}^3}{3\sigma^2} - A\overline{x} + B = 0 \end{array}\right\} \implies \left.\begin{array}{r} A = \dfrac{\overline{x}^2}{3\sigma^2} \\ B = 0 \end{array}\right\} \implies m(x) = \frac{\overline{x}^2 x - x^3}{3\sigma^2} = \frac{x}{3}\frac{\overline{x}^2 - x^2}{\sigma^2} = \frac{x}{3}\mathbb{E}[\tau|x] \tag{A.259}$$

Unconditional means. Using the occupancy measure, we obtain the means conditional on the last rest point:

$$\overline{\mathbb{E}}^-[x] \;=\; \frac{m(-x^*)}{\overline{\mathbb{E}}^-[\tau]} \;=\; -\frac{x^*}{3}; \qquad \overline{\mathbb{E}}^+[x] \;=\; \frac{m(x^*)}{\overline{\mathbb{E}}^+[\tau]} \;=\; \frac{x^*}{3}, \tag{A.260}$$

where $\overline{\mathbb{E}}^{-}[\tau] = \mathbb{E}[\tau|-x^*]$ and $\overline{\mathbb{E}}^{+}[\tau] = \mathbb{E}[\tau|x^*]$.

Unconditional mean. By symmetry, $\mathbb{E}[x] = 0$. To show this formally, we use the conditional means and the renewal distribution:

$$\mathbb{E}[x] \;=\; \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^{+}[x] \;+\; \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^{-}[x] \;=\; \frac{1}{2}\left(\frac{x^*}{3}\right) + \frac{1}{2}\left(\frac{-x^*}{3}\right) \;=\; 0. \tag{A.261}$$

A.9.6 Cross-sectional variances

Unconditional variance. Since $\mathbb{E}[x] = 0$, then $\mathbb{V}ar[x] = \mathbb{E}[x^2]$. Using the cross-sectional distribution, the second moment equals:

$$\begin{aligned}
\mathbb{V}ar[x] &= \int_{-\overline{x}}^{\overline{x}} x^2 g(x)\,\mathrm{d}x \\
&= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\int_{-\overline{x}}^{-x^*} x^2(\overline{x}+x)\,\mathrm{d}x \;+\; (\overline{x}-x^*)\int_{-x^*}^{x^*} x^2\,\mathrm{d}x \;+\; \int_{x^*}^{\overline{x}} x^2(\overline{x}-x)\,\mathrm{d}x\right] \\
&= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\left(\frac{x^3\overline{x}}{3} + \frac{x^4}{4}\right)\Big|_{-\overline{x}}^{-x^*} \;+\; (\overline{x}-x^*)\frac{x^3}{3}\Big|_{-x^*}^{x^*} \;+\; \left(\frac{x^3\overline{x}}{3} - \frac{x^4}{4}\right)\Big|_{x^*}^{\overline{x}}\right] \\
&= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\frac{-x^{*3}\overline{x}}{3} + \frac{x^{*4}}{4} + \frac{\overline{x}^{*4}}{3} - \frac{\overline{x}^4}{4} + (\overline{x}-x^*)\frac{x^{*3}+x^{*3}}{3} + \frac{\overline{x}^4}{3} - \frac{\overline{x}^4}{4} - \frac{x^{*3}\overline{x}}{3} + \frac{x^{*4}}{4}\right] \\
&= \frac{1}{(\overline{x}^2 - x^{*2})}\left(\frac{\overline{x}^4 - x^{*4}}{6}\right) = \frac{1}{(\overline{x}^2 - x^{*2})}\left(\frac{(\overline{x}^2 - x^{*2})(\overline{x}^2 + x^{*2})}{6}\right) \\
&= \frac{\overline{x}^2 + x^{*2}}{6}
\end{aligned} \tag{A.262}$$

Conditional variance. Let $m_2(x) \equiv \mathbb{E}\left[\int_0^\tau x_s^2\,\mathrm{d}s | x_0 = x\right]$. It solves the HJB with borders:

$$0 = x^2 + \frac{\sigma^2}{2}m''(x), \quad m_2(\overline{x}) = m_2(-\overline{x}) = 0. \tag{A.263}$$

Solving for $m_2(x)$:

$$m_2''(x) = -\frac{2x^2}{\sigma^2}, \qquad m_2'(x) = -\frac{2x^3}{3\sigma^2} + A, \qquad m(x) = -\frac{x^4}{6\sigma^2} + Ax + B. \tag{A.264}$$

Evaluating at the border conditions, we obtain values for A and B:

$$\left.\begin{array}{r} -\dfrac{\overline{x}^4}{6\sigma^2} + A\overline{x} + B = 0 \\ -\dfrac{\overline{x}^4}{6\sigma^2} - A\overline{x} + B = 0 \end{array}\right\} \implies \left.\begin{array}{r} A = 0 \\ B = \dfrac{\overline{x}^4}{6\sigma^2} \end{array}\right\} \implies m_2(x) = \frac{\overline{x}^4 - x^4}{6\sigma^2} = \frac{\overline{x}^2 + x^2}{6}\mathbb{E}[\tau|x]. \tag{A.265}$$

Using the occupancy measure, the second moments conditional on the last rest point are:

$$\overline{\mathbb{E}}^{+}[x^2] \;=\; \frac{m_2(x^*)}{\mathbb{E}[\tau|x^*]} \;=\; \frac{\overline{x}^2 + x^{*2}}{6}; \qquad \overline{\mathbb{E}}^{-}[x^2] \;=\; \frac{m_2(-x^*)}{\mathbb{E}[\tau|-x^*]} \;=\; \frac{\overline{x}^2 + x^{*2}}{6}. \tag{A.266}$$

Finally, by definition of the variance, we have that:

$$\overline{\mathbb{V}ar}^{+}[x] \;=\; \overline{\mathbb{E}}^{+}[x^2] - (\overline{\mathbb{E}}^{+}[x])^2 \;=\; \frac{\overline{x}^2 + x^{*2}}{6} - \frac{x^{*2}}{9} \tag{A.267}$$

$$\overline{\mathbb{V}ar}^{-}[x] \;=\; \overline{\mathbb{E}}^{-}[x^2] - (\overline{\mathbb{E}}^{-}[x])^2 \;=\; \frac{\overline{x}^2 + x^{*2}}{6} - \frac{x^{*2}}{9} \tag{A.268}$$

where the conditional means are computed in (A.260).

Variance decomposition. According to the law of total variance:

$$\underbrace{\mathbb{V}ar[x]}_{total} = \underbrace{\mathbb{E}[\mathbb{V}ar[x|\Delta x]]}_{within} + \underbrace{\mathbb{V}ar[\mathbb{E}[x|\Delta x]]}_{between}. \tag{A.269}$$

The *within variance* is computed as:

$$\begin{aligned} \mathbb{E}[\mathbb{V}ar[x|\Delta x]] &= \frac{\mathcal{N}^+}{\mathcal{N}} \frac{\overline{\mathbb{E}}^+[\tau]}{\overline{\mathbb{E}}[\tau]} \mathbb{V}ar^+[x] + \frac{\mathcal{N}^-}{\mathcal{N}} \frac{\overline{\mathbb{E}}^-[\tau]}{\overline{\mathbb{E}}[\tau]} \mathbb{V}ar^-[x] \\ &= 2\left(\frac{1}{2}\right)(1)\left(\frac{\overline{x}^2 + x^{*2}}{6} - \frac{x^{*2}}{9}\right) = \frac{\overline{x}^2 + x^{*2}}{6} - \frac{x^{*2}}{9}. \end{aligned} \tag{A.270}$$

The *between variance* is computed as:

$$\begin{aligned} \overline{\mathbb{V}ar}[\mathbb{E}[x|\Delta x]] &= \frac{\mathcal{N}^+}{\mathcal{N}} \frac{\overline{\mathbb{E}}^+[\tau]}{\overline{\mathbb{E}}[\tau]} (\mathbb{E}^+[x] - \mathbb{E}[x])^2 + \frac{\mathcal{N}^-}{\mathcal{N}} \frac{\overline{\mathbb{E}}^-[\tau]}{\overline{\mathbb{E}}[\tau]} (\mathbb{E}^-[x] - \mathbb{E}[x])^2 \\ &= \frac{1}{2}(1)\left(\frac{x^*}{3}\right)^2 + \frac{1}{2}(1)\left(-\frac{x^*}{3}\right)^2 = \frac{x^{*2}}{9}. \end{aligned} \tag{A.271}$$

In the benchmark cases we decompose the total variance as:

$$\mathbb{V}ar[x] = \begin{cases} \underbrace{\frac{\overline{x}^2}{6}}_{total} = \underbrace{\frac{\overline{x}^2}{6}}_{within} + \underbrace{0}_{between} & \text{if} \quad \tilde{p} = 0 \\ \underbrace{\frac{\overline{x}^2}{3}}_{total} = \underbrace{\frac{2\overline{x}^2}{9}}_{within} + \underbrace{\frac{\overline{x}^2}{9}}_{between} & \text{if} \quad \tilde{\theta} = 0. \end{cases} \tag{A.272}$$

A.9.7 Aggregate q.

From (32) and (59) aggregate q with taxes and without drift equals:

$$q = \frac{1}{\tilde{r}} \left[\frac{1-t^c}{1-t^d} \frac{\alpha A \hat{Y}}{p \hat{K}} + \frac{\sigma^2}{2} - \frac{\overline{\mathbb{C}ov}\left[\Delta \hat{k}, \mathcal{P}(\Delta \hat{k})\right]}{\overline{\mathbb{E}}[\tau]} \right]. \tag{A.273}$$

We first compute the aggregate productivity term and the irreversibility terms.

Aggregate productivity. The numbers refer to the steps below. In equality (1) we start from the approximation of $\hat{Y}/\hat{K}$; in equality (2) we substitute $\mathbb{E}[\hat{k}] = \hat{k}^{ss} + \mathbb{E}[x] = \frac{1}{1-\alpha} \log\left(\frac{1-t^c}{1-t^d} \frac{\alpha A}{p \hat{\mathcal{U}}}\right) + \mathbb{E}[x]$ and $\mathbb{V}ar[\hat{k}] = \mathbb{V}ar[x]$; in equality (3) we simplify, in equality (4) we do a Taylor approximation to the exponential function, and in equality (5) we substitute $\mathbb{E}[x]=0$:

$$\begin{aligned}
\frac{\hat{Y}}{\hat{K}} &=^{(1)} \exp\left\{-(1-\alpha)\left(\mathbb{E}[\hat{k}]+\frac{\alpha}{2}\mathbb{V}ar[\hat{k}]\right)\right\} \\
&=^{(2)} \exp\left\{-(1-\alpha)\left(\frac{1}{1-\alpha}\log\left(\frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{\mathcal{U}}}\right)+\mathbb{E}[x]+\frac{\alpha}{2}\mathbb{V}ar[x]\right)\right\} \\
&=^{(3)} \frac{p\tilde{\mathcal{U}}}{\alpha A}\frac{1-t^d}{1-t^c}\exp\left\{-(1-\alpha)\mathbb{E}[x]-\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right\} \\
&=^{(4)} \frac{p\tilde{\mathcal{U}}}{\alpha A}\frac{1-t^d}{1-t^c}\left(1-(1-\alpha)\mathbb{E}[x]-\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right) \\
&=^{(5)} \frac{p\tilde{\mathcal{U}}}{\alpha A}\frac{1-t^d}{1-t^c}\left(1-\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right).
\end{aligned} \tag{A.274}$$

Irreversibility term. The numerator in the irreversibility term equals the covariance of adjustment size and the auxiliary pricing function $\mathcal{P}(\Delta x)$. We compute this term using conditioning on the last reset point and then averaging with the conditional renewal distribution:

$$\begin{aligned}
\overline{\mathbb{C}ov}\left[\Delta x,\mathcal{P}(\Delta x)\right] &= \overline{\mathbb{E}}\left[\Delta x\mathcal{P}(\Delta x)\right]-\overline{\mathbb{E}}\left[\Delta x\right]\overline{\mathbb{E}}\left[\mathcal{P}(\Delta x)\right] \\
&= \frac{1}{2}\left(\overline{\mathbb{E}}^{-}\left[\Delta x\mathcal{P}(\Delta x)\right]+\overline{\mathbb{E}}^{+}\left[\Delta x\mathcal{P}(\Delta x)\right]\right) \\
&= \frac{1}{2}\left((\overline{x}-x^*)\left(\frac{p^{buy}}{p}-1\right)+(x^*-\overline{x})\left(\frac{p^{sell}}{p}-1\right)\right) \\
&= \frac{(\overline{x}-x^*)}{2}\left[\frac{p^{buy}-p^{sell}}{p}\right] \\
&= (\overline{x}-x^*)\frac{\tilde{\mathcal{U}}}{\alpha}\tilde{p}.
\end{aligned}$$

In the last line we use the relationship between effective and fundamental price wedge in (A.350) and the assumption of a symmetric wedge:

$$\frac{p^{buy}-p^{sell}}{p}=\frac{\tilde{\mathcal{U}}}{\alpha}(\tilde{p}^{buy}-\tilde{p}^{sell})=\frac{2\tilde{\mathcal{U}}\tilde{p}}{\alpha}. \tag{A.275}$$

The denominator of the irreversibility term is the expected duration of inaction in (A.256): $\overline{\mathbb{E}}[\tau]=(\overline{x}^2-x^{*2})/\sigma^2$. Therefore, the irreversibility term for aggregate q equals:

$$-\frac{\overline{\mathbb{C}ov}\left[\Delta x,\mathcal{P}(\Delta x)\right]}{\overline{\mathbb{E}}[\tau]} = -\frac{(\overline{x}-x^*)\tilde{\mathcal{U}}\tilde{p}}{\alpha}\frac{\sigma^2}{\overline{x}^2-x^{*2}} = -\frac{\tilde{\mathcal{U}}}{\alpha}\frac{\tilde{p}\sigma^2}{\overline{x}+x^*}=-\frac{\overline{x}x^*}{3}\tilde{\mathcal{U}}(1-\alpha)<0. \tag{A.276}$$

where the last equality uses the first condition for the optimal policy in (A.223). Finally, substituting the aggregate productivity term (A.274) and the irreversibility term (A.276) into the expression for q (without drift), simplifying, and using the driftless aftertax user cost $\tilde{\mathcal{U}} = \tilde{\rho} - \sigma^2$ and the driftless after-tax discount $\tilde{r} = \tilde{\rho} - \sigma^2/2$:

$$\begin{aligned}
q &= \frac{1}{\tilde{r}}\left[\frac{1-t^c}{1-t^d}\frac{\alpha A}{p}\frac{\hat{Y}}{\hat{K}} + \frac{\sigma^2}{2} - \frac{\overline{x}x^*}{3}\tilde{\mathcal{U}}(1-\alpha)\right] \\
&= \frac{1}{\tilde{r}}\left[\frac{1-t^c}{1-t^d}\frac{\alpha A}{p}\frac{1-t^d}{1-t^c}\frac{p\tilde{\mathcal{U}}}{\alpha A}\left(1 - \frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right) + \frac{\sigma^2}{2} - \frac{\overline{x}x^*}{3}\tilde{\mathcal{U}}(1-\alpha)\right] \\
&= \frac{1}{\tilde{r}}\left[\tilde{\mathcal{U}}\left(1 - \frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right) + \frac{\sigma^2}{2} - \frac{\overline{x}x^*}{3}\tilde{\mathcal{U}}(1-\alpha)\right] \\
&= \frac{\tilde{\mathcal{U}}+\sigma^2/2}{\tilde{r}} - \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x] - \frac{\overline{x}x^*}{3\tilde{r}}\tilde{\mathcal{U}}(1-\alpha) \\
&= 1 - \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\left[\mathbb{V}ar[x] + \frac{2}{\alpha}\frac{\overline{x}x^*}{3}\right].
\end{aligned}$$

In the benchmark cases:

$$q = \begin{cases} 1 - \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x] & \text{if } \tilde{p} = 0, \\ 1 - \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\left(1+\frac{2}{\alpha}\right)\mathbb{V}ar[x] & \text{if } \tilde{\theta} = 0, \end{cases} \tag{A.277}$$

where for the case $\tilde{\theta} = 0$ we note that $\mathbb{V}ar[x] = \overline{x}^{*2}/3$.

A.9.8 CIR

From (38) and (62), the CIR without drift equals:

$$\frac{\text{CIR}(\delta)}{\delta} = \frac{\mathbb{V}ar[x]}{\sigma^2} - \frac{\overline{\mathbb{C}ov}[\Delta\hat{k}, \mathcal{M}(\Delta\hat{k})]}{\overline{\mathbb{E}}[\tau]} + o(\delta). \tag{A.278}$$

Cumulative deviations. Recall the values for the unconditional probabilities of a negative and a positive adjustment $\mathbb{E}[\mathbb{P}^+] = \mathbb{E}[\mathbb{P}^-] = 1/2$ in (A.245) and (A.249), and the conditional probabilities of switching adjustment sign $\mathbb{P}^+(-x^*) = \mathbb{P}^-(x^*) = (\overline{x} - x^*)/2\overline{x}$ in (A.246) and (A.250). Substituting these probabilities, the conditional means $\mathbb{E}^-[x] = -x^*/3$ and $\mathbb{E}^-[x] = x^*/3$ in (A.260), and the conditional durations $\mathbb{E}^-[\tau] = \mathbb{E}^+[\tau] = (\overline{x}^2 - x^{*2})/\sigma^2$ in (A.255) into the definition of cumulative deviations $\mathcal{M}^{buy}$ in (60) and $\mathcal{M}^{sell}$ in (61) yields:

$$\mathcal{M}^{buy} = \mathbb{E}[\mathbb{P}^-]\frac{1}{\mathbb{P}^+(-x^*)}(\mathbb{E}^-[x] - \mathbb{E}[x])\mathbb{E}^-[\tau] = \frac{1}{2}\left(\frac{2\overline{x}}{\overline{x}-x^*}\right)\left(-\frac{x^*}{3} - 0\right)\frac{\overline{x}^2 - x^{*2}}{\sigma^2} = -\frac{x^*\overline{x}(\overline{x}+x^*)}{3\sigma^2}, \tag{A.279}$$

$$\mathcal{M}^{sell} = \mathbb{E}[\mathbb{P}^+]\frac{1}{\mathbb{P}^-(x^*)}(\mathbb{E}^+[x] - \mathbb{E}[x])\mathbb{E}^+[\tau] = \frac{1}{2}\left(\frac{2\overline{x}}{\overline{x}-x^*}\right)\left(\frac{x^*}{3} - 0\right)\frac{\overline{x}^2 - x^{*2}}{\sigma^2} = \frac{x^*\overline{x}(\overline{x}+x^*)}{3\sigma^2}. \tag{A.280}$$

Irreversibility term. The irreversibility term for the CIR equals the covariance of the adjustment size and the auxiliary capital-deviation deviation function $\mathcal{M}(\Delta x)$ defined in (39). Recall $x^* = \overline{x} - \Delta x$ for $\Delta x < 0$ and $-x^* = -\overline{x} - \Delta x$ for $\Delta x > 0$ and by symmetry $\overline{\mathbb{E}}[\Delta x] = 0$. The numerator of the irreversibility term equals:

$$\begin{aligned}
\overline{\mathbb{C}ov}[\Delta x, \mathcal{M}(\Delta x)] &= \overline{\mathbb{E}}[\Delta x\mathcal{M}(\Delta x)] - \overline{\mathbb{E}}[\Delta x]\,\overline{\mathbb{E}}[\mathcal{M}(\Delta x)] \\
&= \frac{1}{2}\left[\overline{\mathbb{E}}^-[\Delta x\mathcal{M}(\Delta x)] + \overline{\mathbb{E}}^+[\Delta x\mathcal{M}(\Delta x)]\right] \\
&= \frac{1}{2}\left[(\overline{x} - x^*)\mathcal{M}^{buy} + (x^* - \overline{x})\mathcal{M}^{sell}\right] = (\overline{x} - x^*)\mathcal{M}^{buy}, \\
&= -(\overline{x} - x^*)\frac{x^*\overline{x}(\overline{x}+x^*)}{3\sigma^2} = -\frac{x^*\overline{x}}{3}\left(\frac{\overline{x}^2 - x^{*2}}{\sigma^2}\right) \\
&= -\frac{x^*\overline{x}}{3}\mathbb{E}[\tau].
\end{aligned}$$

Therefore, the irreversibility term of the CIR equals:

$$-\frac{\overline{\mathbb{C}ov}[\Delta x, \mathcal{M}(\Delta x)]}{\overline{\mathbb{E}}[\tau]} \;=\; \frac{x^*\overline{x}}{3} > 0. \tag{A.281}$$

Finally, substituting the expression for the cross-sectional variance in (A.262) and the irreversibility term in (A.281) into the CIR yields:

$$\frac{\text{CIR}(\delta)}{\delta} \;=\; \frac{\overline{x}^2 + x^{*2}}{6\sigma^2} \;+\; \frac{x^*\overline{x}}{3} \;+\; o(\delta). \tag{A.282}$$

In the benchmark cases:

$$\text{CIR} \;=\; \begin{cases} \frac{\overline{x}^2}{6\sigma^2} \;+\; o(\delta) & \text{if} \quad \tilde{p} = 0, \\ \\ \frac{(1+\sigma^2)\overline{x}^{*2}}{3\sigma^2} \;+\; o(\delta) & \text{if} \quad \tilde{\theta} = 0, \end{cases} \tag{A.283}$$

A.10 Proof of Proposition 5

This proposition computes the optimal investment policy and the macro outcomes for models with very large drift relative to idiosyncratic costs. Assume $\nu \to \infty$. If $\nu \to \infty$ then economy converges to an economy with $\nu > 0$ and $\sigma \to 0$ since idiosyncratic shocks are small relative to the drift—see Álvarez, Beraja, Gonzalez-Rozada and Neumeyer (2018) for details in the context of price-setting. The firm investment problem is given by:

$$\mathcal{V}(x) \;=\; \max_{\tau,\Delta x} \; \mathbb{E}\left[\int_0^\tau e^{-\tilde{r}\tau}(x_s^\alpha - \alpha x_s)\,\mathrm{d}s + e^{-\tilde{r}\tau}\left(-\tilde{\theta} \;+\; \mathcal{V}(x_\tau)\right)\Big| x_0 = x\right], \tag{A.284}$$

$$\mathrm{d}x_s \;=\; -\nu x_s\,\mathrm{d}s. \tag{A.285}$$

A.10.1 Sufficient optimality conditions

This problem was studied by Sheshinski and Weiss (1977). They show that the optimal policy consists of a one-sided inaction region with lower threshold x^- and a reset point x^*. Since there are no idiosyncratic shock, there is no mass above x^* and the cross-sectional distribution is uniform in the range $[x^-, x^*]$.

Let $h(x) \equiv x^\alpha - \alpha x$. Following Sheshinski and Weiss (1977), the optimal policy satisfies the following conditions (in our notation):

$$h(x^*) \;-\; h(x^-) \;=\; \tilde{r}\tilde{\theta}, \tag{A.286}$$

$$\int_{e^{x^-}}^{e^{x^*}} h'(x)\; x^{\frac{\tilde{r}}{\nu}}\,\mathrm{d}x \;=\; 0. \tag{A.287}$$

The first optimality condition equals:

$$e^{\alpha x^*} - \alpha e^{x^*} = \tilde{r}\tilde{\theta} + e^{\alpha x^-} - \alpha e^{x^-}. \tag{A.288}$$

The second optimality condition equals:

$$\int_{e^{x^-}}^{e^{x^*}} \alpha(x^{\alpha-1} - 1)x^{\frac{\tilde{r}}{\nu}}\,\mathrm{d}x \;=\; \frac{x^{\tilde{r}/\nu} + \alpha +}{\tilde{r}/\nu + \alpha +} - \frac{x^{\tilde{r}/\nu} + 1}{r/\nu + 1}\Bigg|_{e^{x^-}}^{e^{x^*}} = 0, \tag{A.289}$$

or equivalently:

$$\frac{e^{(\tilde{r}/\nu+\alpha)x^*} - e^{(\tilde{r}/\nu+\alpha)x^-}}{r/\nu + \alpha} \;=\; \frac{e^{(\tilde{r}/\nu+1)x^*} - e^{(\tilde{r}/\nu+1)x^-}}{r/\nu + 1}. \tag{A.290}$$

A.10.2 Optimal investment policy

Next we approximate the decision problem. As in (A.210), we do a second-order Taylor approximation of the flow profits $\pi(x) = e^{\alpha x} - \alpha e^{x}$ around $x = 0$: $\pi(x) = (1-\alpha) - \frac{\alpha(1-\alpha)}{2}x^2$. Applying this approximation to the first optimality condition in (A.288) yields:

$$x^{*2} - x^{-2} = -\frac{2\tilde{r}\tilde{\theta}}{\alpha(1-\alpha)}. \tag{A.291}$$

Since the cross-sectional distribution is uniform in the range $[x^-, x^*]$, it has the following moments:

$$\mathbb{V}ar[x] = \frac{(x^* - x^-)^2}{12}; \qquad \mathbb{E}[x] = \frac{x^* + x^-}{2}. \tag{A.292}$$

Thus we can write the first optimality condition in (A.291) as:

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} = -\frac{\tilde{r}\tilde{\theta}}{2\sqrt{3}\alpha(1-\alpha)}. \tag{A.293}$$

Next, consider a third-order approximation to the following exponential function:

$$\frac{e^{\beta x}}{\beta} = 1 + x + \frac{\beta}{2}x^2 + \frac{\beta^2}{6}x^2. \tag{A.294}$$

Apply this approximation to both sides of the second optimality condition, using $\beta \in \{\tilde{r}/\nu + \alpha, \tilde{r}/\nu + 1\}$, and rearrange to obtain:

$$\left[\frac{(\tilde{r}/\nu + \alpha) - (\tilde{r}/\nu + 1)}{2}\right](x^{*2} - x^{-2}) + \left[\frac{(\tilde{r}/\nu + \alpha)^2 - (\tilde{r}/\nu + 1)^2}{6}\right](x^{*3} - x^{-3}) = 0 \tag{A.295}$$

$$\frac{(x^* + x^-)}{2} + \frac{(2\tilde{r}/\nu + \alpha + 1)}{6}((x^* + x^-)^2 - x^* x^-) = 0. \tag{A.296}$$

Note the following relationship:

$$-x^* x^- = \frac{(x^* - x^-)^2 - (x^* + x^-)^2}{4} \tag{A.297}$$

Substituting back this expression into (A.296) and also $\mathbb{E}[x] = (x^* + x^-)/2$ we get:

$$\mathbb{E}[x] + \frac{(2\tilde{r}/\nu + \alpha + 1)}{6}\left((x^* + x^-)^2 + \frac{(x^* - x^-)^2 - (x^* + x^-)^2}{4}\right) = 0 \tag{A.298}$$

$$6\mathbb{E}[x] + (2\tilde{r}/\nu + \alpha + 1)\left(\frac{3(x^* + x^-)^2 + (x^* - x^-)^2}{4}\right) = 0 \tag{A.299}$$

$$\mathbb{E}[x] = -\left(\frac{\tilde{r}}{\nu} + \frac{\alpha + 1}{2}\right)\mathbb{E}[x^2]. \tag{A.300}$$

Therefore, together with (A.292), the optimal policy (x^-, x^*) is characterized by the following 2×2 non-linear system:

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} = -\frac{\tilde{r}\tilde{\theta}}{\sqrt{12}\alpha(1-\alpha)}; \qquad \frac{\mathbb{E}[x]}{\mathbb{V}ar[x] + \mathbb{E}[x]^2} = -\left(\frac{\tilde{r}}{\nu} + \frac{\alpha + 1}{2}\right). \tag{A.301}$$

A.10.3 Macro outcomes

(i) Misallocation. The cross-sectional variance is the variance of a uniform distribution in the range $[x^-, x^*]$:

$$\mathbb{V}ar[x] = \frac{(x^* - x^-)^2}{12}. \tag{A.302}$$

(ii) Aggregate q. Let $\tilde{\rho} \equiv \rho(1-t^p)/(1-t^g)$. Without idiosyncratic shocks, the after-tax user cost is $\tilde{\mathcal{U}} = \tilde{\rho} + \xi^k$ and the after-tax discount is $\tilde{r} = \tilde{\rho} - \mu$ and simplifying:

$$
\begin{aligned}
q &= \frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{r}}\frac{\hat{Y}}{\hat{K}} - \frac{\nu}{\tilde{r}} \\
&= \frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{r}}\frac{1-t^d}{1-t^c}\frac{p\tilde{\mathcal{U}}}{\alpha A}\left(1 \; - (1-\alpha)\mathbb{E}[x] \; - \frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right) - \frac{\nu}{\tilde{r}} \\
&= 1 \; - \frac{\tilde{\rho}+\xi^k}{\tilde{\rho}-\mu}(1-\alpha)\left(\mathbb{E}[x] + \frac{\alpha}{2}\mathbb{V}ar[x]\right)
\end{aligned}
$$

(iii) CIR: From Corollary 2 in Baley and Blanco (2021):

$$
\frac{\text{CIR}(\delta)}{\delta} = 0. \tag{A.303}
$$

A.11 Illustrative example.

Consider an economy in which most firms make frequent upward adjustments. The durations of inaction are $\overline{\mathbb{E}}[\tau] = 2$, $\overline{\mathbb{E}}^{-}[\tau] = 1.5$, and $\overline{\mathbb{E}}^{+}[\tau] = 4$, and the frequencies are $\mathcal{N} = 0.5$, $\mathcal{N}^{-} = 0.4$, and $\mathcal{N}^{+} = 0.1$.[24] The shares of upward and downward adjustments are $\mathcal{N}^{-}/\mathcal{N} = 0.8$ and $\mathcal{N}^{+}/\mathcal{N} = 0.2$, and the relative durations are $\overline{\mathbb{E}}^{-}[\tau]/\overline{\mathbb{E}}[\tau] = 0.75$ and $\overline{\mathbb{E}}^{+}[\tau]/\overline{\mathbb{E}}[\tau] = 2$. While only 20% of adjustments are downward, they happen after longer inactions spells with twice the average duration, implying that the underlying states $\hat{k}$ generating those adjustments are occupied for longer periods of time. To account for this higher occupancy, the implied duration-modified frequencies, $\mathcal{N}^{-}\,\overline{\mathbb{E}}^{-}[\tau] = 0.6$ and $\mathcal{N}^{+}\,\overline{\mathbb{E}}^{+}[\tau] = 0.4$, are the appropriate weights to recover the unconditional distribution of firms as $g = 0.6\ g^{-} + 0.4\ g^{+}$. Thus the effective share of downward adjustments increases from 0.2 to 0.4.

A.12 Proof of Proposition 1

This proposition expresses the parameters of the stochastic process (ν, σ^2) as functions of the data. The proof follows the strategy in Baley and Blanco (2021), but taking into account the two reset points arising from partial irreversibility. Let $\hat{k}_s$ follow a Brownian motion with drift ν and volatility σ and two reset states $\hat{k}^{*-} < \hat{k}^{*+}$, where $\hat{k}^{*-} = \hat{k}^*(\Delta\hat{k} > 0)$ and $\hat{k}^{*+} = \hat{k}^*(\Delta\hat{k} < 0)$.

A.12.1 Drift

Evaluating the law of motion $\hat{k}_s = \hat{k}^*(\Delta\hat{k}) - \nu t + \sigma W_t$ at a stopping time $s = \tau$,

$$\hat{k}_\tau - \hat{k}^*(\Delta\hat{k}) + \nu\tau \;=\; \sigma W_\tau. \tag{A.304}$$

Since the stopped capital depends on the adjustment sign, we take conditional using the distribution of adjusters $H(\Delta\hat{k}, \tau)$:

$$\overline{\mathbb{E}}\left[\hat{k}_\tau \middle| \Delta\hat{k}\right] - \hat{k}^*(\Delta\hat{k}) + \nu\overline{\mathbb{E}}\left[\tau \middle| \Delta\hat{k}\right] \;=\; 0. \tag{A.305}$$

Taking expectation again with H and using the law of iterating expectation, we obtain

$$\underbrace{\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\hat{k}_\tau \middle| \Delta\hat{k}\right]\right]}_{\overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right] \;-\; \overline{\mathbb{E}}[\Delta\hat{k}]} \;-\; \overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right] \;+\; \nu\underbrace{\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\tau \middle| \Delta\hat{k}\right]\right]}_{\overline{\mathbb{E}}[\tau]} \;=\; 0 \tag{A.306}$$

To compute the first term, we substitute the relationship between stopped capital, reset state and adjustment size, $\hat{k}_\tau = \hat{k}^*(\Delta\hat{k}) - \Delta\hat{k}$, and then use the Markovian property of adjustments:

$$\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\hat{k}_\tau \middle| \Delta\hat{k}\right]\right] \;=\; \overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k}') \middle| \Delta\hat{k}\right]\right] - \overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\Delta\hat{k}' \middle| \Delta\hat{k}\right]\right] \;=\; \overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right] - \overline{\mathbb{E}}[\Delta\hat{k}]. \tag{A.307}$$

To compute the second term, we use the law of iterated expectations and obtain:

$$\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\tau' \middle| \Delta\hat{k}\right]\right] = \overline{\mathbb{E}}\left[\tau\right]. \tag{A.308}$$

Substituting (A.307) and (A.308) into (A.306) and rearranging, we obtain the result

$$\nu \;=\; \frac{\overline{\mathbb{E}}[\Delta\hat{k}]}{\overline{\mathbb{E}}[\tau]}. \tag{A.309}$$

[24] Note that $\overline{\mathbb{E}}[\tau] = 1/\mathcal{N}$ but $\overline{\mathbb{E}}^{\pm}[\tau] \neq 1/\mathcal{N}^{\pm}$.

A.12.2 Idiosyncratic volatility

Let $Y_s \equiv (\hat{k}_s + \nu s)^2$. Applying Itō's lemma to Y_s,

$$\text{(A.310)} \qquad \mathrm{d}Y_s \;=\; 2(\hat{k}_s + \nu s)(\mathrm{d}\hat{k}_s + \nu\,\mathrm{d}s) \;+\; (\mathrm{d}\hat{k}_s)^2 \;=\; 2(\hat{k}_s + \nu s)\sigma\,\mathrm{d}W_s + \sigma^2\,\mathrm{d}s$$

We integrate both sides from 0 to τ and take conditional expectations with respect to the initial condition $\hat{k}_0 = \hat{k}^*(\Delta\hat{k})$, which is a function of Δk. Then, we use the OST to set the martingale to zero, $\overline{\mathbb{E}}[\int_0^\tau (\hat{k}_s + \nu s)\,\mathrm{d}W_s] = 0$, and obtain:

$$\text{(A.311)} \qquad \overline{\mathbb{E}}\left[Y_\tau(\Delta k) - Y_0 \Big| \Delta\hat{k}\right] \;=\; 2\sigma\overline{\mathbb{E}}\left[\int_0^\tau (\hat{k}_s + \nu s)\,\mathrm{d}W_s \Big| \Delta k\right] + \sigma^2\overline{\mathbb{E}}\left[\int_0^\tau 1\,\mathrm{d}s \Big| \Delta k\right] \;=\; \sigma^2\overline{\mathbb{E}}[\tau|\Delta k]$$

Substituting $Y_\tau \equiv (\hat{k}_\tau(\Delta k) + \nu\tau)^2$ and $Y_0 \equiv \hat{k}^*(\Delta\hat{k})^2$, and taking expectations again to average across positive and negative adjustments, we get:

$$\text{(A.312)} \qquad \overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[(\hat{k}_\tau(\Delta\hat{k}') + \nu\tau')^2|\Delta\hat{k}\right]\right] - \mathbb{E}\left[\hat{k}^*(\Delta\hat{k}')^2 \Big| \Delta k\right] \;=\; \sigma^2\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\tau' \Big| \Delta k\right]\right].$$

By the Markovian property of Δk and τ, the previous expression simplifies to:

$$\text{(A.313)} \qquad \overline{\mathbb{E}}[(\hat{k}_\tau(\Delta\hat{k}) + \nu\tau)^2] - \overline{\mathbb{E}}[\hat{k}^*(\Delta\hat{k})^2] \;=\; \sigma^2\overline{\mathbb{E}}[\tau]$$

Rearranging, we obtain the mapping to σ^2:

$$\text{(A.314)} \qquad \sigma^2 \;=\; \frac{\overline{\mathbb{E}}[(\hat{k}_\tau(\Delta\hat{k}) + \nu\tau)^2] - \overline{\mathbb{E}}[\hat{k}^*(\Delta\hat{k})^2]}{\overline{\mathbb{E}}[\tau]}.$$

A.12.3 Reset states

To characterize the reset states as a function of the data, we first apply the envelope theorem.

Envelope theorem. Start from the recursive definition of $v(\hat{k})$ in (C.388) for an arbitrary initial condition $\hat{k}_0$ and evaluate it at the optimal policy $(\tau^*, \hat{k}^*)$

$$\text{(A.315)} \qquad v(\hat{k}_0) \;=\; \mathbb{E}\left[\int_0^{\tau^*} Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s \;+\; e^{-r\tau^*}\left(-\theta - p(\Delta\hat{k})(e^{\hat{k}^*} - e^{\hat{k}_{\tau^*}}) + v(\hat{k}^*)\right)\right]$$

Substitute the evolution of log capital-to-productivity ratios $\hat{k}_s = \hat{k}_0 - \nu s + \sigma W_s$ and take the derivative with respect to the initial state $\hat{k}_0$ (note that the terms $-\theta - p(\Delta\hat{k})e^{\hat{k}^*} + v(\hat{k}^*)$ are independent of $\hat{k}_0$) to get

$$\text{(A.316)} \qquad \begin{aligned} v'(\hat{k}_0) &= \mathbb{E}\left[\int_0^{\tau^*} \frac{\mathrm{d}Ae^{-rs+\alpha(\hat{k}_0-\nu s+\sigma W_s)}}{\mathrm{d}\hat{k}_0}\,\mathrm{d}s \quad + p(\Delta\hat{k})e^{-r\tau^*}\frac{\mathrm{d}e^{\hat{k}_0-\nu\tau^*+\sigma W_{\tau^*}}}{\mathrm{d}\hat{k}_0}\right] \\ &= \mathbb{E}\left[\int_0^{\tau^*} \alpha Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k})e^{-r\tau^*+\hat{k}_{\tau^*}}\right] \end{aligned}$$

In the second line apply the envelope condition for arbitrary choice sets (Milgrom and Segal, 2002) to ignore the derivative of the optimal policies with respect to the initial condition.

Because there are two reset points, at this step we must condition on the appropriate initial condition to evaluate (A.316). If the last reset point is $\hat{k}_0 = \hat{k}^{*-}$ (there was a capital purchase), then we use the optimality condition $v'(\hat{k}^{*-}) = p^{buy}e^{\hat{k}^*}$ in the LHS to get:

$$\text{(A.317)} \qquad p^{buy}e^{\hat{k}^*} \;=\; \mathbb{E}^-\left[\int_0^{\tau^*} \alpha Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k})e^{-r\tau^*+\hat{k}_{\tau^*}}\right].$$

Analogously, if the last reset point is $\hat{k}_0 = \hat{k}^{*+}$ (there was a capital sale), then we use the optimality condition

$v'(\hat{k}^{*+}) = p^{sell} e^{\hat{k}^*}$ in the LHS to get:

$$p^{sell} e^{\hat{k}^{*+}} \;=\; \mathbb{E}^{+}\left[\int_0^{\tau^*} \alpha A e^{-rs+\alpha \hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k})\; e^{-r\tau^*+\hat{k}_{\tau^*}}\right]. \tag{A.318}$$

Reset state without partial irreversibility. We depart from the optimality condition

$$p e^{\hat{k}^*} = \mathbb{E}\left[\int_0^{\tau} \alpha A e^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s + p e^{-r\tau+\hat{k}_\tau}\Big|\hat{k}_0 = \hat{k}^*\right] \tag{A.319}$$

Now we write the RHS as a function of observable data. Define $Y_s = e^{-rs+\alpha\hat{k}_s}$ and apply Ito's lemma:

$$\mathrm{d}Y_s \;=\; Y_s\left(-r\,\mathrm{d}s + \alpha\,\mathrm{d}\hat{k}_s + \frac{\alpha^2}{2}\,\mathrm{d}\hat{k}_s^2\right) \;=\; \left(\frac{\alpha^2\sigma^2}{2} - r - \alpha\nu\right) Y_s\,\mathrm{d}s + \alpha\sigma Y_s\,\mathrm{d}B_s. \tag{A.320}$$

Let $\phi \equiv r + \alpha\nu - \alpha^2\sigma^2/2$. Integrating both sides from 0 to τ, taking expectations conditional on adjustment, i.e., with respect to the initial condition $\hat{k}_0 = \hat{k}^*$, and using the OST to set the expectation of martingales to zero, we obtain:

$$\overline{\mathbb{E}}\left[\int_0^{\tau} \mathrm{d}Y_s\,\mathrm{d}s\right] \;=\; -\phi\overline{\mathbb{E}}\left[\int_0^{\tau} Y_s\,\mathrm{d}s\right] + \alpha\sigma\,\underbrace{\overline{\mathbb{E}}\left[\int_0^{\tau} Y_s\,\mathrm{d}W_s\right]}_{=0}. \tag{A.321}$$

Since $Y_\tau = e^{-r\tau+\alpha\hat{k}_\tau} = e^{-r\tau+\alpha(\hat{k}^*-\Delta\hat{k})}$ and $Y_0 = e^{\alpha\hat{k}^*}$

$$\overline{\mathbb{E}}\left[Y_\tau - Y_0\right] \;=\; -\phi\overline{\mathbb{E}}\left[\int_0^{\tau} e^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s\right] \iff e^{\alpha\hat{k}^*}\frac{1-\overline{\mathbb{E}}[e^{-r\tau-\alpha\Delta\hat{k}}]}{\phi} = \overline{\mathbb{E}}\left[\int_0^{\tau} Y_s\,\mathrm{d}s\right]. \tag{A.322}$$

Since $\hat{k}_\tau = \hat{k}^* - \Delta\hat{k}$

$$\mathbb{E}\left[p e^{-r\tau+\hat{k}_\tau}|\hat{k}_0 = \hat{k}^*\right] = p e^{\hat{k}^*}\,\mathbb{E}\left[e^{-r\tau-\Delta\hat{k}}|\hat{k}_0 = \hat{k}^*\right]. \tag{A.323}$$

From equations (A.319) to (A.323),

$$\begin{aligned} p e^{\hat{k}^*} &= \mathbb{E}\left[\int_0^{\tau} \alpha A e^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s + p e^{-r\tau+\hat{k}_\tau}|\hat{k}_0 = \hat{k}^*\right] \\ &= \alpha A e^{\alpha\hat{k}^*}\frac{1-\overline{\mathbb{E}}[e^{-r\tau-\alpha\Delta\hat{k}}]}{\phi} + p e^{\hat{k}^*}\,\overline{\mathbb{E}}\left[e^{-r\tau-\Delta\hat{k}}\right] \end{aligned} \tag{A.324}$$

Operating from the previous equation

$$\hat{k}^* = \frac{1}{1-\alpha}\left[\log\left(\frac{\alpha A}{r+\alpha\nu-\alpha^2\sigma^2/2}\right) - \log(p) + \log\left(\frac{1-\overline{\mathbb{E}}[e^{-r\tau-\alpha\Delta\hat{k}}]}{1-\overline{\mathbb{E}}\left[e^{-r\tau-\Delta\hat{k}}\right]}\right)\right] \tag{A.325}$$

Finally, letting $\Phi \equiv \log\left(\frac{\alpha A}{\phi}\right) = \log\left(\frac{\alpha A}{(r+\alpha\nu-\alpha^2\sigma^2/2)}\right)$, we obtain the result

$$\hat{k}^* = \frac{1}{1-\alpha}\left[\Phi - \log(p) + \log\left(\frac{1-\overline{\mathbb{E}}[e^{-r\tau-\alpha\Delta\hat{k}}]}{1-\overline{\mathbb{E}}\left[e^{-r\tau-\Delta\hat{k}}\right]}\right)\right]. \tag{A.326}$$

Reset states with partial irreversibility. We follow similar steps but taking into account that adjustments happen at two different reset points. Consider the optimality condition of a firm that has sold capital at price p^{sell}

and resetting to $\hat{k}^{*+}$

$$p^{\text{sell}}e^{\hat{k}^{*+}} = \alpha A\overline{\mathbb{E}}^{+}\left[\int_0^{\tau} e^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s\right] + \overline{\mathbb{E}}^{+}\left[p(\Delta\hat{k})e^{-r\tau+\hat{k}_\tau}\right] \tag{A.327}$$

$$p^{\text{sell}}e^{\hat{k}^{*+}} = e^{\Phi}e^{\alpha\hat{k}^{*+}}\left(1-\overline{\mathbb{E}}^{+}[e^{-r\tau+\alpha(\hat{k}_\tau-\hat{k}^{*+})}]\right) + p^{\text{sell}}e^{\hat{k}^{*+}}\overline{\mathbb{E}}^{+}\left[\frac{p(\Delta\hat{k})}{p^{\text{sell}}}e^{-r\tau+\hat{k}_\tau-\hat{k}^{*+}}\right]$$

$$e^{(1-\alpha)\hat{k}^{*+}} = \frac{e^{\Phi}}{p^{\text{sell}}}\frac{1-\overline{\mathbb{E}}^{+}[e^{-r\tau+\alpha(\hat{k}_\tau-\hat{k}^{*+})}]}{1-\mathbb{E}^{+}\left[\frac{p(\Delta\hat{k})}{p^{\text{sell}}}e^{-r\tau+\hat{k}_\tau-\hat{k}^{*+}}\right]}$$

Taking logs, we obtain the reset point for negative investments:

$$\hat{k}^{*+} = \frac{1}{1-\alpha}\left[\Phi-\log(p^{\text{sell}})+\log\left(\frac{1-\overline{\mathbb{E}}^{+}[e^{-r\tau+\alpha(\hat{k}_\tau-\hat{k}^{*+})}]}{1-\mathbb{E}^{+}\left[\frac{p(\Delta\hat{k})}{p^{\text{sell}}}e^{-r\tau+\hat{k}_\tau-\hat{k}^{*+}}\right]}\right)\right]. \tag{A.328}$$

With analogous steps we obtain the reset point for positive investments:

$$\hat{k}^{*-} = \frac{1}{1-\alpha}\left[\Phi-\log(p^{\text{buy}})+\log\left(\frac{1-\overline{\mathbb{E}}^{-}[e^{-r\tau+\alpha(\hat{k}_\tau-\hat{k}^{*-})}]}{1-\mathbb{E}^{-}\left[\frac{p(\Delta\hat{k})}{p^{\text{buy}}}e^{-r\tau+(\hat{k}_\tau-\hat{k}^{*-})}\right]}\right)\right]. \tag{A.329}$$

A.13 Proof of Proposition 2

This proposition expresses the cross-sectional moments of $\hat{k}$ and joint moments of $(\hat{k},a)$ as functions of the data. The proof follows the strategy in Baley and Blanco (2021), but taking into account the two reset points arising from partial irreversibility.

A.13.1 Moments of $\hat{k}$.

We apply Itō's lemma to $\hat{k}_s^n$ for $n\geq 2$:

$$\mathrm{d}\hat{k}_s^{n+1} = -\nu(n+1)\hat{k}_s^n\,\mathrm{d}s + \sigma(n+1)\hat{k}_s^n\,\mathrm{d}s + \frac{\sigma^2 n(n+1)}{2}\hat{k}_s^{n-1}\,\mathrm{d}s. \tag{A.330}$$

We integrate this expression from 0 to τ and take conditional expectations with respect to the initial condition $\hat{k}_0=\hat{k}^*$, which is a function of Δk:

$$\overline{\mathbb{E}}^{\pm}\left[\hat{k}_\tau^{n+1}-(\hat{k}^*)^{n+1}\right] = -\nu(n+1)\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau}\hat{k}_s^n\,\mathrm{d}s\right] + \frac{\sigma^2 n(n+1)}{2}\overline{\mathbb{E}}^{\pm}\left[\int_0^{\tau}\hat{k}_s^{n-1}\,\mathrm{d}s\right]. \tag{A.331}$$

Next, take law of iterated expectation to average $\overline{\mathbb{E}}^{-}$ and $\overline{\mathbb{E}}^{+}$, divide both sides by $\overline{\mathbb{E}}[\tau]$, and then use Auxiliary Theorem 2 to recover steady-state moments using the occupancy measure:

$$\frac{\overline{\mathbb{E}}\left[\hat{k}_\tau^{n+1}-(\hat{k}^*)^{n+1}\right]}{\overline{\mathbb{E}}[\tau]} = -\nu(n+1)\frac{\overline{\mathbb{E}}\left[\int_0^{\tau}\hat{k}_s^n\,\mathrm{d}s\right]}{\overline{\mathbb{E}}[\tau]} + \frac{\sigma^2 n(n+1)}{2}\frac{\overline{\mathbb{E}}\left[\int_0^{\tau}\hat{k}_s^{n-1}\,\mathrm{d}s\right]}{\overline{\mathbb{E}}[\tau]} \tag{A.332}$$

$$= -\nu(n+1)\mathbb{E}[\hat{k}^n] + \frac{\sigma^2 n(n+1)}{2}\mathbb{E}[\hat{k}^{n-1}]. \tag{A.333}$$

Solving for $\mathbb{E}[\hat{k}^n]$ and substituting $\nu=\overline{\mathbb{E}}[\Delta\hat{k}]/\overline{\mathbb{E}}[\tau]$:

$$\mathbb{E}[\hat{k}^n] = \frac{1}{n+1}\frac{\overline{\mathbb{E}}\left[(\hat{k}^*)^{n+1}-\hat{k}_\tau^{n+1}\right]}{\overline{\mathbb{E}}[\Delta\hat{k}]} + \frac{\sigma^2 n}{2\nu}\mathbb{E}[\hat{k}^{n-1}]. \tag{A.334}$$

Applying similar steps as before, we compute the centered moments:

$$\mathbb{E}[(\hat{k}-\mathbb{E}[\hat{k}])^n] \;=\; \frac{1}{n+1}\frac{\overline{\mathbb{E}}[(\hat{k}^*-\mathbb{E}[\hat{k}])^{n+1}]-\overline{\mathbb{E}}\left[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^{n+1}\right]}{\overline{\mathbb{E}}[\Delta\hat{k}]}+\frac{\sigma^2 n}{2\nu}\mathbb{E}[(\hat{k}-\mathbb{E}[\hat{k}])^{n-1}]. \tag{A.335}$$

Mean of $\hat{k}$. Evaluate expression (A.334) at $n=1$ and use the formula for the drift (A.309) to obtain:

$$\mathbb{E}[\hat{k}] \;=\; \frac{\overline{\mathbb{E}}\left[(\hat{k}^*)^2-\hat{k}_\tau^2\right]}{2\overline{\mathbb{E}}[\Delta\hat{k}]}+\frac{\sigma^2}{2\nu}. \tag{A.336}$$

Without a price wedge (only fixed costs), there is a unique reset point $\hat{k}^*$. Factorize the quadratic difference $(\hat{k}^*)^2-\hat{k}_\tau^2$ and substitute $\Delta\hat{k}=\hat{k}^*-\hat{k}_\tau$ to obtain:

$$\mathbb{E}[\hat{k}]=\frac{\overline{\mathbb{E}}\left[(\hat{k}^*+\hat{k}_\tau)(\hat{k}^*-\hat{k}_\tau)\right]}{2\overline{\mathbb{E}}[\Delta\hat{k}]}+\frac{\sigma^2}{2\nu} \;=\; \overline{\mathbb{E}}\left[\frac{\hat{k}^*+\hat{k}_\tau}{2}\left(\frac{\Delta k}{\overline{\mathbb{E}}[\Delta\hat{k}]}\right)\right]+\frac{\sigma^2}{2\nu}. \tag{A.337}$$

Conditional means:

$$\mathbb{E}^-[\hat{k}] \;=\; \overline{\mathbb{E}}^-\left[\frac{\hat{k}^{*-}+\hat{k}_\tau}{2}\left(\frac{\Delta k}{\overline{\mathbb{E}}^-[\Delta\hat{k}]}\right)\right] \;+\; \frac{\sigma^2}{2}\frac{\overline{\mathbb{E}}^-[\tau]}{\overline{\mathbb{E}}^-[\Delta\hat{k}]} \tag{A.338}$$

$$\mathbb{E}^+[\hat{k}] \;=\; \overline{\mathbb{E}}^+\left[\frac{\hat{k}^{*+}+\hat{k}_\tau}{2}\left(\frac{\Delta k}{\overline{\mathbb{E}}^+[\Delta\hat{k}]}\right)\right] \;+\; \frac{\sigma^2}{2}\frac{\overline{\mathbb{E}}^+[\tau]}{\overline{\mathbb{E}}^+[\Delta\hat{k}]} \tag{A.339}$$

Variance of $\hat{k}$. To compute the variance of $\hat{k}$, we evaluate expression for centered moments (A.335) at $n=2$:

$$\mathbb{V}ar[\hat{k}] \;=\; \frac{\overline{\mathbb{E}}[(\hat{k}^*-\mathbb{E}[\hat{k}])^3]-\overline{\mathbb{E}}\left[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^3\right]}{3\overline{\mathbb{E}}[\Delta\hat{k}]} \tag{A.340}$$

Without a price wedge (only fixed costs), there is a unique reset point $\hat{k}^*$ and we can simplify the previous expression. Inside the first term, add and subtract $\hat{k}_\tau$ and substitute $\Delta\hat{k}=\hat{k}^*-\hat{k}_\tau$. Expand the cubic polynomials, cancel terms, and rearrange:

$$\begin{aligned}
\mathbb{V}ar[\hat{k}] &= \frac{\overline{\mathbb{E}}[(\hat{k}_\tau-\mathbb{E}[\hat{k}]+\Delta\hat{k})^3-(\hat{k}_\tau-\mathbb{E}[\hat{k}])^3]}{3\overline{\mathbb{E}}[\Delta\hat{k}]}\\
&= \frac{\overline{\mathbb{E}}[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^3+3(\hat{k}_\tau-\mathbb{E}[\hat{k}])^2\Delta\hat{k}+3(\hat{k}_\tau-\mathbb{E}[\hat{k}])\Delta\hat{k}^2+\Delta\hat{k}^3-(\hat{k}_\tau-\mathbb{E}[\hat{k}])^3]}{3\overline{\mathbb{E}}[\Delta\hat{k}]}\\
&= \frac{\overline{\mathbb{E}}[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^2\Delta\hat{k}+(\hat{k}_\tau-\mathbb{E}[\hat{k}])\Delta\hat{k}^2]}{\overline{\mathbb{E}}[\Delta\hat{k}]}+\frac{\overline{\mathbb{E}}[\Delta\hat{k}^3]}{3\overline{\mathbb{E}}[\Delta\hat{k}]}\\
&= \frac{\overline{\mathbb{E}}[\Delta\hat{k}(\hat{k}_\tau-\mathbb{E}[\hat{k}])(\hat{k}^*-\mathbb{E}[\hat{k}])+\Delta\hat{k}(\hat{k}^*-\hat{k}_\tau)^2/3]}{\overline{\mathbb{E}}[\Delta\hat{k}]}\\
&= \overline{\mathbb{E}}\left[\left((\hat{k}_\tau-\mathbb{E}[\hat{k}])(\hat{k}^*-\mathbb{E}[\hat{k}])+\frac{(\hat{k}^*-\hat{k}_\tau)^2}{3}\right)\left(\frac{\Delta\hat{k}}{\overline{\mathbb{E}}[\Delta\hat{k}]}\right)\right].
\end{aligned}$$

A.13.2 Joint moments of $(\hat{k}, a)$.

To compute the joint moments of $\hat{k}$ and its age a, we consider the function $Y_s = (\hat{k}_s - \mathbb{E}[\hat{k}])^{n+1}s$. Applying Itō's lemma, we obtain:

$$\text{(A.341)} \qquad \begin{aligned} \mathrm{d}Y_s \; = \; & (\hat{k}_s - \mathbb{E}[\hat{k}])^n \, \mathrm{d}s \; - \; \nu n(\hat{k}_s - \mathbb{E}[\hat{k}])^{n-1} s \, \mathrm{d}s \; + \; \sigma n(\hat{k}_s - \mathbb{E}[\hat{k}])^{n-1} s \, \mathrm{d}W_s \\ & + \; \frac{\sigma^2}{2} n(n-1)(\hat{k}_s - \mathbb{E}[\hat{k}])^{n-2} s \, \mathrm{d}s. \end{aligned}$$

We integrate this expression from 0 to τ and take conditional expectations with respect to the initial condition $\hat{k}_0 = \hat{k}^*$, which is a function of Δk.

$$\text{(A.342)} \qquad \frac{\overline{\mathbb{E}}\left[(\hat{k}_\tau - \mathbb{E}[\hat{k}])^{n+1}\tau\right]}{\overline{\mathbb{E}}[\tau]} = \mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^{n+1}] - \nu(n+1)\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^n a] + \frac{\sigma^2}{2} n(n+1)\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^{n-1} a]$$

Rearranging:

$$\text{(A.343)} \qquad \mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^n a] = \frac{1}{\nu(n+1)} \left[\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^{n+1}] - \frac{\overline{\mathbb{E}}[(\hat{k}_\tau - \mathbb{E}[\hat{k}])^{n+1}\tau]}{\overline{\mathbb{E}}[\tau]} + \frac{\sigma^2}{2} n \mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^{n-1} a] \right]$$

Finally, to compute the covariance between $(\hat{k}, a)$, we evaluate expression (A.343) at $n = 1$ to obtain

$$\text{(A.344)} \qquad \mathbb{C}ov[\hat{k}, a] = \frac{1}{2\nu} \left(\mathbb{V}ar[\hat{k}] - \frac{\overline{\mathbb{E}}[(\hat{k}_\tau - \mathbb{E}[\hat{k}])^2 \tau]}{\overline{\mathbb{E}}[\tau]} + \frac{\sigma^2}{2} \frac{\overline{\mathbb{E}}[\tau]}{2} (1 + \mathbb{CV}^2[\tau]) \right)$$

A.14 Disentangling investment frictions with microdata

The previous analysis teaches us that the the nature of adjustment frictions matters for the macroeconomy. How can we disentangle their relative importance? Here we show that a decomposition of the allocation of capital provides relevant information. Using the law of total variance, we decompose $\mathbb{V}ar[x]$ into two terms that condition on the last adjustment:

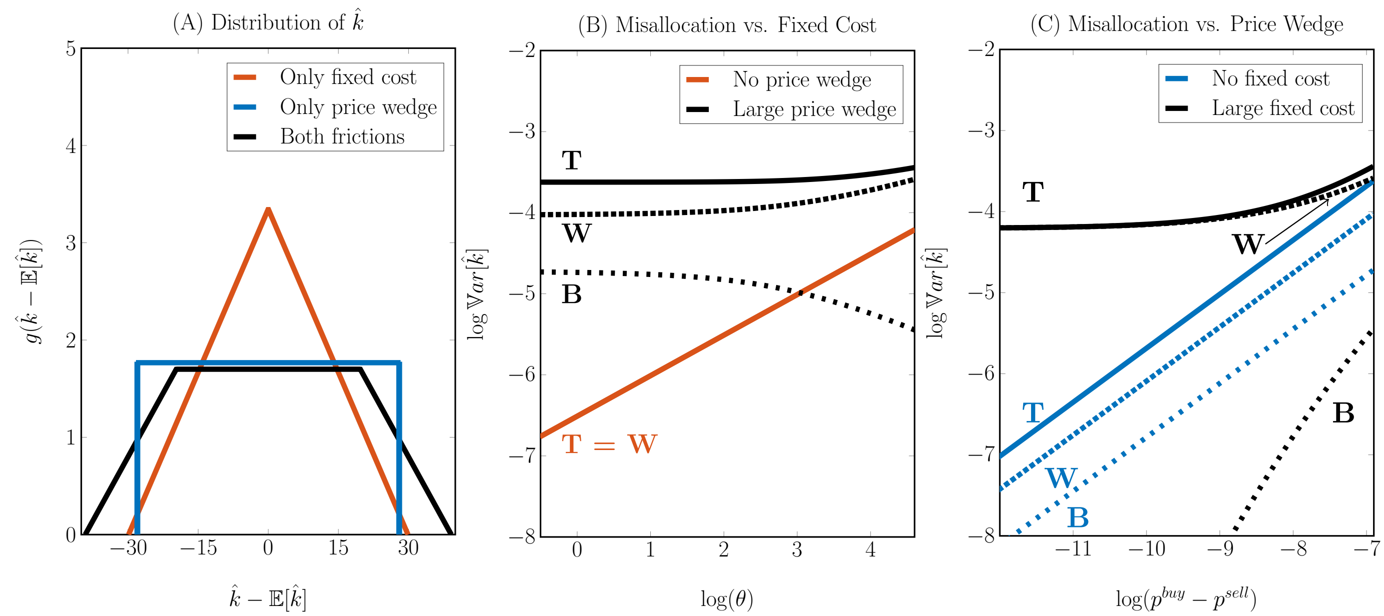
$$\text{(A.345)} \qquad \underbrace{\mathbb{V}ar[x]}_{\text{total}} \; = \; \underbrace{\mathbb{E}\left[\mathbb{V}ar[x|\Delta x]\right]}_{\text{within}} \; + \; \underbrace{\mathbb{V}ar\left[\mathbb{E}[x|\Delta x]\right]}_{\text{between}}.$$

The first term is the average of the variance *within* each conditional distribution g^+ and g^-, that is, the average of $\mathbb{V}ar^-[\hat{k}]$ and $\mathbb{V}ar^+[\hat{k}]$, where each variance is computed conditioning on the sign of $\Delta\hat{k}$ and using the conditional renewal measure as in (47)). Both investment frictions add to this dispersion. The second term reflects the distance *between* the conditional means $\mathbb{E}^-[\hat{k}]$ and $\mathbb{E}^+[\hat{k}]$, where each mean is computed conditioning on the sign of $\Delta\hat{k}$ and using the conditional renewal measure. This term arises exclusively from the price wedge that generates two different means. The larger the price wedge, the further apart are the conditional means and the larger the between variance. Note that this term is zero when only fixed costs are present as there is a unique reset point.

Next, we show that the relative size of frictions affects the response of misallocation to an increase in these frictions. Figure A.1 illustrates the effects of each adjustment friction on the total, within, and between variances of capital-productivity ratios. To sharpen the exposition, we assume zero drift and a symmetric price wedge. Panel A plots the stationary density $g(\hat{k})$. Panels B and C are log-log plots of misallocation against one friction, setting the other friction either at zero or at a large value. We use a log-log scale to highlight the linearity that arises. We also mark within (W), between (B), and total (T) variances.

Consider an environment where the fixed cost is the only investment friction (orange lines). The density is a triangle that concentrates at the unique reset point and decreases linearly toward the boundaries of the inaction region. A higher fixed cost widens the inaction region and increases misallocation in a log-log linear way (Panel B). Now consider an environment where the price wedge is the only investment friction (blue lines). The density is a rectangle between the two reset/inaction points. In this case, a higher price wedge increases all components of misallocation (within and between) in a log-log linear way (Panel C).

Figure A.1 – Misallocation and Investment Frictions



Notes: Figures assume zero drift and symmetric price wedge. Panel A plots the steady-state distribution of capital-productivity ratios, normalized by their mean. Triangle = only fixed costs; Rectangle = only price wedge; Parallelogram = both frictions. Panel B plots misallocation against the fixed cost for a zero (orange) and a large (black) price wedge. Panel C plots misallocation against the price wedge for a zero (blue) and a large (black) fixed cost. Variances: total (T), within (W) and between (B).

With both frictions active (black lines), the density is a trapezoid. The relationship between misallocation and frictions is now flattened in the following sense. Consider the case with fixed costs and a large price wedge. Misallocation is at a higher level but the relationship between misallocation and the fixed cost flattens out. A higher fixed cost still widens the distance between inaction thresholds, increasing the within variance, but simultaneously reduces the distance between the reset points, decreasing the between variance. These opposing forces compensate each other cancelling the effects on misallocation (see the dotted and dashed black lines in Panel B which move in opposite directions). Now consider the case with partial irreversibility and a large fixed cost. Again, misallocation is at a higher level and the relationship between misallocation and the price wedge flattens out. The between variance disappears (its log becomes very negative).

A.15 Proof of Proposition 6

This proposition expresses the effective investment frictions in terms of fundamental parameters and taxes. Then it computes the derivatives of after-tax frictions with respect to taxes.

A.15.1 Effective frictions

Let $\tilde{\mathcal{U}} \equiv \frac{(1-t^p)}{(1-t^g)}\rho + \xi^k - \sigma^2$ be the after-tax user cost of capital. Substitute the static policy in (25), given by

$$\hat{k}^{ss} \;=\; \frac{1}{1-\alpha} \log\left(\frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{\mathcal{U}}},\right) \tag{A.346}$$

into the after-tax static profits and into the after-tax profit-capital ratio:

$$(1-t^c)Ae^{\alpha\hat{k}^{ss}} = (1-t^c)A\left(\frac{1-t^c}{1-t^c z}\frac{\alpha A}{p\tilde{\mathcal{U}}}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{(1-t^c)A}{(1-t^c z)^\alpha}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{p\tilde{\mathcal{U}}}\right)^{\frac{\alpha}{1-\alpha}} \tag{A.347}$$

$$\frac{(1-t^c)}{(1-t^c z)}Ae^{(\alpha-1)\hat{k}^{ss}} = \frac{(1-t^c)}{(1-t^c z)}A\left(\frac{1-t^c}{1-t^c z}\frac{\alpha A}{p\tilde{\mathcal{U}}}\right)^{\frac{\alpha-1}{1-\alpha}} = \frac{(1-t^c)}{(1-t^c z)}A\frac{(1-t^c z)p\tilde{\mathcal{U}}}{(1-t^c)\alpha A} = \frac{p\tilde{\mathcal{U}}}{\alpha}. \tag{A.348}$$

Therefore, the scaled investment frictons are equal to:

$$\tilde{\theta} \equiv \frac{1-t^c z}{1-t^c}\frac{\theta}{Ae^{\alpha\hat{k}^{ss}}} = \left(\frac{(1-t^d)^\alpha}{(1-t^c)A}\right)^{\frac{1}{1-\alpha}}\left(\frac{p\tilde{\mathcal{U}}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}(1-t^c z)\theta = \left(\frac{(1-t^c z)}{(1-t^c)A}\right)^{\frac{1}{1-\alpha}}\left(\frac{p\tilde{\mathcal{U}}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}\theta, \tag{A.349}$$

$$\tilde{p}^{\text{buy}} - \tilde{p}^{\text{sell}} \equiv \frac{1-t^c z}{1-t^c}\frac{\omega p(1-t^c)}{Ae^{(\alpha-1)\hat{k}^{ss}}} = \frac{\alpha}{\tilde{\mathcal{U}}}\omega(1-t^c). \tag{A.350}$$

A.15.2 Derivatives of frictions wrt. taxes

Now we compute the derivatives with respect to taxes $\mathbb{T}$. Assume $t^c > 0$, $\tilde{\mathcal{U}} > 0$ and p fixed. To correctly isolate the tax effects, we define the following parameters:

$$t^\rho \equiv \frac{1-t^p}{1-t^g}, \qquad \tilde{\xi}^d \equiv \frac{\xi^d}{t^\rho\rho + \xi^d}, \qquad t^d = t^c\tilde{\xi}^d. \tag{A.351}$$

To ease the computations, we sign the tax effects on the log of $\tilde{\theta}$:

$$\log\tilde{\theta} = \frac{1}{1-\alpha}\log\left(1-\frac{t^c\xi^d}{t^\rho\rho+\xi^d}\right) - \frac{1}{1-\alpha}\log(1-t^c)A + \frac{\alpha}{1-\alpha}\log R + \frac{\alpha}{1-\alpha}\log\left(\frac{p}{\alpha}\right) + \log\theta. \tag{A.352}$$

- Effects of taxes on the effective fixed cost in (A.349).

$$\begin{aligned}
\frac{\partial\log\tilde{\theta}}{\partial t^c} &= \frac{1}{(1-\alpha)(1-t^c)}\left[1-\frac{\xi^d(1-t^c)}{t^\rho\rho+\xi^d(1-t^c)}\right] > 0,\\
\frac{\partial\log\tilde{\theta}}{\partial \xi^d} &= \frac{1}{1-\alpha}\left[\frac{1-t^c}{t^\rho\rho+\xi^d(1-t^c)} - \frac{1}{t^\rho\rho+\xi^d}\right] < 0,\\
\frac{\partial\log\tilde{\theta}}{\partial t^p} &= -\frac{1}{1-\alpha}\frac{\rho}{1-t^g}\left[\frac{1}{t^\rho\rho+\xi^d(1-t^c)} - \frac{1}{t^\rho\rho+\xi^d} + \frac{1}{R}\right] < 0,\\
\frac{\partial\log\tilde{\theta}}{\partial t^g} &= \frac{1}{1-\alpha}\frac{\rho(1-t^p)}{(1-t^g)^2}\left[\frac{1}{t^\rho\rho+\xi^d(1-t^c)} - \frac{1}{t^\rho\rho+\xi^d} + \frac{1}{R}\right] > 0.
\end{aligned}$$

- Effects of taxes on the effective price wedge in (A.350)

$$\begin{aligned}
\frac{\partial(\tilde{p}^{\text{buy}}-\tilde{p}^{\text{sell}})}{\partial t^c} &= 0,\\
\frac{\partial(\tilde{p}^{\text{buy}}-\tilde{p}^{\text{sell}})}{\partial \xi^d} &= 0,\\
\frac{\partial(\tilde{p}^{\text{buy}}-\tilde{p}^{\text{sell}})}{\partial t^p} &= \frac{\alpha}{pR^3}\frac{\rho}{1-t^g}(p^{\text{buy}}-p^{\text{sell}}) > 0,\\
\frac{\partial(\tilde{p}^{\text{buy}}-\tilde{p}^{\text{sell}})}{\partial t^g} &= -\frac{\alpha}{pR^3}\frac{\rho(1-t^p)}{(1-t^g)^2}(p^{\text{buy}}-p^{\text{sell}}) < 0.
\end{aligned}$$

B Generalized Hazard Model

In the main text, we specialize investment frictions to a symmetric adjustment cost θ paid indistinctly for positive and negative investments, and a price wedge that gives rise to partial irreversibility. We examine this model mainly for pedagogical reasons, as it simplifies the exposition of the theory. In this section, we expand the scope of the analysis and present an asymmetric generalized hazard model, which follows the contributions by Caballero and Engel (1999, 2007) and examined in contemporaneous work by Álvarez, Lippi and Oskolkov (2020), which may accommodate other empirically-relevant frictions.

The generalized hazard function depends mainly on the assumption over the fixed cost of adjustment. For that reason, in this section, we assume technology and shocks as as in Section 2. Moreover, the firm can control its capital stock through buying and selling investment goods at prices p^{buy} and p^{sell}, with $p^{\text{buy}} > p^{\text{sell}}$. For simplicity, we assume away the tax system.

Adjustment costs. The first step generalizes the adjustment cost structure. For every investment $i = \Delta k$, the firm must pay an adjustment cost θ_s proportional to current productivity u_s and measured in consumption units (Caballero and Engel, 1999):

$$\theta_s = \Theta(i_s, \mathrm{d}N_s^-, \mathrm{d}N_s^+, \vartheta_s^-, \vartheta_s^+)u_s, \tag{B.353}$$

where the function $\Theta(\cdot) > 0$ is described by

$$\Theta(i, \mathrm{d}N^+, \mathrm{d}N^-, \vartheta^-, \vartheta^+) = \begin{cases} 0 & \text{if} \quad i = 0 \\ \bar{\theta}^+(1 - \mathrm{d}N) + \mathrm{d}N\vartheta^+ & \text{if} \quad i < 0 \\ \bar{\theta}^-(1 - \mathrm{d}N) + \mathrm{d}N\vartheta^- & \text{if} \quad i > 0. \end{cases} \tag{B.354}$$

Let us describe each element in equation (B.354).

(i) N_s^+ and N_s^- follows Poisson counter with unit increments and arrival rates λ^+ and λ^-;

(ii) $\bar{\theta}^+$ and $\bar{\theta}^-$ are non negative number; and

(iii) ϑ_s^+ and ϑ_s^- are *i.i.d.* random variables with support $\mathbb{S}upp(\vartheta^+) = [0, \bar{\vartheta}^+]$ and $\mathbb{S}upp(\vartheta^-) = [0, \bar{\vartheta}^-]$. We assume that $\vartheta^- \leq \bar{\theta}^-$ and $\vartheta^+ \leq \bar{\theta}^+$. Define $J^+(x) \equiv \Pr(\vartheta^+ < x)$ and $J^-(x) \equiv \Pr(\vartheta^- < x)$ the cumulative distribution for each random variable.

Relationship to the literature. The stochastic process of fixed cost in (B.354) can derive the majority of lumpy adjustment models used previous work.

1. Setting $\lambda^+ = \lambda^- = 0$ and $\bar{\theta}^+ = \bar{\theta}^-$ yields the standard fixed cost model of adjustment, originally proposed by Scarf (1959) in an inventory model and Sheshinski and Weiss (1977) in a pricing context.
2. Setting $\lambda^+ = \lambda^- > 0$ and $\mathbb{S}upp(\vartheta^+) = \mathbb{S}upp(\vartheta^-) = \{0\}$, and $\bar{\theta}^+ = \bar{\theta}^- > 0$ yields the CalvoPlus model proposed by Nakamura and Steinsson (2010), which nests the standard fixed cost model and the time-dependent Calvo model.
3. Under this case, if $\bar{\theta}^+ \neq \bar{\theta}^-$, then we have the Bernoulli fixed cost model or asymmetric Bernoulli fixed cost model if $\lambda^+ \neq \lambda^-$, see Baley and Blanco (2021).
4. Finally, setting $\lambda^+ = \lambda^- > 0$ and $\bar{\vartheta}^+ = \bar{\vartheta}^- = \bar{\theta}^- = \bar{\theta}^+$ yields the generalized hazard model originally proposed by Caballero and Engel (1993).

Value. Let $V(k, u)$ denote the value of a firm with capital stock k and productivity u. Given initial conditions (k_0, u_0), the firm chooses a sequence of adjustment dates $\{T_h\}_{h=1}^{\infty}$ and investments $\{i_{T_h}\}_{h=1}^{\infty}$, where h counts the number of adjustments, to maximize its expected discounted stream of profits. The sequential problem is

$$V(k_0, u_0) = \max_{\{T_h, i_{T_h}\}_{h=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} e^{-\rho s}\pi_s \, \mathrm{d}s - \sum_{h=1}^{\infty} e^{-\rho T_h}\left(\theta_{T_h} + p\left(i_{T_h}\right) i_{T_h}\right)\right], \tag{B.355}$$

subject to the production technology (1), the idiosyncratic productivity shocks (2), the investment price function (4), the law of motion for the capital stock (6), and the stochastic process of adjustment cost in (B.354).

Capital-productivity ratios $\hat{k}$. As in the main text, it is easy to show that $v(k,u) = uv(\hat{k})$ where

$$v(\hat{k}) = \max_{\tau,\Delta\hat{k}} \mathbb{E}\left[\int_0^\tau Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s + e^{-r\tau}\left(-\theta_\tau(\Delta\hat{k}) - p(\Delta\hat{k})(e^{\hat{k}_\tau+\Delta\hat{k}} - e^{\hat{k}_\tau}) + v(\hat{k}_\tau + \Delta\hat{k})\right)\Big|\hat{k}_0 = \hat{k}\right]. \quad \text{(B.356)}$$

Here, $\theta_\tau(\Delta\hat{k})$ is a random variable instead of a number and it is a function of the sign of adjustment—similarly to he price of investment.

Optimal investment policy. The optimal investment policy is characterize by four numbers $\mathcal{K} \equiv \{\hat{k}^- \leq \hat{k}^{*-} \leq \hat{k}^{*+} \leq k^+\}$, and a hazard rate of adjustment $\Lambda(\hat{k})$. $\hat{k}^-$ and k^+ correspond to the lower and upper borders of the inaction region

$$\mathcal{R} \;=\; \left\{\hat{k} : \hat{k}^- < \hat{k} < \hat{k}^+\right\}, \quad \text{(B.357)}$$

and $\hat{k}^{*-} < \hat{k}^{*+}$ to the two reset points following a positive and a negative investment, respectively. $\Lambda(\hat{k}) : \overline{\mathcal{R}} \to \mathbb{R}^+$ is a non-negative function corresponding to the arrival rate of a new Poisson counter N^Λ. Given $\mathcal{R}$ and N^Λ, the optimal adjustment dates are

$$T_h \;=\; \inf\left\{s \geq T_{h-1} : \hat{k}_s \notin \mathcal{R} \text{ or } \mathrm{d}N_s^\Lambda(\hat{k}) = 1\right\} \quad \text{with} \quad T_0 = 0. \quad \text{(B.358)}$$

Following Øksendal and Sulem (2005) and Oksendal (2007), Lemma C.3 establishes the optimality conditions that characterize (B.356).

Lemma C.3. *The value function $v(\hat{k})$ and the policy $\mathcal{K} \equiv \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ satisfy:*

(i) For all $\hat{k} \in \mathcal{R}$, $v(\hat{k})$ solves the HJB equation:

$$\begin{aligned} rv(\hat{k}) \;&=\; Ae^{\alpha\hat{k}} \;-\; \nu v'(\hat{k}) \;+\; \frac{\sigma^2}{2}v''(\hat{k}) \\ &+ \lambda^- \int_0^{\bar{\vartheta}^-} \max\left\{v^{buy}(\hat{k}) - \vartheta, 0\right\}\mathrm{d}J^-(\vartheta) + \lambda^+ \int_0^{\bar{\vartheta}^+} \max\left\{v^{sell}(\hat{k}) - \vartheta, 0\right\}\mathrm{d}J^+(\vartheta) \end{aligned} \quad \text{(B.359)}$$

where the values v^{buy} and v^{sell} are defined as follows:

$$v^{buy}(\hat{k}) \;\equiv\; v(\hat{k}^-) - v(\hat{k}) - p^{buy}(e^{\hat{k}^-} - e^{\hat{k}}), \quad \text{(B.360)}$$

$$v^{sell}(\hat{k}) \;\equiv\; v(\hat{k}^+) - v(\hat{k}) - p^{sell}(e^{\hat{k}^+} - e^{\hat{k}}). \quad \text{(B.361)}$$

(ii) At the borders of the inaction region, $v(\hat{k})$ satisfies the value-matching conditions:

$$v^{buy}(\hat{k}^-) = \bar{\theta}^-; \qquad v^{sell}(\hat{k}^+) = \bar{\theta}^+; \quad \text{(B.362)}$$

(iii) At the borders of the inaction region and the two reset states, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$\frac{\mathrm{d}v^{buy}(\hat{k})}{\mathrm{d}\hat{k}} \;=\; p^{buy}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \quad \text{(B.363)}$$

$$\frac{\mathrm{d}v^{sell}(\hat{k})}{\mathrm{d}\hat{k}} \;=\; p^{sell}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \quad \text{(B.364)}$$

Hazard rate of adjustment $\Lambda(\hat{k})$. We are now ready to define $\Lambda(\hat{k})$, which gives the probability of adjustment $\Lambda(\hat{k})\,\mathrm{d}t$ in a time period $\mathrm{d}t$ a firm with $\hat{k} \in \mathcal{R}$. The hazard rate of adjustment is given by

$$\Lambda(\hat{k}) = \lambda^- J^-\left(v^{\text{buy}}(\hat{k})\right)\mathbb{1}_{\{\hat{k}\in(\hat{k}^-,\hat{k}^{*-})\}} + \lambda^+ J^+\left(v^{\text{sell}}(\hat{k})\right)\mathbb{1}_{\{\hat{k}\in(\hat{k}^{*+},\hat{k}^+)\}}. \quad \text{(B.365)}$$

The hazard function $\Lambda(\hat{k})$ satisfies the following properties:

1. $\Lambda(\hat{k}) = 0$ in the inner inaction region, i.e., for all $\hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+})$,
2. $\Lambda(\hat{k})$ is weakly decreasing in $(\hat{k}^{-}, \hat{k}^{*-})$ and weakly increasing in $(\hat{k}^{*+}, \hat{k}^{+})$;
3. If $J^{-}(0) > 0$ then $\Lambda(\hat{k})$ is bounded below in the domain $(\hat{k}^{-}, \hat{k}^{*-})$ by $\Lambda(\hat{k}) = \lambda^{-} J^{-}(0)$.
4. If $J^{+}(0) > 0$ then $\Lambda(\hat{k})$ is bounded below in the domain $(\hat{k}^{*+}, \hat{k}^{+})$ by $\Lambda(\hat{k}) = \lambda^{+} J^{+}(0)$

C A General Equilibrium Framework

This section provides a general equilibrium model that microfounds the parsimonious investment model presented allows to examine the macroeconomic effects of corporate tax reforms in an equilibrium setting. Its backbone components are: a small open economy with an exogenous interest rate, a corporate tax structure as in Summers (1981), and "capital quality shocks" as in Baley and Blanco (2021).

C.1 Economic environment

Time is continuous and it extends forever. Five types of agents live in the economy: (i) A representative household, (ii) a capital–goods producer, (iii) a final–good producer, (iv) a unit mass of intermediate–good firms indexed by $f \in [0,1]$ who are subject to capital adjustment frictions, and (v) a government.

(i) Representative household. The household chooses the stochastic processes for consumption C_s, risk free bonds B_s and equality for each firm E_{fs} subject to the law of motion of nominal wealth $W_s = \int_0^1 E_{ft}\,\mathrm{d}f + B_t$

$$\mathrm{d}B_s + \int_0^1 P_{fs}\,\mathrm{d}E_{fs}\,\mathrm{d}f \;=\; (\mathcal{Y}_s - C_s)\,\mathrm{d}s - t^g \int_0^1 \mathrm{d}P_{fs}E_{fs}\,\mathrm{d}f, \tag{C.366}$$

where P_{fs} is the price of equity for firm f, t^g is the capital gain tax, and $\mathcal{Y}_s$ is after tax available income given by

$$\mathcal{Y}_s \;=\; R_s + (1-t^p)\left(\int_0^1 D_{fs}E_{fs}\,\mathrm{d}f + \tilde{\rho}_s B_s\right). \tag{C.367}$$

Here, R_s are lump-sum transfers for the government, D_{fs} are firm's f dividend payments, t^p denotes the personal income tax, and $\tilde{\rho}_s$ is the world interest rate. We omit the profits of the final–good producer and the capital–good since they have constant returns to scale and do not generate profits for the household. Thus, we omit those sectors' profits from the household budget constraint. Taking the prices of equity $\{P_{fs}\}_{fs}$, the real interest rate $\tilde{\rho}_s$, and taxes $(t^p, t^g, \{R_s\}_s)$ as given, the household's problem is to maximize her expected utility (discounted at the rate χ)

$$\max_{\{C_s,B_s,\{E_{ft}\}_f\}_{t=0}^{\infty}} \mathbb{E}_0\left[\int_0^\infty e^{-\tilde{\chi}t}\log C_s\,\mathrm{d}s\right], \tag{C.368}$$

subject to her budget constraints in (C.366) and (C.367) and initial conditions B_0 and $\{E_{f0}\}_f$.

(ii) Capital–good producer. The capital–good producer manufactures firm-specific investment goods $\{i_{ft}\}_{f\in[0,1]}$ in competitive market, according to a linear technology

$$\int_0^1 \left(\frac{\varphi(i_{ft})i_{fs}}{u_{fs}}\right)\,\mathrm{d}f \;=\; i_s, \tag{C.369}$$

where

$$\varphi(i_{ft}) = \begin{cases} \varphi^- & \text{if } i_{ft} > 0 \\ \varphi^+ & \text{if } i_{ft} \le 0 \end{cases}. \tag{C.370}$$

We refer to u_{fs} as capital quality shocks. The parameters φ^- and φ^+ measure the level of partial irreversibility with $\varphi- > \varphi^+$. Taking the prices of firm-specific investment goods p^k_{js} as given, the capital–good firm problem maximizes her profits

$$\max_{\{i_{ft},i_t\}_{t=0}^{\infty}} (1-t^c)\left(\int_0^1 p^k_{fs}i_{fs}\,\mathrm{d}f - i_s\right), \tag{C.371}$$

subject to the technology described in (C.369). Here, t^c denotes the corporate income tax and i_s is the aggregate investment to produce capital. Note that i_{ft} may be positive or negative as there is no technological constraint on its sign.

(iii) Final–good producer. The final–good producer assembles output Y_s using intermediate inputs $\{\hat{y}_{fs}\}_{f\in[0,1]}$ according to a linear aggregator

$$Y_s \;=\; \int_0^1 \left(\frac{\hat{y}_{fs}}{u_{fs}}\right)\,\mathrm{d}f, \tag{C.372}$$

where capital quality u_{fs} decreases the marginal product of the intermediate good f. Taking the prices of intermediate inputs p_{ft} as given, the producer's problem entails choosing final–good supply Y_s and input demands $\hat{y}_{fs}$ to maximize her profits

$$\max_{Y_s,\hat{y}_{fs}} (1-t^c)\left(Y_t - \int_0^1 p_{ft}\hat{y}_{ft}\,\mathrm{d}f\right), \tag{C.373}$$

subject to the aggregator in (C.372).

(iv) Intermediate-good firms. These are the most important agents in the economy for our question as they make investment choices subject to adjustment costs. Intermediate–good firm $f \in [0,1]$ produces output y_{fs} using capital k_{fs} according to a production function with decreasing returns to scale

$$y_{fs} = u_s^{1-\alpha}k_{fs}^{\alpha}, \quad \alpha < 1. \tag{C.374}$$

Firm's total productivity is driven by an idiosyncratic components

$$\mathrm{d}\log(u_{fs}) = \mu\,\mathrm{d}s + \sigma\,\mathrm{d}W_{fs} \qquad W_{fs} \sim \mathit{Wiener}, \tag{C.375}$$

where the processes W_{ft} are independent across intermediate–good firms.

The firm pays the corporate income tax rate t^c on its cash flow $p_{fs}y_{fs}$ net of deductions $\xi^d k_{fs}$, where ξ^d denotes the deduction rate. Since the physical and the legal depreciation rates differ, we distinguish deductions from the capital stock and denote these with d_{fs}. The state space now includes deductions (k,u,d). The corporate income tax and deductions jointly determine the after–tax profit rate

$$\pi_{fs} \;=\; p_{fs}y_{fs} - t^c(p_{fs}y_{fs} - \xi^d d_{fs}) \;=\; (1-t^c)p_{fs}y_{fs} + t^c\xi^d d_{fs}, \tag{C.376}$$

and the evolution of deductions

$$\log d_{fs} \;=\; \log d_0 - \xi^d s + \sum_{h:T_{fh}\le s}\left(1 + \frac{p^k(\hat{i}_{fT_{fh}})\hat{i}_{fT_{fh}} + \theta_{fT_h}}{d_{fT_{fh}^-}}\right). \tag{C.377}$$

Here, $\theta_s = \theta u_s$ where $\theta > 0$ is constant and $\hat{i}$ is firm's investment.

Taking the prices of the intermediate goods p_{ft}, the marginal investor discount factor Q_t, and firm-specific capital goods $p^k_{ft}(\hat{i})$ as given, together with the adjustment friction θ_{ft}, each firm f chooses a sequence of capital adjustment dates $\{T_{fh}\}_{h=1}^{\infty}$ and investments $\{i_{f,T_{fh}}\}_{h=1}^{\infty}$ to maximize its expected discounted stream of profits

$$\max_{\{T_{fh},\hat{i}_{f,T_{fh}}\}_{h=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} Q_t\pi_{ft}\;\mathrm{d}s - \sum_{h\,=\,1}^{\infty} Q_{T_{fh}}p^k_{f,T_{fh}}\left(\theta_{fT_{fh}} + p^k_{fT_{fh}}(\hat{i}_{fT_{fh}})\hat{i}_{fT_{fh}}\right)\right], \tag{C.378}$$

subject to the profits function in (C.376) and the law of motion for its capital stock

$$\log(k_{ft}) \;=\; \log(k_{f0}) - \zeta t + \sum_{h:T_{fh}\le t}\log\left(1 + \frac{\hat{i}_{f,T_{fh}}}{k_{T_{fh}^-}}\right). \tag{C.379}$$

(v) Government. Since Ricardian equivalence holds, we assume the government follows a period by period balance budget without loss of generality. The period expenditures, given by the lump-sum transfers R_s, has to be equal to the revenue from the firms $\int_0^1 T_{js}\,\mathrm{d}f$ and the household T_s^h

$$\begin{aligned} R_s &= \int_0^1 T_{fs}\,\mathrm{d}f + T_s^h, \\ T_{fs} &= t^c \left(p_{fs} y_{fs} - \xi^d d_{js}\right), \\ T_s^h &= t^p \left(\int_0^1 D_{fs} E_{fs}\,\mathrm{d}f + \tilde{\rho}_s B_s\right) + t^g \int_0^1 P_{fs} E_{fs}\,\mathrm{d}f. \end{aligned} \tag{C.380}$$

Market structure. There are three type of goods (respectively, markets) in the economy: (i) final good, (ii) intermediate goods, and (iii) firm-specific investment goods. There are two assets: (i) risk-free bonds and (ii) equity. All good and asset markets are competitive. We assume equity can only be hold by the representative household, thus we have segmented equity market, and the bond market is freely trade across countries. The market clearing conditions, respectively, are as follows:

$$E_{fs} = 1 \quad \text{for all } t \text{ and } f, \tag{C.381}$$

$$\hat{y}_{fs} = y_{fs} \quad \text{for all } s \text{ and } f, \tag{C.382}$$

$$\hat{i}_{fs} = i_{fs} \quad \text{for all } s \text{ and } f. \tag{C.383}$$

Equilibrium. *Given a stochastic processes for capital quality $\{u_{fs}\}_{fs}$, and adjustment costs θ_{ft}, an equilibrium is a set of stochastic processes for prices $\{\tilde{\rho}_s, \{p_{fs}, p^k_{fs}(i), P_{fs}\}_{f\in[0,1]}\}_{s=0}^{\infty}$, the household's policy $\{C_s, B_s, \{E_{fs}\}_{f\in[0,1]}\}_{t=0}^{\infty}$, the final–good producer's policy $\left\{Y_s, \{\hat{y}_{fs}\}_{f\in[0,1]}\right\}_{t=0}^{\infty}$, the capital–good producer's policy $\left\{\{i_{fs}\}_{f\in[0,1]}, i_s\right\}_{t=0}^{\infty}$, and the intermediate–good firms' policy $\left\{\{T_{fh}, i_{f,T_{hf}}\}_{h=1}^{\infty}\right\}$ such that:*

(i) Given prices $\{r_s, P_{fs}\}$, the household solves (C.368).

(ii) Given prices $\{p^k_{fs}\}$, the capital–good producer solves (C.371).

(iii) Given prices $\{p_{fs}\}$, the final–good producer solves (C.373).

(iv) Given prices $\{Q_s, p_{fs}, p^k_{fs}\}$, intermediate–good firms solve (C.378).

(v) Market clears in (C.381) to (C.383).

C.2 Equilibrium characterization

We now describe equilibrium determination of prices and quantities, in that order. For now on, we assume that world interest rate is constant: $\tilde{\rho}_s = \tilde{\rho}$. We derive the three macroeconomic outcomes (e.g., capital allocation, valuations, and dynamics) departing from first principles.

Equilibrium determination of prices. The household's optimality conditions over bonds and equity are

$$\begin{aligned} \tilde{\rho}(1-t^p)\,\mathrm{d}s &= \chi\,\mathrm{d}s - \frac{\mathrm{d}(1/C_s)}{1/C_s} \quad \forall s \\ \frac{(1-t^g)\mathbb{E}[dP^i_{fs}] + (1-t^p)D^i_{fs}\,\mathrm{d}s}{P^i_{js}} &= \chi\,\mathrm{d}s - \frac{\mathrm{d}(1/C_s)}{1/C_s} \quad \forall s, f \end{aligned} \tag{C.384}$$

The differential equations in (C.384) jointly imply an unique equilibrium for the price of equity. Under the equilibrium condition of an unit supply of equity in (C.381), we that

$$V_0 \;=\; P_0 \;=\; \frac{1-t^p}{1-t^g}\;\mathbb{E}_0\left[\int_0^\infty e^{-\tilde{\rho}\frac{1-t^p}{1-t^g}s} D_s\,\mathrm{d}s\right]. \tag{C.385}$$

Finally, the zero profit conditions for the final– and capital–good producers imply the following relationships for the input and output prices of the respective goods:

$$p_{ft} \;=\; \frac{1}{u_{ft}} \quad ; \quad p^k_{ft}(i) \;=\; p_{ft}\varphi(i), \tag{C.386}$$

where $\varphi(i) = \varphi^{+}\mathbb{1}_{\{i<0\}} + \varphi^{-}\mathbb{1}_{\{i>0\}}$ and $p^{k}_{ft}(i)$ stands for the relative price of capital.

Equilibrium policy of intermediate good firms. With these facts about equilibrium prices established, we turn to the problem facing an individual intermediate–good firm. Let $V(k, u, d)$ be the value of a firm with capital k, productivity u, and capital depreciation deductions d. The sufficient optimality conditions satisfied by a firm's policy are: (i) the HJB equation valid during periods of inactivity, (ii) the value matching conditions, and (iii) the smooth pasting conditions. The firm policy consists of an inaction region $\mathcal{R} \equiv \{(k, u, d) : k^{-}(u, d) \le k \le k^{+}(u, d)\}$, where $k^{-}(u, d)$ and $k^{+}(u, d)$ are the lower and upper inaction thresholds, together with a reset capitals $k^{*-}(u, d)$ and $k^{*+}(u, d)$ for positive and negative investments upon adjustment.

Define the PDV of depreciation deductions:

$$z \equiv \frac{\xi^{d}}{\tilde{\rho}\frac{1-t^{p}}{1-t^{g}} + \xi^{d}}. \tag{C.387}$$

Also let $r \equiv \tilde{\rho}(1 - t^{p})(1 - t^{g}) - \mu$ (without subtracting $\sigma^{2}/2$, in contrast to the main text) be the adjusted discount factor and let $v(\hat{k}) : \mathbb{R} \to \mathbb{R}$ be a function of the log capital-productivity ratio equal to

(C.388)
$$v(\hat{k}) = \max_{\tau,\Delta\hat{k}} \mathbb{E}\left[\int_{0}^{\tau} (1 - t^{c})e^{-rs+\alpha\hat{k}_{s}}\,\mathrm{d}s + e^{-r\tau}\left(-\theta(1 - t^{c}z) - p(\Delta\hat{k})(e^{\hat{k}_{\tau}+\Delta\hat{k}} - e^{\hat{k}_{\tau}}) + v(\hat{k}_{\tau} + \Delta\hat{k})\right)\Big|\hat{k}_{0} = \hat{k}\right]$$

where the price function with taxes now given by:

$$p\left(i_{s}\right) = \left(1 - t^{c}z\right)\left(\varphi^{-}\mathbb{1}_{\{i_{s}>0\}} + \varphi^{+}\mathbb{1}_{\{i_{s}<0\}}\right). \tag{C.389}$$

Then the firm value equals $V_{0} = \frac{1-t^{p}}{1-t^{g}}\left[v(\hat{k}_{0}) + t^{c}zd_{0}\right]$.

A few remarks about the firms' investment policy are in place. The formulations of the capital quality shocks and the adjustment costs allow us to collapse the state-space of the firms (k, u) into the capital-to-productivity ratio $\hat{k} = k/u$. Note, that the value of the firm $v(\hat{k}_{0})$ is not scale by the level of productivity to recover the time-0 value V_{0}.The prices of intermediate goods p_{ft} and capital goods p^{k}_{ft}, as well as the adjustment costs θ_{ft}, are proportional to capital quality u_{ft}, making profits and investment scaled by total productivity the relevant variables for the firm.

Equilibrium determination of macroeconomic outcomes. With equilibrium prices and firms' policies, we can determine equilibrium aggregate quantities. Lemma D.4 characterizes the equilibrium detrended aggregate quantities: It shows that all aggregates are functions of the distribution of capital-to-productivity ratios.

Lemma D.4. *Let $g(\hat{k})$ be the density of capital-to-productivity ratios and define the following expectations:* $\mathbb{E}\left[\exp(\hat{k})\right] \equiv \int_{\hat{k}^{-}}^{\hat{k}^{+}} \exp(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}$. *Then, the equilibrium aggregate output is*

$$\hat{Y}_{t} \equiv \int_{0}^{1} p_{ft}y_{ft}\,\mathrm{d}f = \mathbb{E}\left[\exp(\alpha\hat{k}_{t})\right], \tag{C.390}$$

In principle, we cannot sum firms' capital since they are different goods, but as in the main text, the only input to determine output in this economy is capital–productivity. Thus, we define aggregate capital as

$$\hat{K}_{t} = \mathbb{E}\left[\exp(\hat{k}_{t})\right]. \tag{C.391}$$

Equations (C.390) and (C.391) shows an important property in this economy: Without fixed cost of adjustment and partial irreversibility, the supply side of this model collapses to a neoclassical firm with technology $\hat{Y} = \hat{K}^{\alpha}$. Doing a first order approximation on equation (C.391), it follows the CIR(δ) definition in 37.

With these facts over aggregate quantities, we now describe misallocation and the Tobin's q in our economy. Misallocation is define as the dispersion of the log of productivity weighted marginal revenue given by

$$\mathbb{V}\left[\log\left((1 - t^{c})u_{f}\frac{\mathrm{d}p_{f}y_{f}}{\mathrm{d}k_{f}}\right)\right] = \mathbb{V}\left[\log\left((1 - t^{c})\right) + \log\left(\frac{u_{f}}{k_{f}}\left(\frac{k_{f}}{u_{f}}\right)^{\alpha}\right)\right] = (1 - \alpha)\mathbb{V}\left[\hat{k}\right]. \tag{C.392}$$

The argument of why we need weight according to capital capital quality comes from the technology to produce investment. If the rate of transformation from consumption goods to firm–specific investment good is one, then there is no need to weight with capital capital. This is not the case, in our economy since the rate of transformation from consumption goods to firm–specific investment good is given by u_f. This the main reason we need to weight by the idiosyncratic capital quality shocks to obtain misallocation or productivity in this economy.

With a similar discussion as before, the aggregate Tobin's q is given by

$$\frac{\int_0^1 u_f \frac{\mathrm{d}V(k_f,u_f,d_f)}{\mathrm{d}k_f}}{p\hat{K}} = \mathbb{E}[v'(\hat{k})]. \tag{C.393}$$

C.3 Remarks on the economic framework

General equilibrium structure. Capital quality u_{ft} was first used in Baley and Blanco (2021) in the investment context. In the pricing literature, an analogous formulation was first used by Woodford (2009) to keep the tractability of their model. It is also used by Midrigan (2011), Álvarez and Lippi (2014), Baley and Blanco (2019), and Blanco (2020), among others. This formulation implies that aggregate feasibility only depends on firms' capital-to-productivity ratios instead of capital and productivity separately. As a result, capital quality shocks reduce the dimensionality of the aggregate state space from the joint distribution of capital and productivity to the distribution of their ratio.

Partial irreversibility. The price wedge is a technological constraint in the capital–good producer, and therefore, it is exogenous. We choose this modeling strategy to focus on its consequences for capital allocation, capital valuation, and capital dynamics. This formulation follows Veracierto (2002) and Khan and Thomas (2008). Alternative, partial irreversibility could be the outcome of distortionary taxation. For example, Chen *et al.* (2019) use China's 2009 VAT reform to study changes in the level of partial irreversibility. It would be easy to extend our framework to micro-found partial irreversibility as an outcome of a tax system as in Chen *et al.* (2019). See Lanteri (2018) for a model that endogenies partial irreversibility.

Financial markets. We assume that the representative household can trade in the bond market, but the economy is closed to the equity market. That is, only the household in the small open economy is the firms' owner, and it provides the firms' discount factor. While these are extreme assumptions, they are a reasonable approximation of small open economies. On the empirical ground, it is well known that central banks, firms, and households in emerging economies tend to save in dollar-denominated risk–free assets (e.g., T-bill). Moreover, it is also well known that despite the globalization of finance and financial institutions, market participants put the majority of their wealth into assets from their own country. This "home bias" might be due to regulatory constraints, information and transaction costs, but some are regarded as a matter of taste. While the current version of the model is an extreme version of the "home bias" facts, it provides a starting point to analyze the macroeconomic consequence of corporate taxation without considering the effect of personal taxation in other countries though its effect in the firms discount factor.

Tractability. Given the novelty of the general equilibrium framework, a further discussion of the assumptions and economic adjusting mechanisms is warranted. The tractability of our framework arises from three features. First, all aggregate variables are express in terms of the distribution of capital-to-productivity ratios. This result comes directly from how we introduce capital quality shocks and the shape of capital adjustment costs. Second, the model generates a constant real interest rate due to the small open economy assumption. Third, by the close equity market assumption, we can determine the first discount factor as a function of the world interest rate and the personal taxation in the economy.

The theory developed in the main text takes as given that the cross-sectional distribution of capital-to-productivity ratios is the relevant aggregate state, exogenous interest rate, and taxation in general equilibrium. The general equilibrium framework presented here provides a microfoundation to analyze the macroeconomics consequences of corporate tax changes.

D Data Appendix

This Appendix describes the Chilean tax and investment data.

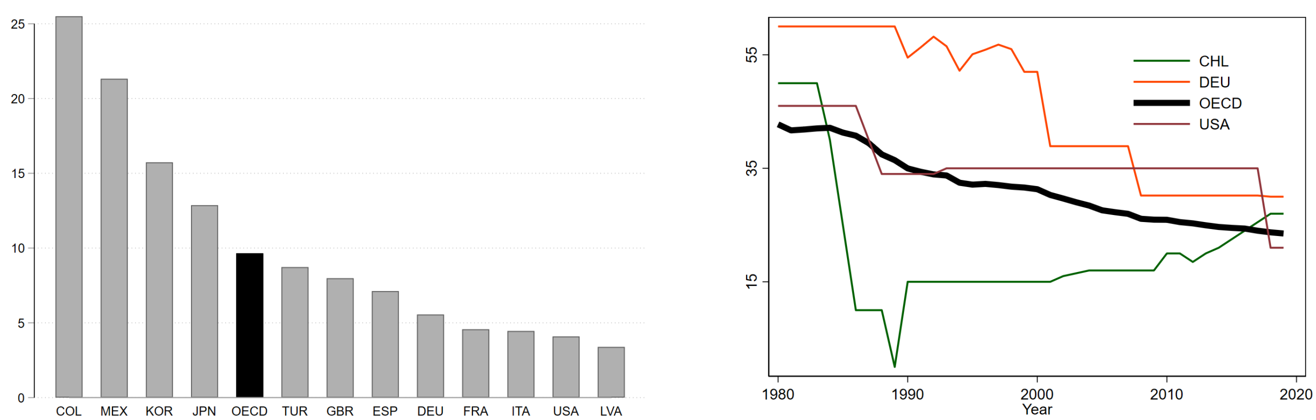
D.1 OECD Statistics

Among OECD countries, corporate income tax revenue in 2018 accounted for an average of 10% of total tax revenue, ranging from 3.4% in Latvia to 25% in Colombia (Panel A in Figure D.2). The importance of corporate taxation remains large, despite a generalized falling trend in tax rates over the last four decades; in particular, the median corporate income tax rate has decreased from 42% in 1980 to 25% in 2020. At the country level, corporate tax reforms happen infrequently and thus are very persistent. In the US, for instance, only two reforms in the corporate income tax rate have occurred in the last 40 years, in 1986 and 2018. (Panel B in Figure D.2)

Figure D.2 – Corporate Taxes in OECD Countries

(A) Corporate income tax revenue
(% of total tax revenue)

(B) Corporate income tax rate
(%, highest marginal rate)



Source: OECD Revenue Statistics Database. Corporate income tax revenue includes corporate income tax and capital gains tax revenue. Data for the largest OECD countries in terms of GDP and the countries with the lowest and the highest value in the sample.

D.2 Corporate Tax Policy in Chile

Policy Context.

- The 1974 tax reform introduced different corporate income tax rates depending on the type of enterprise, at least up to 1984. We decided to take to keep rates applicable to corporations (*sociedades anónimas)* as those were the highest ones, including surtaxes.
- At least since 1984, Chile maintains an **integrated tax system**. It owes its name to the fact that the system integrates corporate income, capital gains and personal incomes.
- In 2014, a tax reform implemented during Bachelet's second presidency created the so-called **semi-integrated tax system** for corporate income. Under this scheme, which came into force in 2017, business owners pay taxes over rents or distributed rents. Taxes over regular enterprise profits may be deferred if those earnings are reinvested. Another important distinction is that corporation can not longer take the total amount of corporate taxes as credit for the global complementary tax. Finally, marginal rates are higher under this system. As a consequence, for 2017, 2018 and 2019 we decided to take those rates as our observations for corporate income tax rate.

Tax rates.

- **Corporate income tax t^c.** Corporate income tax rates apply to business owners over the profits (in broad sense, including capital gains) generated by the enterprises they own (*impuesto de primera categoría*).
- **Personal income tax t^p.** Personal income (for example, salary, fees, pension) is also taxed, progressively according to income level (*impuesto de segunda categoría).*
- **Global complementary tax.** Both sources of income (corporate and personal) constitute the base of the global complementary tax (*Impuesto Global Complementario*). By that means, ultimately all sources of income are taxed under a single scheme. Consequently, to avoid double taxation, the amount payed as corporate income tax can be taken as tax credit when facing the annual global complementary tax[25].
- **Capital gains tax t^g.** Chile does not features a specific tax for capital gains, which are taxed by the corporate income tax. Ultimately personal income tax rates apply to all sources of income. Therefore, in our data set capital gains rates coincides with personal income tax rates every year.

D.2.1 Data sources

We searched for major tax reforms in Chile starting from 1974. We cross-checked several sources: Arenas de Mesa (2016), Barris et al. (1994), Boylan (1996), Cheyre (1986), Fairfield (2015), Focanti et al. (2016), Mahon (2004), Marfán (2001) and Weyland (1997). Compiling data included in those sources, we managed to construct new time series for corporate and personal income tax rates. We always keep the top marginal rate to ensure consistency.

Table C.II – Mapping of data sources for Table C.I

1	Arenas de Mesa (2016)	6	Focanti et al. (2016)
2	Boylan (1996)	7	Mahon (2004)
3	Carciofi et al. (1994)	8	Marfan (2001)
4	Cheyre (1986)	9	Weyland (1997)
5	Fairfield (2015)	10	OECD data

[25]Non residents are taxed by *Impuesto Adicional* instead of *Impuesto Global Complementario*. The former features a flat rate that nowadays it is set on 35%.

We present a table below with specific sources that give support to each observation for Chilean time series.

Table C.I – Sources of Chilean data

year	vat_tr	$t^p = t^g$	t^c
1975	3,4,7	4	3,4
1976	3,4	4	3,4
1977	3,4	4	3,4
1978	3,4	4	3,4
1979	3,4	4	3,4
1980	3,4	4	3,4
1981	3,4	4	3,4
1982	3,4	4	3,4
1983	3,4	4	3,4
1984	3,4	4	3,4
1985	3	3	3
1986	3	3	3
1987	3	3	3
1988	2,3,8	3	3
1989	1,3,6	2,6,8	2,8
1990	1,3,5,6,8	2,6,8,9	2,3,5,6,8,9
1991	1,5,6,8	2,6,8	1,3,6,8
1992	1,5,6,8	2,6,8	1,3,6,8
1993	1,5,6,8	2,6,8	1,3,6,8
1994	1,5,6,8	1,6,8	1,5,6
1995	1,5,6,8	6,8	1,5,6
1996	1,5,6,8	6,8	1,5,6
1997	1,5,6,8	6,8	1,5,6
1998	1,5,6,8	6,8	1,5,6
1999	1,5,6	6	1,5,6
2000	1,5,6	5,6	1,5,6
2001	1,5,6	5,6	1,5,6
2002	1,5,6	6	1,5,6
2003	1,5,6	6	1,5,6
2004	1,5,6	6	1,5,6
2005	1,5	5,10	1,5
2006	1,5	5,10	1,5
2007	5	5,10	1,5
2008	5	5,10	1,5
2009	5	5,10	1,5
2010	5	5,10	1,5
2011	5	5,10	1,5
2012	5	5,10	1,5
2013	5	5,10	1,5
2014	1	1,10	1
2015	1	1,10	1
2016	1	1	1
2017	1	1	1
2018	10	1	1
2019	10	10	10

D.2.2 Time series of corporate income tax rate in Chile

One important feature of our data is that it diverges from Vegh and Vuletin (2015) inputs, specially before the year 2000. Below, we transcribe the two series and present some plots to point out differences between data sets.

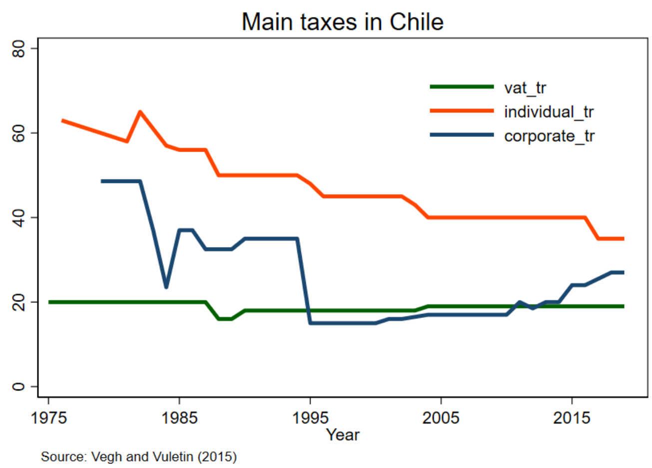


Figure D.3 – Top marginal rates in Chile using data by Vegh and Vuletin (2015) data

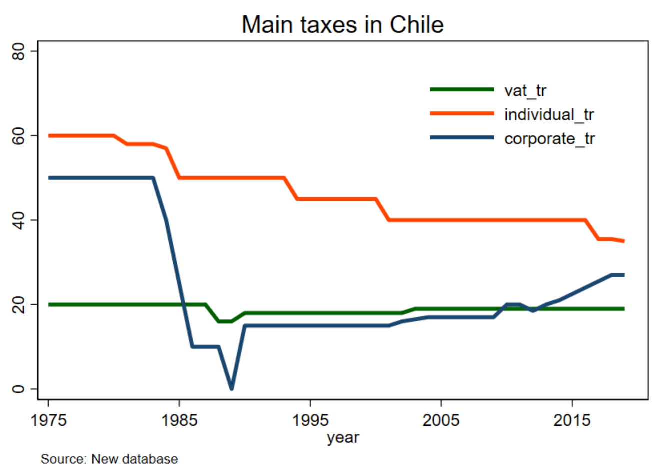


Figure D.4 – Top marginal rates in Chile using sources listed in Table C.II

D.3 Establishment-level investment data

This section describe the sources, the construction of variables, and the filters we apply to clean the data in order to construct the investment series at the firm level in order

D.3.1 Source, description and data cleaning

Data come from the *Encuesta Nacional Industrial Anual* (ENIA). The sample period covers 31 years, from 1980 to 2011, with an average of 543 manufacturing plants per year. We have a total number of plant-year observations of 154,591.

1. First, we drop the 3,984 permanently small firms (i.e. with less than 10 workers throughout the sample period, 4% of sample). This filter is motivated by the lack of good quality data with respect to these firms since ENIA is directed to plant with more than 10 workers.
2. Second, we drop 5,343 observations with non-positive total value of book capital, wage bill or sales.
3. Third, we drop 12,161 observations that had a frequency of non-zero investment lower than 10% of the sample period.
4. Finally, we drop 5,556 plants with less than 3 years of coverage (6% of the sample). Note that we consider as new plants (and give a new ID) those that disappear from the sample more than three years and reappear in the sample after that.
5. In total, we drop about 18% of the year-plant observations and keep 127,631 observations. Within this remaining sample, a balanced panel would maintain 101,160 plant-years.

Table D.3 summarizes the cleaning process and shows the number of observations dropped at each step.

Table D.3 – Data cleaning

Description	**Chile**
Start year	1980
End year	2011
Avg. number of plants per year	543
Plant-year observations	154,591
Cleaning	**Removed observations**
Less than 10 employees	3,984
Non-positive wage bill, capital, or sales	5,343
Frequency of non-zero investment less than 10	12,161
Less than 3 years of coverage	5,556
Remaining observations	127,631
% of total	82.6
With more than 10 years of data	101,160
% of remaining observations	79.3

Sources: Authors' calculations using establishment-level survey data from Chile. Less than 10 employees refers to plants with less than 10 employees for all the years in the sample.

D.3.2 Perpetual Inventory Method

In order to deal with reporting and measurement errors in the surveys, we construct capital series using the standard perpetual inventory method (PIM) with the addition of an investment price wedge.

Capital stocks. Let firm's i stock of capital on year t be given by:

$$k_{i,t} = (1-\xi^k)k_{i,t-1} + \frac{I_{i,t}}{p(I_t)D_t} \quad \text{for } k_{i,t_0} \text{ given.} \tag{D.394}$$

We consider the following elements to construct the capital series:

- Capital types considered are $j \in \{\text{structures, machinery and equipment, vehicles}\}$.
- Gross investment: $I_{i,j,t}$ is gross nominal investment into capital of type j at time t, and it is based on the information on purchases, reforms and improvements, and sales of fixed assets reported by each plant in the ENIA and EAM data sets.

  $$I_{i,j,t} = puchases_{i,j,t} + reforms_{i,j,t} + improvements_{i,j,t} - sales_{i,j,t} \tag{D.395}$$

- Depreciation rate: $\xi^k = 0.09$ is a the depreciation rate.
- Price deflators: $D_{j,t}$ are gross fixed capital formation deflators by capital type from Penn World Tables (PWT).
- Investment prices and wedge.
- Initial capital: K_{i,j,t_0} is given by:

  $$K_{i,j,t_0} = \frac{\tilde{K}_{i,j,t_0}}{D_{t_0}} \quad \text{if } \tilde{K}_{i,j,t_0} \geq 0, \tag{D.396}$$

  where $\tilde{K}_{i,j,t_0}$ is firm i's self-reported nominal stock of capital of type j at current prices on the starting year $t_0 = t_{0,i,j}$, which is the first year in which firm i reports a non-negative capital stock of type j.

Investment rates Once we construct the investment and capital stock series, we generate the investment rate $i_{i,j,t}$ by dividing investment by initial capital:

$$i_{i,j,t} = \frac{I_{i,j,t}}{K_{i,j,t-1}}, \tag{D.397}$$

Outliers. In both countries, once we generate the series of investment rates, we eliminate investment rates below the 2nd percentile and above the 98th percentile of the investment rate distribution.

Figure D.5 plots the aggregate capital stock computed with the perpetual inventory method and compare it to the reported book value. We observe that, in the aggregate, the reported book value is consistent with the PIM series, for each capital-type and for the total stock. This shows the good quality of the micro-data. Moreover, given the similarity in the series, we validate our choice of using the initial book value reported by the a plant as the initial condition for the PIM construction.

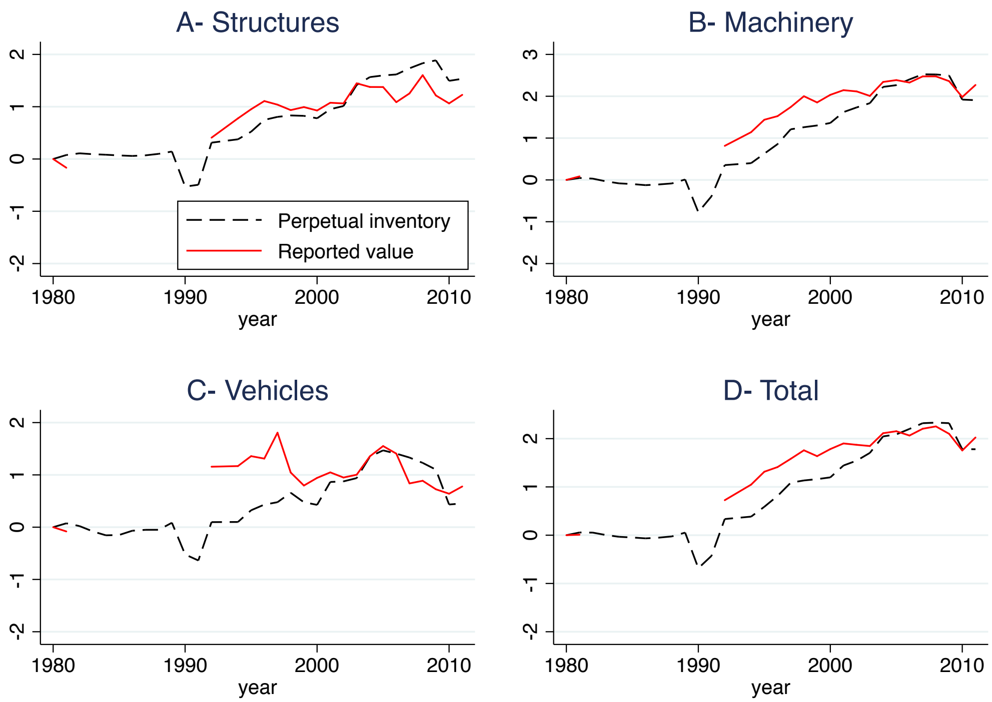
D.3.3 Comparison with National Accounts

This section verifies that the information contained in the survey data is consistent with aggregated information from National Accounts.

The national account office in Chile uses the ENIA survey as a source to compute several indices, such as variations in inventories or value added by type of industry. Nevertheless, National Accounts does not use ENIA to compute total investment or investment in the manufacturing sector; for that purpose, it uses sources related to the supply of capital goods (i.e., balance of payments, national statistical institute, Corporacion de Desarrollo Tecnologico de Bienes de Capital). Therefore, National Accounts serve as an orthogonal source to verify that the micro-data from the survey is consistent with the total investment in the manufacturing sector.

Panel A of Figure D.6 describes total nominal investment constructed from the ENIA (dashed black line) and the total nominal investment in the manufacturing sector constructed using National Accounts (solid blue line), in

Figure D.5 – Chile: Reported Book Value vs Perpetual Inventory



Notes: Aggregate capital stock in Chile's manufacturing sector reported by plants and computed through the PIM. All the variables are in logs and real terms, normalized to zero in 1980.

current millions of pesos. As we can see, the two series are very close to each other with a correlation of 0.62. Total investment from the micro-data for the period 2005-2009 seems to growth at a much faster rate than National Accounts, but we found that this is mainly explained by a few outliers. For example, if we drop observations with investment rates larger than 5% of aggregate investment (dashed-dotted red line), then the fit between the aggregate investment from the micro-data and the national account is much better, both in levels and cyclicality. Finally, Panel B of D.6 describes the proportion of total investment that is done in the manufacturing sector, which represents on average 7% in the sample period, but has been declining.

For the year 2003-2009, the National Accounts calculates the distribution of investment by capital types at the sector level. Table D.4 describes the composition of capital across different types from the ENIA and from National Accounts. We find that the proportions invested in structures are similar between National Accounts and in ENIA, but the decomposition between machinery and vehicles is different across datasets.

Table D.4 – Chile: Distribution of Investment Across types of Capital

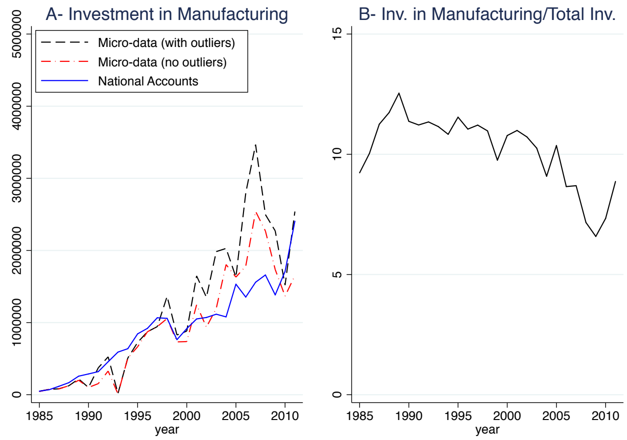
	Structures	Machinery	Vehicles
National Accounts	35.4	51.4	13.1
ENIA	29.1	68.6	2.1

Notes: Proportion of investment across different types of capital in the ENIA and national accounts.

D.3.4 Inputs from data

In this section, we construct cross-sectional statistics using the panel data on $(\Delta x, \tau)$ and then use them as inputs into our observation formulas to recover parameters, steady state moments and the CIR. First, we explain how to

Figure D.6 – Chile: Micro-data vs. National Accounts



Notes: Panels A describes investments in the manufacturing sector. Blacked dashed line plots aggregate nominal investment constructed from ENIA, red dashed-dotted line plots the same variable but dropping outliers (i.e., investment larger than 5% of aggregate investment), and the blue solid line plots the total investment from National Accounts. Panels B describes investment in the manufacturing sector over total investment. Nominal investment from national account uses the concatenated investments from the base year 2015.

use the data on investment rates to construct changes in capital gaps, and second, we explain how to construct the stopping time distribution and some challenges.

Capital gaps. Recall that the change in capital gaps is given by the log difference in the capital stock between an adjustment date $\tau_{i,j,t}$ and immediately before adjustment $\tau^-_{i,j,t}$:

$$\Delta x_{i,j,t} = \hat{x} - x_{\tau^-_{i,j,t}} = \log\left(K_{\tau_{i,j,t}}/K_{\tau^-_{i,j}}\right) = \log\left(1 + i_{i,j,t}\right) \tag{D.1}$$

Using the information on investment rates, we construct capital gaps as:

$$\Delta x_{i,j,t} = \begin{cases} \log\left(1 + i_{i,j,t}\right) & \text{if} \quad |i_{i,j,t}| > \underline{i} \\ 0 & \text{if} \quad |i_{i,j,t}| < \underline{i}, \end{cases} \tag{D.2}$$

where $\underline{i} > 0$ is a parameter that captures the idea that small maintenance investments do not incur the fixed cost of investment. Following Cooper and Haltiwanger (2006), we set $\underline{i} = 0.01$, such that all investments smaller than 1% in absolute value are excluded and considered as part of the inaction frequency.

D.4 Mappings from microdata to macro outcomes

This section describes the application of the theory with producer level investment data. Let I_{ft} be the nominal investment in period t for firm f. The sequence of steps describe next, describe the steps to compute all the macroeconomic outcomes describe in Table II.

Construction of $(\Delta\hat{k}_{fh}, \tau_{fh})_{fh}$ and weights ω_f: We follow Baley and Blanco (2021) Online Appendix L to drop observations that do not satisfy a set of criteria (e.g., positive sales, wage bill, minimum number of year, etc.). We apply perpetual inventory to construct the capital stock k_{ft} of firm f in period t

$$k_t \;=\; (1-\xi^k)k_{t-1} \;+\; I_t/(p(I_t)D_t), \tag{D.3}$$

where ξ^k is the physical depreciation rate; I_{ft} is the nominal value of investment; $p(I_{ft})$ is the investment pricing function, which considers different prices for capital purchases and sales; D_t is the gross fixed capital formation deflator, and k_{f0} is a plant's self-reported nominal capital stock at current prices for the first year enter in the data or its firms investment level. We construct the change in the capital-productivity ratio upon action $\Delta\hat{k}_h$ as

$$\Delta\hat{k}_{fh} \;=\; \begin{cases} \log\left(1+\frac{I_{ft}/(p(I_{ft})D_t)}{k_{ft-1}}\right) & \text{if} \quad |\iota_h| > \underline{\iota}, \\ 0 & \text{if} \quad |\iota_h| < \underline{\iota}. \end{cases} \tag{D.4}$$

With $\Delta\hat{k}_{fh}$, we construct τ^{fh} as the number of periods between non-zero investments.

Unobserved heterogeneity in the frequency of non-zero investment can generate a bias in the estimation of moment relevant for the theory. For that reason, we construct firms' weights using the inverse of the frequency of non-zero investments. By definition of weights

$$\sum_{fh}\omega_f = 1. \tag{D.5}$$

Estimation of $(\hat{k}^{*-}, \hat{k}^{*+}, \sigma)$: First we estimate the drift as:

$$\nu = \frac{\sum_{fh}\Delta\hat{k}_{fh}\omega_f}{\sum_{fh}\tau_{fh}\omega_f}, \tag{D.6}$$

With the estimate of the drift we design an iterative method to estimate $(\hat{k}^{*-}, \hat{k}^{*+}, \sigma^2)$. The method constructs a sequence $(\hat{k}^{*-}_j, \hat{k}^{*+}_j, \sigma^2_j)_{j=0}^{\infty}$ that converges to the solution of the implicit equations (51), (54), and (55) from Section 3.

0. Fix a convergence parameter $\xi > 0$ and a dumpening parameter $\Gamma \in (0,1)$.
1. Construct $(\hat{k}^{*-}_0, \hat{k}^{*+}_0, \sigma_0)$ assuming no partial irreversibility: Under the assumption of no partial irreversibility, we can estimate σ^2 as

$$\sigma_0^2 = \frac{\sum_{fh}\Delta\hat{k}^2_{fh}\omega_f}{\sum_{fh}\tau_{fh}\omega_f} - 2\nu\left(\frac{\sum_{fh}\Delta\hat{k}_{fh}\omega_f}{2}(1-\mathbb{CV}[\tau]) + \frac{\mathbb{C}ov[\Delta hk,\tau]}{\sum_{fh}\tau_{fh}\omega_f}\right) \tag{D.7}$$

$$\mathbb{CV}[\tau] := \frac{\sum_{fh}(\tau_{fh}-\bar{\tau})^2\omega_f}{\left(\sum_{fh}\tau_{fh}\omega_f\right)^2}, \quad \text{where} \quad \bar{\tau} = \sum_{fh}\tau_{fh}\omega_f, \tag{D.8}$$

$$\mathbb{C}ov[\Delta hk,\tau] := \sum_{fh}(\tau_{fh}-\bar{\tau})(\Delta\hat{k}_{fh} - \bar{\Delta\hat{k}})\omega_f, \quad \text{where} \quad \bar{\Delta\hat{k}} = \sum_{fh}\Delta\hat{k}_{fh}\omega_f. \tag{D.9}$$

With σ_0^2, we compute

$$\Phi = \log\left(\frac{\alpha(1-t^c)}{\tilde{r} + \alpha\nu - \frac{\sigma_0^2\alpha^2}{2}}\right) \tag{D.10}$$

$$\hat{k}_0^{*-} = \frac{1}{1-\alpha}\left(\Phi - \log((1-t^d)p^{\text{buy}}) + \log\left(\frac{1-\mathcal{N}um_0^-}{1-\mathcal{D}en_0^-}\right)\right) \tag{D.11}$$

$$\hat{k}_0^{*+} = \frac{1}{1-\alpha}\left(\Phi - \log((1-t^d)p^{\text{buy}}) + \log\left(\frac{1-\mathcal{N}um_0^+}{1-\mathcal{D}en_0^+}\right)\right) \tag{D.12}$$

where the numerators and denominators in the last terms are computed as:

$$\mathcal{N}um_0^+ = \frac{\sum_{fh}\exp(-\tilde{r}\tau_{fh} - \alpha\Delta\hat{k}_{fh})I(\Delta\hat{k}_{fh-1} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh-1} < 0)\omega_f} \tag{D.13}$$

$$\mathcal{D}en_0^+ = \frac{\sum_{fh}\exp(-\tilde{r}\tau_{fh} - \Delta\hat{k}_{fh})I(\Delta\hat{k}_{fh-1} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh-1} < 0)\omega_f} \tag{D.14}$$

$$\mathcal{N}um_0^- = \frac{\sum_{fh}\exp(-\tilde{r}\tau_{fh} - \alpha\Delta\hat{k}_{fh})I(\Delta\hat{k}_{fh-1} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh-1} > 0)\omega_f} \tag{D.15}$$

$$\mathcal{D}en_0^- = \frac{\sum_{fh}\exp(-\tilde{r}\tau_{fh} - \Delta\hat{k}_{fh})I(\Delta\hat{k}_{fh-1} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh-1} > 0)\omega_f} \tag{D.16}$$

Define $\hat{k}_0^*(\Delta\hat{k})$ as:

$$\hat{k}_0^*(\Delta\hat{k}) = \hat{k}_0^{*-}\mathbb{1}_{\{\Delta\hat{k}>0\}} + \hat{k}_0^{*+}\mathbb{1}_{\{\Delta\hat{k}<0\}}, \tag{D.17}$$

2. For $j = 1, 2, 3, \dots, J$, compute $(\tilde{\hat{k}}_j^{*-}, \tilde{\hat{k}}_j^{*+}, \tilde{\sigma}_j^2)$ as

$$\tilde{\sigma}_j^2 = \frac{\sum_{fh}(\hat{k}_{j-1}^*(\Delta\hat{k}_{fh}) - \Delta\hat{k}_{fh} + \nu\tau_{fh})\omega_f - \sum_{fh}\hat{k}_{j-1}^*(\Delta\hat{k}_{fh})\omega_f}{\sum_{fh}\tau_{fh}\omega_f} \tag{D.18}$$

$$\Phi_j = \log\left(\frac{\alpha(1-t^c)}{\tilde{r} + \alpha\nu - \frac{\sigma_{j-1}^2\alpha^2}{2}}\right) \tag{D.19}$$

$$\tilde{\hat{k}}_j^{*-} = \frac{1}{1-\alpha}\left(\Phi - \log((1-t^d)p^{\text{buy}}) + \log\left(\frac{1-\mathcal{N}um_j^-}{1-\mathcal{D}en_j^-}\right)\right) \tag{D.20}$$

$$\tilde{\hat{k}}_j^{*+} = \frac{1}{1-\alpha}\left(\Phi - \log((1-t^d)p^{\text{sell}}) + \log\left(\frac{1-\mathcal{N}um_j^+}{1-\mathcal{D}en_j^+}\right)\right) \tag{D.21}$$

where the numerators and denominators in the last terms are computed as:

$$\mathcal{N}um_j^+ = \frac{\sum_{fh} \exp\left(-\tilde{r}\tau_{fh} + \alpha\left(\hat{k}^*_{j-1}(\Delta\hat{k}_{fh}) - \hat{k}^{*+}_{j-1} - \Delta\hat{k}_{fh}\right)\right) I(\Delta\hat{k}_{fh-1} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh-1} < 0)\omega_f} \tag{D.22}$$

$$\mathcal{D}en_j^+ = \frac{\sum_{fh} \frac{p(\Delta\hat{k}_{fh})}{p(\Delta\hat{k}_{fh-1})} \exp\left(-\tilde{r}\tau_{fh} + \hat{k}^*_{j-1}(\Delta\hat{k}_{fh}) - \hat{k}^{*+}_{j-1} - \Delta\hat{k}_{fh}\right) I(\Delta\hat{k}_{fh-1} < 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh-1} < 0)\omega_f} \tag{D.23}$$

$$\mathcal{N}um_j^- = \frac{\sum_{fh} \exp\left(-\tilde{r}\tau_{fh} + \alpha\left(\hat{k}^*_{j-1}(\Delta\hat{k}_{fh}) - \hat{k}^{*-}_{j-1} - \Delta\hat{k}_{fh}\right)\right) I(\Delta\hat{k}_{fh-1} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh-1} > 0)\omega_f} \tag{D.24}$$

$$\mathcal{D}en_j^- = \frac{\sum_{fh} \frac{p(\Delta\hat{k}_{fh})}{p(\Delta\hat{k}_{fh-1})} \exp\left(-\tilde{r}\tau_{fh} + \hat{k}^*_{j-1}(\Delta\hat{k}_{fh}) - \hat{k}^{*+}_{j-1} - \Delta\hat{k}_{fh}\right) I(\Delta\hat{k}_{fh-1} > 0)\omega_f}{\sum_{fh} I(\Delta\hat{k}_{fh-1} > 0)\omega_f} \tag{D.25}$$

Update $(\hat{k}_j^{*-}, \hat{k}_j^{*+}, \sigma_j^2)$ with the dampening parameter Γ

$$\sigma_j^2 = (1-\Gamma)\tilde{\sigma}_j^2 + \Gamma\sigma_{j-1}^2 \tag{D.26}$$

$$\hat{k}_j^{*-} = (1-\Gamma)\tilde{\hat{k}}_j^{*-} + \Gamma\hat{k}^{*+}_{j-1} \tag{D.27}$$

$$\hat{k}_j^{*+} = (1-\Gamma)\tilde{\hat{k}}_j^{*+} + \Gamma\hat{k}^{*+}_{j-1} \tag{D.28}$$

3. Repeat step 2 until there is a j, such that, $||\sigma_j^2 - \sigma_{j-1}^2, \hat{k}_j^{*-} - \hat{k}^{*-}_{j-1}, \hat{k}_j^{*+} - \hat{k}^{*+}_{j-1})|| < \xi$.

We construct the reset points $\hat{k}^*(\Delta\hat{k})$ and stopped capitals $\hat{k}_\tau(\Delta\hat{k})$ as:

$$\hat{k}^*(\Delta\hat{k}) = \hat{k}^{*-}\mathbb{1}_{\{\Delta\hat{k}>0\}} + \hat{k}^{*+}\mathbb{1}_{\{\Delta\hat{k}<0\}} \tag{D.29}$$

$$\hat{k}_\tau(\Delta\hat{k}) = \hat{k}^*(\Delta\hat{k}) - \Delta\hat{k} \tag{D.30}$$

Estimation of misallocation: To estimate misallocation, first we estimate the mean level of capital–productivity ratio as

$$\mathbb{E}[\hat{k}] = \frac{\sum_{fh} \hat{k}^*(\Delta\hat{k}_{fh})^2\omega_f - \sum_{fh} \hat{k}_\tau(\Delta\hat{k}_{fh})^2\omega_f}{2\sum_{fh} \Delta\hat{k}_{fh}\omega_f} + \frac{\sigma^2}{2\nu} \tag{D.31}$$

$$\mathbb{V}[\hat{k}] = \frac{\sum_{fh}(\hat{k}^*(\Delta\hat{k}_{fh}) - \mathbb{E}[\hat{k}])^3\omega_f - \sum_{fh}(\hat{k}_\tau(\Delta\hat{k}_{fh}) - \mathbb{E}[\hat{k}])^3\omega_f}{3\sum_{fh} \Delta\hat{k}_{fh}\omega_f} \tag{D.32}$$

Estimation of aggregate q: We estimate with and without the approximation of the output–capital ratio, and they give similar magnitude. The computation of the output–capital ratio is immediately with equations (D.31) and (D.32). We also computed without the approximation. In this case, capital and output are given by

$$\hat{Y} = \frac{\sum_{fh} \exp\left(\alpha\hat{k}^*(\Delta\hat{k}_{fh})\right)\omega_f - \sum_{fh} \exp\left(\alpha\hat{k}_\tau(\Delta\hat{k}_{fh})\right)\omega_f}{(\nu\alpha - \sigma^2\alpha^2/2)\sum_{fh}\tau_{fh}\omega_f}, \tag{D.33}$$

$$\hat{K} = \frac{\sum_{fh} \exp\left(\hat{k}^*(\Delta\hat{k}_{fh})\right)\omega_f - \sum_{fh} \exp\left(\hat{k}_\tau(\Delta\hat{k}_{fh})\right)\omega_f}{(\nu - \sigma^2/2)\sum_{fh}\tau_{fh}\omega_f}. \tag{D.34}$$

We estimate irreversibility term in q using the following expression:
(D.35)

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{P}(\hat{k}_s)\omega(\hat{k}_s))\right]\right] = \frac{\sum_{fh} \exp\left(\hat{k}^*(\Delta\hat{k}_{fh})\right)\left(\frac{p(\Delta\hat{k}_{fh})}{p} - 1\right)\omega_f - \sum_{fh} \exp\left(\hat{k}_\tau(\Delta\hat{k}_{fh})\right)\left(\frac{p(\Delta\hat{k}_{fh})}{p} - 1\right)\omega_f}{\hat{K}\sum_{fh}\tau_{fh}\omega_f}$$

From equation (D.33), (D.34), and (D.35), we estimate the Tobin's q as

$$q = \frac{1}{\tilde{r}} \left[\frac{\alpha(1-t)\hat{Y}}{p\hat{K}} + \frac{\sigma^2}{2} - \nu + \mathbb{E}\left[\frac{1}{\mathrm{d}s} \mathbb{E}_s \left[\mathrm{d}(\mathcal{P}(\hat{k}_s)\omega(\hat{k}_s)) \right] \right] \right]. \tag{D.36}$$

Estimation of CIR(δ): To estimate the CIR(δ), we need to estimate the covariance and the partial irreversibility term in (38). We estimate the covariance using the sample moment of (58) given by

$$\mathbb{C}ov[\hat{k}, a] = -\frac{\sum_{fh} (\hat{k}_\tau(\Delta \hat{k}_{fh}) - \mathbb{E}[\hat{k}])^2 \tau_{fh} \omega_f}{2\nu\bar{\tau}} + \frac{\mathbb{V}[\hat{k}]}{2\nu} + \frac{\sigma^2}{2\nu} \bar{\tau}(1 + \mathbb{CV}(\tau)), \tag{D.37}$$

where $\mathbb{CV}(\tau)$ and $\bar{\tau}$ are estimated using (D.8), $\mathbb{V}[\hat{k}]$ is estimated using (D.32), and $\mathbb{E}[\hat{k}]$ is estimated using (D.31). We estimate the irreversibility term following its sample counter-part. First, we estimate $\mathcal{M}^{buy}$ and $\mathcal{M}^{buy}$ as:

$$\mathcal{M}^{buy} = (\overline{\mathbb{E}}^-[\hat{k}] - \mathbb{E}[\hat{k}]) \overline{\mathbb{E}}^-[\tau] \frac{\mathbb{E}[\mathbb{P}^+]}{\mathbb{P}^{-+}} < 0, \tag{D.38}$$

$$\mathcal{M}^{sell} = (\overline{\mathbb{E}}^+[\hat{k}] - \mathbb{E}[\hat{k}]) \overline{\mathbb{E}}^+[\tau] \frac{\mathbb{E}[\mathbb{P}^-]}{\mathbb{P}^{+-}} > 0. \tag{D.39}$$

where we compute conditional means as:

$$\overline{\mathbb{E}}^-[\hat{k}] = \frac{\hat{k}^{*-} - \sum_{fh} \hat{k}_\tau (\Delta \hat{k}_{fh})^2 \mathbb{1}_{\{\Delta \hat{k}_{fh-1} > 0\}} \omega_f}{2 \sum_{fh} \Delta \hat{k}_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh-1} > 0\}} \omega_f} + \frac{\sigma^2}{2\nu}, \tag{D.40}$$

$$\overline{\mathbb{E}}^+[\hat{k}] = \frac{\hat{k}^{*+} - \sum_{fh} \hat{k}_\tau (\Delta \hat{k}_{fh})^2 \mathbb{1}_{\{\Delta \hat{k}_{fh-1} < 0\}} \omega_f}{2 \sum_{fh} \Delta \hat{k}_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh-1} < 0\}} \omega_f} + \frac{\sigma^2}{2\nu}, \tag{D.41}$$

conditional durations of inaction as:

$$\mathbb{E}^-[\tau] = \frac{\sum_{fh} \tau_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh} > 0\}} \omega_f}{\sum_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh} > 0\}} \omega_f}, \tag{D.42}$$

$$\mathbb{E}^+[\tau] = \frac{\sum_{fh} \tau_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh} < 0\}} \omega_f}{\sum_{fh} \tau_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh} < 0\}} \omega_f}, \tag{D.43}$$

and probabilities as:

$$\mathbb{P}^{-+} = \frac{\sum_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh} < 0\}} \mathbb{1}_{\{\Delta \hat{k}_{fh-1} > 0\}} \omega_f}{\sum_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh-1} > 0\}} \omega_f}, \tag{D.44}$$

$$\mathbb{P}^{+-} = \frac{\sum_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh} > 0\}} \mathbb{1}_{\{\Delta \hat{k}_{fh-1} < 0\}} \omega_f}{\sum_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh-1} < 0\}} \omega_f}, \tag{D.45}$$

$$\mathbb{E}[\mathbb{P}^+] = \frac{\sum_{fh} \tau_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh} < 0\}} \omega_f}{\bar{\tau}}, \tag{D.46}$$

$$\mathbb{E}[\mathbb{P}^-] = \frac{\sum_{fh} \tau_{fh} \mathbb{1}_{\{\Delta \hat{k}_{fh} > 0\}} \omega_f}{\bar{\tau}}. \tag{D.47}$$

With these result, we obtain the irreversibility term in the CIR(δ) given by

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s} \mathbb{E}_s \left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s) \right] \right] = \frac{\sum_{fh} \left(\mathcal{M}^{buy} \mathbb{1}_{\{\Delta \hat{k}_{fh} > 0\}} + \mathcal{M}^{sell} \mathbb{1}_{\{\Delta \hat{k}_{fh} < 0\}} \right) \Delta \hat{k}_{fh} \omega_f}{\sum_{fh} \tau_{fh} \omega_f}, \tag{D.48}$$

and the CIR(δ) given by

$$\frac{\mathrm{CIR}(\delta)}{\delta} = \frac{\mathbb{V}ar[\hat{k}]}{\sigma^2} + \frac{\nu \mathbb{C}ov[\hat{k}, a]}{\sigma^2} + \mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)]\right] + o(\delta). \tag{D.49}$$

D.5 Results for Chile

Next, we present yearly averages of cross-sectional statistics. Inaction frequency is defined as the fraction of observations with investment below 1% in absolute value; positive spikes are investments above 20% and negative spikes below −20%.

Table D.5 presents the yearly average of cross-sectional statistics by capital category for a balanced panel within the ENIA establishment-level survey data for Chile. Note that the column total considers the statistics for the total capital stock, and it is not the average of the statistics by capital type. For comparison, we include information for the US in Cooper and Haltiwanger (2006) and Zwick and Mahon (2017). Following these papers, investment rates reported in this table are computed as Investment divided by Initial Capital. We use perpetual inventories to compute capital stock.

Table D.5 – Investment Rate Statistics (Chile: by capital type)

	Structures	**Machinery**	**Vehicles**	**Total**	**US I**	**US II**
Average Investment	7.3	17.0	17.1	15.8	12.2	10.4
Positive Fraction ($i > 1\%$)	22.2	54.9	25.5	56.8	81.5	—
Negative Fraction ($i < -1\%$)	0.5	1.5	5.3	3.1	10.4	—
Inaction rate ($\|i\| <= 1\%$)	77.3	43.7	69.2	40.1	8.1	23.7
Spike rate ($\|i\| > 20\%$)	8.9	23.2	21.2	22.8	20.4	14.4
Positive spikes ($i > 20\%$)	8.9	23.2	18.7	22.7	18.6	—
Negative spikes ($i < -20\%$)	0.0	0.0	2.5	0.1	1.8	—
Serial correlation	0.0	0.0	0.0	0.0	0.1	0.4

Notes: Authors calculations using establishment-level survey data for Chile (balanced panel). US I shows data from Cooper and Haltiwanger (2006) and US II shows data reported in Zwick and Mahon (2017) for the balanced panel. Following these papers, investment rates reported in this table are computed as investment divided by initial capital. We use the perpetual inventory method to compute capital stocks. We eliminate investment rates below the 1st percentile and above the 99th percentile of the investment rate distribution.

D.6 Comparative Statics

This section conducts a comparative statics exercise with respect to the irreversibility wedge ω and the returns to scale α. Other parameters are described in Table I.

Table D.6 – Aggregate Capital Behavior: Comparative Statics

		$\omega = 0.15$		$\alpha = 0.5$	
	Benchmark	$\alpha = 0.4$	$\alpha = 0.6$	$\omega = 0.05$	$\omega = 0.25$
Productivity process					
ν	0.11	0.11	0.11	0.12	0.11
σ	0.23	0.23	0.24	0.23	0.24
Investment Policy					
Difference in reset capitals $(\hat{k}^{*+} - \hat{k}^{*-})$	0.568	0.472	0.697	0.221	0.914
Exogenous price wedge	0.325	0.271	0.406	0.102	0.575
Endogenous response	0.243	0.201	0.291	0.118	0.339
Capital Allocation					
Variance	0.098	0.097	0.099	0.096	0.098
Capital Valuation					
Tobin's q	1.06	1.07	1.05	1.07	1.05
Productivity	1.09	1.10	1.08	1.08	1.10
Irreversibility	-0.03	-0.03	-0.03	-0.01	-0.05
Capital Fluctuations					
CIR	3.07	3.71	2.50	3.40	2.62
Responsiveness	2.29	2.37	2.17	2.51	1.93
Irreversibility	0.77	1.33	0.33	0.89	0.69

Notes: Objects recovered from theory mappings applied to establishment-level data from Chile. Comparative statics with respect to the wedge ω and the returns to scale α. Other parameters are described in Table I.