Industry Dynamics and Search in the Labor Market

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October 6, 2008

Abstract

The paper proposes a model of on- and off-the-job search that combines convex hiring costs and directed search. Firms permanently differ in productivity levels, their production function features constant or decreasing returns to scale, and search costs are convex in search intensity. Wages are determined in a competitive manner, as firms advertise wage contracts (expected discounted incomes) so as to balance wage costs and search costs (queue length). An important assumption is that a firm is able to sort out its coordination problems with their employees in such a way that the on-the-job search behavior of workers maximizes the match surplus. Our model has several interesting features. First, it is close in spirit to the competitive model, with a tractable and unique equilibrium, and is therefore useful for empirical testing. Second, the resulting equilibrium gives rise to an efficient allocation of resources. Third, the equilibrium is characterized by a job ladder: unemployed workers search for low-productivity, low-wage firms. Workers in low-wage firms search for firms slightly higher on the productivity/ ladder, and so forth up to the workers in the second most productive firms who only apply to the most productive firms. Finally, the model rationalizes empirical regularities of on-the-job search and labor turnover. First, job-to job mobility falls with average firm tenure and firm size. Second, wages increase with firm size, and wage growth is larger in fast-growing firms.

1 Introduction

In the real economy, firm-and industry dynamics play an important role. Firms are born, expand and contract. Resources are allocated from less productive to more productive firms, and thereby improve the allocation of resources. There is substantial evidence that reallocation of resources on firms is important for economic growth, and Baily, Hulten and Campbell (1992) argues the about half of overall productivity growth in the U.S. manufacturing in the 80ies can be attributed to this. Existing empirical evidence also shows that industry dynamics is associated with large worker flows, not only in and out of unemployment, but even more importantly as direct job to job movements (Haltiwanger, 1999; Foster et al., 2007; Bartelsman et al; 2005). Lentz and Mortensen (2005, 2006) decompose the effect of firm selection on the growth rate, and then estimate that it accounts for 58 percent of the growth rate

Several recent papers analyze models of industry dynamics (Hopenayn, 1992; Hopenayn and Rogerson, 1993, Melitz 2003, Klette and Kortum 2004). However, these papers typically do not take into account that the factor markets, and in particular the labor market, may contain frictions. (An exception here is Lents and Mortensen (2007), who do include a frictional labour market in a Klette-Kortum model of innovation-driven industry dynamics).

This paper studies the joint determination of worker flows and firm dynamics with on the job search. The model contains three key elements. First, it applies the *competitive search equilibrium* concept, initially proposed by Moen (1997). Thus, firms post wages and post a number of vacancies so as to minimize search -and waiting costs. Furthermore, the labor market is endogenously separated into submarkets so that in each submarket, all agents at the same side of the market are identical.

Second, we assume that firms have access to a search technology with *convex hiring costs* (Bertola and Cabalero, 1992; Bertola and Garibaldi, 2001). In the traditional search model (Mortensen and Pissarides, 1994) adjustment costs are linear. Together with constant returns to scale in production, this implies that the size of the firms typically is undefined. Our assumption of convex hiring costs allows firms with different productivity and with constant-returns to scale technology to coexist in the market.

Third, we follow Moen and Rosen (2004) and allow for *efficient contracting.* The contracts are thus designed so as to resolve any agency problems between employers and employees so that their joint income is maximized. In particular, this implies that the workers' on-the-job search behavior maximizes the joint surplus of the worker and the firm. This assumption simplifies the model enormously. Without this assumption, a worker's current wage will influence his search behavior. As shown by Shimer (2006), this opens up for multiple equilibria and generally makes on-the-job search models intractable.

Our analysis thus delivers a tractable model of on-the-job search, closely related to the competitive model, in which on-the-job search and wage differentials for identical workers is an optimal response to search frictions and hetroegenous firms. As a tool for empirical analysisour model is interesting because it includes, in a simple way, the effects of search frictions for industry dynamics. Finally, as our model gives rise to a (constrained) efficient allocation of resources, and hence is well suited as a benchmark for welfare analysis.

The equilibrium of the model is characterized by a sluggish employment growth toward a steady-state employment level. Low productivity firms pay low wages, face high turnover rates, and grow slowly towards a steady state with low employment. More efficient firms pay higher wages, post more vacancies, and grow more quickly to a steady state with a higher employment level. The equilibrium features a job ladder: unemployed workers disproportionately search for firms with the lowest productivity. Workers employed in these firms, in turn, search only for firms with higher productivity. Hence our model easily explain a set of stylized facts about industry dynamics and worker flows: 1) productivity differences between firms are large and persistent, 2) workers move from low-wage to high-wage occupations, 3) more productive firms are larger and pay higher wages than less productive firms, 4) job-to-job mobility falls with average firm size and worker tenure, 5) wages increase with firm size, and 6) wages are higher in fast-growing firms .

Pissarides (1994) was the first paper that studied on-the-job search in a Diamond-Mortensen-Pissarides type of matching model. As show by Shimer (2006), a problem with this model is that the bargaining is not convex, and this may give rise to a continuum of wages. Current papers by Bagger and Lentz, Lise et. al. (2008) still uses this model, and get around the problem of a non-convex bargaining set by introducing competitive bidding for the worker after successful on-the-job search.

Maybe the most used model of on-the-job search in empirical research is Burdett and Mortensen (1998) with its many follow-ups, for instance Postel-Vinay and Robin (2002). Moen and Rosen (2004) are the first to analyse competitive on-the-job search and the first to assume efficient on-the-job search. Shi (2008) studies competitive on-the-job search in a model with wage tenure contracts. Menzio and Shi (2008), in a paper written simulataneously and independently of the current paper, study the effects of business cycle fluctuations in a model with competitive on-the-job search.

Kiyotaki and Lagos (2006) study optimal assignment of workers to jobs in a model where matches differ in quality, but without entry of firms. Delacroix and Shi (2006) analyzes on-the-job search in an urn-ball type of model of the labor market, and also obtain a job ladder in a similar way as we do. However, in their model all agents on both side of the market are homogenous, and firms at most hire one worker, hence their model is ill suited to analyze industry dynamics.

The paper proceeds as flows. Section 2 briefly describes the empirical regularities we are interested and reviews the relevant literature. Section 3 introduces the structure of the model while sections 4 and 5 derive the main formulation of the model for different type of firms. Section 6 introduces the general equilibrium and spells out some key results. Section 7 presents the baseline simulation.

2 A Brief Look at some empirical regularities

We briefly review some of the key empirical regularities linked to industry dynamics and worker flows. We are certainly not meant to be exhaustive in this review, and the selection of facts outlined in closely associated to the theoretical approach we will propose in the rest of the paper.

1. *In any industry there is a large scale reallocation of input and output across producers.* The work of Davis and Haltiwanger (1999) summarize much of the literature on gross job flows that was carried during the last decade; they note that in the United States more than 1 in 10 jobs is created in a given year and more than 1 in 10 jobs is destryoed. Much of this reallocation reflects reallocation within narrowly defined sectors. Note that not only labor is being reallocated but also capital and output (Haltiwanger, 2000).A large fraction of the input and output gross creation is associated with entry of firms and a large fraction output and input gross destruction is associated to firm.

2. *Productivity differences between firms are large and persistent.* Bartelsman and Dome (2000) summarize most of the evidence based on longitudinal micro data on firm level productivity diffferential. They clearly argue that the most significant finding of this vast literature is the heterogeneity across establishments and firms in productivity in nearly all industries examined. Bernard et al. (2003) report the distribution across plants of value added per worker relative to the overall mean, and show that a substantial number of plants have productivity either less than a fourth or more than four times the average. These differences are also very persistent over time. Danish data analysed by Mortensen (2007) provide similar patterns.

3. *More productive firms are larger and pay higher wages.* Wage differentials across observationally equivalent workers are both sizable and persistent . The employer size-wage effect is perhaps the strongest such stylized fact: larger firms or plants pay higher wages. The literature includes especially influential work by Krueger and Summers (1988) and Brown and Medoff (1989), and has been surveyed by Oi and Idson (1999).

4. *Job to job mobility falls with worker tenure.* There is a well established relationship between job duration and tenure. Farber (1999) carefully reviews this literature and reports that a monotonically declining survival rates is one of the mot robust stylized facts in the labour market. Such monotonicity holds regardless of the reasons beyond job termination and is particularly significant for voluntary job to job movements.

5. *Job to job movements are associated to wage gains* . Bartel and Borjas (1978) early work showed that young men who quit experience significant wage gains compared both to to job stayers and to their own wage growth prior to the job change. More recently Light (2005) summarize empirical evidence based on the National Longitudinal Survey of Youth. He shows that the the typical worker holds about five jobs in the first 8 years of the career, but that workers vary considerably in their mobility rates. He also reports evidence that workers who change jobs voluntarily thorugh a job to job transition receive significant contemporaneous wage boosts that, on average, are at least as large as the wage gains received by job stayers.

6. *and wages are higher in fast growing firms.* Belzil (2000) uses Danish data and shows that after controlling for individual and business cycle effects, job creation at the firm level is found to increase male wages.

7. *Wage differentials are associated to productivity differentials* . The evidence on this link is more scarce, since dataset able to observe output and wages for a variety of workers are not readily available. Iranzo et al. (2007) using Italian data show that the level of labour productivity is clearly associated to larger wages for both production and non production workers.

3 Technological Structure of the model

The structure of our model is as follows

- Labor is the only factor of production. The labor market is populated by a measure 1 of identical workers. Individuals are neutral, infinitely lived, and discount the future at rate r.
- The technology requires an entry cost equal to K. Conditional upon entry, the firm learns its productivity, which may be either low, y_1, or high, y_2, $y_1 < y_2$, with probabilities α and $1-\alpha$, respectively. The productivity of the firm is a fixed effect throughout its life.
- Firms post vacancies and wages to maximize expected profits. Vacancy costs are convex in the number of vacancies posted, so that $c(v_i()) = \frac{v_i^2()}{2c}$, where c is a constant. We further discuss this assumption at the end of this section.
- Firms die at rate δ and workers exogenously leave the firm at rate s

3.1 Competitive equilibrium wage dynamics

We assume first that the labor market is perfect and in equilibrium there is a single wage paid to the entire workforce. There is full employment but to obtain the equilibrium wage we need to derive labour demand. Type i firm's instantaneous profit is $(y_i - w)N_i - cv_i^{\alpha}()$. Dynamics reads $\dot{N} = v() - sN$. If $w \geq y_i$ the firm leaves the market and obtains zero profit. The firm takes as given the wage and chooses vacancies to maximize profits. The Hamiltonian writes

$$H = (y_i - w)N_i - cv_i^{\alpha}() + \lambda_i(v - sN)$$

First order conditions reads

$$\begin{aligned} v_i() &= (\frac{\lambda_i}{\alpha c})^{\frac{1}{\alpha-1}} \\ \lambda_i &= \frac{y_i - w}{r+s+\delta} \end{aligned}$$

The profit of a type i firm entering the market is

$$\Pi_i = \frac{v_i \lambda_i}{r+\delta} = \frac{(\alpha c)^{\frac{1}{1-\alpha}} \lambda^{\frac{1+\alpha}{r+\delta}}}{r+\delta}$$

The free entry condition uniquely pins down w

$$E\Pi_i = K$$

We can now show the following result:

Proposition 1 *a) Suppose $\alpha \to 1^+$. Then in the limit only the most productive firms are active, and they pay a wage*

$$w = y_{\max} - c(r + s + \delta)$$

b) Suppose $\alpha > 1$. Then firm i is active provided that y_i is sufficiently close to y max.

In the competitive setting with adjustment costs different firms can coexist in the market as long as the productivity differential is not too high. Since the wage is unique there is no links between firms dynamics and wage differentials. There is also no on the job search. This suggests that most of the empirical regularities discussed above can not be rationalized in the competitive setting. In the rest of the paper we show that the combination between convex adjustment costs at the firm level and labour market imperfections do deliver most of such implications.

4 The vacancy-wage trade-off

Before turning to the general equilibrium with imperfect labour market, we analyze the microeconomics of a firm that has some ability to fix wages. This section shows that a wage size effect equires adjustment costs to be convex. We suppose that hiring can be obtained through two means, v and w. In other words the firm has the ability to attract a given amount of workers in two ways. Either by advertising effort v, where v is a measure of efficiency unit of search. Alternatively the firm can attract workers with higher wage w. These feature are common to the competitive search equilibrium that we will be using.

If we assume that the firm needs to hire an amount of labour h, the relationship between h, v and w is given by the following function

$$h = q(w)v$$

where v and w are defined as above. The function $q(w)$ denote the arrival rate of workers per efficiency unit of search, increasing and concave and v, the number of efficiency unit. For a given level of hiring h the previous condition determines a technological trade off between vacancies and wages. Let us assume that the firm needs to hire an amount h and needs to minimize total costs. Labor costs are naturally given by hw and suppose the cost of efficiency units is given by $c(v) = cv^a$ where α is a positive constnat. The formal problem of the firm of obtaining a hiring flow of h is then given by

$$\min hw + cv^{\alpha} \qquad \text{s.t. } q(w)v = h$$

The associated Lagrangian is

$$L = hW + cv^{\alpha} - \lambda[q(w)v - h]$$

with first order conditions

$$
\begin{aligned}
h &= \lambda q'(w)v \\
c\alpha v^{\alpha-1} &= \lambda q(w)
\end{aligned}
$$

or

$$c\alpha v^{\alpha} = h\frac{q(w)}{q'(w)}$$

Substituting in $v = h/q$ gives

$$
\begin{aligned}
c\alpha(h/q)^{a} &= h\frac{q(W)}{q'(W)} \\
\frac{q'}{q^{(1+\alpha)}} &= \frac{h^{1-\alpha}}{c\alpha}
\end{aligned}
$$

which uniquely determines w as a function of α. Assuming $q = w^{\beta}$, the left hand right hand side reads $\frac{q'}{q^{(1+\alpha)}} = \beta w^{-(1+\alpha\beta)}$ so that the wage paid by the firm will be or

$$w = h^{-\frac{1-\alpha}{1+\alpha\beta}}k$$

where $k = (c\alpha)^{\frac{1}{1+\alpha\beta}}$.

We establish an important result

Remark 2 *A positive link between wage and firm size requires α to be less than one.*

The model relies on a convex hiring cost to prevent firms from posting an infinite number of vacancies upon entry in the labor market. This assumption can be justified along several dimension.

The measure v is closer to search effort from the firm standpoint rather than to a measure vacancies. When a firm double its search intensity it does not typically double the number of applicants, in a way similar to diminishing returns. The counterpart of this simple reasoning is a convex hiring cost.

Another way to justify convexity relies on labour market frictions linked to firm's optimal scale. Changing firm scale requires a costly look for talent that is not easily available in local markets. Our modeling strategy can be thus seen as a reduced form of this extremely costly search for talent.

At a more technical level, convex hiring costs can be seen as a generalization of Burdet Mortensen (1999) model with on the job search. In their model, the arrival of workers to firms is exogenously set. Our specification allows for some flexibility, and we let firms to increase this arrival rate through search effort. Note also that the structural estimates provided by Yashiv (2000a,b) are fully consistent with a marginal cost increasing in the stock of vacancies.

Finally, most models of endogenous search effort focus on the worker side. In such models (Pissarides, 2000) workers' cost of effort is typically model as convex function with respect to individual effort. Our function is the analogous approach on the firm side.

5 Firm Dynamics in Imperfect Labor Market

- Unemployed workers have access to an income flow y_0, which may denote unemployment benefits, the value of leisure, or the income when self-employed. A viable labour market clearly requires that $z \leq y_1 < y_2$. Workers search for jobs on and off the jobs at no cost (search intensity is given).

- Search is directed. Firms post vacancies and wages to maximize expected profits. Firms face a relationship between the wage they set and the arrival rate of workers, which is derived from the indifference constraint of workers. Firms set wages so as to maximize profits given this relationship.

- Wage contracts are complete, and resolve any agency problems between employers and employees. In particular, the wage contract ensures efficient on-the-job search.

A submarket is characterized by an aggregate matching functions, brining together the searching workers and the vacant firms. Let the sumbmarket ij denote the market where workers employed in type i firm search for firms of type j. We will naturally require that $j > 1$. As we will see, in our model at most three submarkets operate in equilibrium. In submarket 01, unemployed workers search for low productivity firms. In Submarket 12, high productivity firms are hiring from workers employed in low productivity firms. Finally, there may be a submarket 02 in which high-productivity firms hire directly from the unemployment pool. In any submarket ij the aggregate number of vacancies is equal to $V_j(i) = f_{ji}v_j(i)$, where f_{ji} is the measure of firms operating in that submarket and $v_j(j)$ is the number of vacancies per firm j . If we assume a Cobb Douglas matching function with constant returns to scale and weight β on the workers, the transition rate for workers and for firms is

$$\begin{aligned} p_{ij} &= \theta_{ij}^{1-\beta} \\ q_{ij} &= \theta_{ij}^{-\beta} \end{aligned}$$

where θ_{ij} is the ratio of vacancies to searching workers in submarket ij. Inverting the first of the previous condition one gets that $\theta_{ij} = {p_{ij}}^{\frac{1}{1-\beta}}$ so that the transition rate for vacancies can be expressed as

$$q_{ij} = {p_{ij}}^{-\frac{\beta}{1-\beta}} \tag{1}$$

6 Submarket 01: Unemployed workers and Low-type firms

In this section we analyze optimal search behavior of unemployed workers and low-productivity firms.

6.1 Search behavior of unemployed workers

We start by the search behavior of unemployed workers. The asset valuation of an unemployed worker that searches in a submarket $0j$ where the wages are W_i can be written as

$$rU_{0j} = y_0 + p_{0j}(W_{0j} - U_{0j})$$

where U_{0j} and W_{0j} are the continuation values of being unemployed and employed, respectively. In what follows, we define the expected rent from a job to the worker as the net gain obtain to the worker from a move from unemployment to a job, and its expression is

$$R_{0j} \equiv W_{0j} - U_{0j}$$

Note that the rent is indicated as R_{0j} as it represents a move from a type 0 job (the unemployment) to a type j job. In equilibrium, all firms that attract applicants from the unemployment pool must give the workers the same equilibrium income, which we denote by U. Thus,

$$rU_{0j} = y_0 + p_{0j}R_{0j} = rU$$

This equation is key in competitive search equilibrium, as it defines p_{0j} as a function of R_{0j}, $p_{oj} = p(R_{0j})$ (for a given U). It follows that

$$rU^* - z = p_{0j}(R_{0j})R_{oj}$$

Taking elasticities with respect to R_i gives

$$\begin{aligned} 0 &= \varepsilon_{p_{0j},R_{0j}} + 1 \\ \varepsilon_{p_{0j},R_{0j}} &= -1 \end{aligned} \quad (2)$$

where $\varepsilon_{p_{0j},R_{0j}} = \frac{dp}{dR_{01}}\frac{R}{p}$

6.2 Joint income in Firm j

Let M_1 denote the expected discounted joint income of a worker and a type 1 firm, which is equal to the sum of the NPV of the worker, W_{01} , and the firm, J_1. It follows that

$$M_1 = J_1 - W_{01}$$

$$(r+s)M_1 = y_1 + (s+\delta)U + p_{12}[W_{12} - U_1 - M_1] \quad (3)$$

where p_{12} is the probability that the worker finds a job in high productivity jobs. Note that the joint income is a forward looking concept and does not depend directly from the submarket in which the job was formed. In the previous expression p_{12} is the submakret in which the worker may eventually search on the job. The notation M_1 reflects this feature.

When workers do on-the-job search, they choose between searching in submarkets with different combinations of wages W_{12} and job finding rates. This may potentially give rise to excessive on-the-job search if the workers do not take into account that quitting may lead to a negative externality towards the firm as it looses J_1. In the present model this is not an issue. We assume that the wage contracts are complete, and hence that the workers' on-the-job search behavior maximizes the joint income $M_1$of the worker-firm pair. The are various wage contracts that implement this behavior. For example, the worker pays the firm J_1 up front and then gets a wage equal to y_1. Alternatively, the worker gets a constant wage and pays a quit fee equal to the continuation value of the firm J_1 if a new job is accepted (see Moen and Rosen (2004) for more examples). Importantly, the wages paid to the worker in the current job do not influence her on-the-job search behavior.[1]

Let R_{12} denote the net gain for the worker-firm pair obtained when the workers climbs to firm 2. In that case, the worker gains $W_{12} - W_{01}$, while the firm looses J_1. Thus

$$R_{12} \equiv W_{12} - M_1 \tag{4}$$

It follows that we can write M_1 as

$$M_1 = \frac{y_1 + (s+\delta)U + p_{12}R_{12}}{r+s+\delta} \tag{5}$$

6.3 The Firm's Maximization Problem

The key firm decision concerns the number of vacancies to be opened and the rent to be paid to each worker. Let $q(R_1)$ denote the relationship between the rents offered to the workers and the arrival rate of workers. The firms then solve

$$\begin{aligned} Max_{R_{01},v_1(0)} &= -\frac{v_1^2(0)}{c} + v_1(0)q(R_{01})[M_1 - U - R_{01}] \qquad (6) \\ s.t \qquad M_1 &= \frac{y_1 + (s+\delta)U + p_{12}R_{12}}{r+\delta+s} \end{aligned}$$

and employment dynamics, contingent on the firm's continued existence is

$$\dot{N}_1 = v_1(0)q(R_{01}) - (s + p_{12} + \delta)N_1$$

As we have already pointed out, the worker's on-the-job search is set so as to maximize M_1, independently of R. The first order conditions for the firms' maximization problem can thus be written as

$$\begin{aligned} \frac{v_1(0)}{c} &= (M_1 - U - R_{01})q(R_{01}) \\ -v_1 q(R_{01}) + [M_1 - U - R_{01}]v_1 q'(R_{01}) &= 0 \end{aligned}$$

The latter condition easily becomes

$$R_{01} = \varepsilon_{q_{01},R_{01}}[M_1 - U - R_{01}]$$

[1]It follows from this that a worker in a low-type firm will never search for a job in another low-type firm, as these cannot offer a wage that exceeds the productivity in the current firm.

where $\varepsilon_{q_{01},R_{01}} = \frac{dq}{dR}\frac{R}{q}$. The previous expression can be solved for a value of R_{01} as

$$R_{01} = \frac{\varepsilon_{q_{01},R_{01}}(M_1 - U)}{1 + \varepsilon_{q_{01},R_{01}}} \tag{7}$$

Now

$$\begin{aligned}
\varepsilon_{q_1,R_{01}} &= -\frac{\beta}{1-\beta} p(R_{01})^{-\frac{\beta}{1-\beta}-1} p'(R_{01}) \frac{R_{01}}{q_{01}} \\
&= -\frac{\beta}{1-\beta} p(R_{01})^{-1} p'(R_{01}) R_{01} \\
&= -\frac{\beta}{1-\beta} \varepsilon_{p,R_{01}}
\end{aligned}$$

Since competitive search implies equation 2, we get

$$\varepsilon_{q_{01},R_{01}} = \frac{\beta}{1-\beta} \tag{8}$$

Since competitive search implies equation (2), we get

$$\varepsilon_{q,R_{01}} = \frac{\beta}{1-\beta} \tag{9}$$

Inserted into (7) this gives

$$R_{01} = \beta(M_1 - U)$$

To repeat, the first order conditions to the firm's maximization problem writes

$$\begin{aligned}
R_{01} &= (M_1 - U)\beta && (10) \\
\frac{v_1}{c} &= (1-\beta)(M_1 - U)q(R_{01}) && (11) \\
M_1 &= \frac{y_1 + (s+\delta)U + p_{12}R_{12}}{r + \delta + s} && (12)
\end{aligned}$$

At the firm level the system solves for $v_1(0), R_{01}$ and M_1 while p_{12} and R_{12} are taken as given. The value of a firm that enters the submarket 01 with zero workers and post $v_1(0)$ vacancies reads

$$\begin{aligned}
\Pi_{01}(0, v_1(0)) &= \text{gain from search-costs of vacancies} \\
&= \frac{1}{r+\delta}\left\{ q(R_{01})v_1(M_1 - U)(1-\beta) - \frac{v_1^2(0)}{2c} \right\}
\end{aligned}$$

The first term refers to the gain from search. In words, a firm that posts v_1 vacancies filled them with probability q and enjoys a fraction $(1-\beta)$ of the full surplus. The second term refers to the quadratic cost of vacancies. Using the expression for the vacancies obtained in (11), the value of the profits is

$$
\begin{aligned}
\Pi_{01} &= \frac{1}{r+\delta}\left\{[(M_1-U)(1-\beta)q(R_{01})]^2 c - \frac{[(M_1-U)(1-\beta)q(R_{01})]^2 c^2}{2c}\right\} \\
&= \frac{1}{r+\delta}\frac{[(M_1-U)(1-\beta)q(R_{01})]^2 c}{2}
\end{aligned}
$$

so that a firm that is hiring N workers has profits and employment d

$$
\begin{aligned}
\Pi_{01}(N) &= N(1-\beta)(M_1-U)+\Pi_1(0) \\
\dot{N}_1 &= v_1 q(R_{01})-(s+p)N_1
\end{aligned}
$$

To summarize, firms in this submarket post a measure of vacancies v_1 independently of their employment status. This feature follows directly from the constant returns to scale assumptions. Workers leave the firm exogenously at rate s and the firm may go bankrupt at rate δ. Firms in this submarket face also turnover risk in terms of a direct job transition to a high productivity firm at rate p_{12}. In light of the perfect contract structure of our economy, such event is not associated with inefficient separation. Firms in that survive reach a steady state situation in which hiring $v_1(0)q(p_{01})$ excaly match separation $l(s+p_{12})$. Note that also in steady state the firm is characterized by continuous job turnover, even though employment does not grow. The wage contract and the associated rent R_{01} offered and posted by the firm is also constant throughout the life of the firm and does not feature any transitional dynamics.

7 Submarkets with High Productivity Firms

A high-productivity firm may choose to direct its search towards unemployed workers or workers employed in low-type jobs. We will discuss the two cases in turn

7.1 Submarket 12:Employed worker search

In this subsection we derive the optimal wage policy when the high-type firms search for workers employed in low-type firms. We proceed in the same way as in the previous section. Due to efficient contracting, we know that workers in a low-type job will behave so as to maximize M_1. It follows that for all firms that attract workers, we have that will maximize the value of the match. So that

$$
\begin{aligned}
M_1 &= \frac{y_1+rU_1+p_{12}R_{12}}{r+\delta+s} \\
&= const
\end{aligned}
$$

which defines $p_{12}=p_{12}(R_{12})$. Solving for $p_{12}R(p_{12})$ and taking elasticities of both sides give

$$\varepsilon_{p_{12},R_{12}}=-1$$

Let J_2 denote the asset value of a filled job, and define $M_2 = J_2 + W_{12}$. It follows that

$$M_2 = \frac{y_2 + (\delta + s)U}{r + s + \delta}$$

Firms decide the number of vacancies to be opened and the rents to attatch to them. In other words, the firms' problem is

$$Max_{R_{12}, v_{2(1)}} = -\frac{v_2^2(1)}{c} + v_2(1) q_2(R_{12})[M_2 - M_1 - R_{12}]$$

(note that M_1 is exogenous to the firm). The first order condition with respect to v is

$$\frac{v_{2(1)}}{c} = (M_2 - M_1 - R_{12}) q_2$$

The condition with respect to R_{12} is

$$-vq(R_{12}) + [M_2 - M_1 - R_{12}] v q_2'(R_{12}) = 0$$

this gives a value of R as

$$\begin{aligned} R_{12} &= \frac{\varepsilon_{q_2, R_{12}}(M_2 - M_1)}{1 + \varepsilon_{q_2, R_{12}}} \\ &= \beta(M_2 - M_1) \end{aligned}$$

analogous to the expression for R_1 in (10). To summarize, the first order conditions are given by

$$\begin{aligned} R_{12} &= \beta(M_2 - M_1) \\ \frac{v_2(1)}{c} &= (1 - \beta)(M_2 - M_1) q_2(R_{12}) \\ M_2 &= \frac{y_2 + (s + \delta)U_i}{r + \delta + s} \end{aligned}$$

The value of a type 2 firm is

$$\begin{aligned} \Pi_{12}(0, M_2) &= \frac{1}{r + \delta} \left\{ [(M_2 - M_1)(1 - \beta) q(R_{12})]^2 c - \frac{[(M_2 - M_1)(1 - \beta) q(R_{12})]^2 c^2}{2c} \right\} \\ &= \frac{1}{r + \delta} \frac{[(M_2 - M_1)(1 - \beta) q(R_{12})]^2 c}{2} \end{aligned}$$

To summarize, firms in this submarket post a measure of vacancies v_2 independently of their employment status. The measure of vacancy is proportional to productivity y_2 and is thus likely to be larger than the vacancy posted by firms in submarket 1. Workers leave the firm exogenously at rate s and the firm may go bankrupt at rate δ. In this submarket firms *do not* face the turnover risk associated to a direct job transition. Workers in such job enojoy the best job in the economy and do not search any longer. Firms in that survive reach a steady state situation in which hiring $v_2 q(p_{12})$ excaly match separation ls. Note that also in steady state the firm is characterized by continuous job turnover, even though employment does not grow. The wage contract and the associated rent R_{12} offered and posted by the firm is also constant throughout the life of the firm and does not feature any transitional dynamics.

7.2 Submarket 02: Unemployed-worker search

Finally we will specify the behavior of high-type firms searching for unemployed workers. Let U_{02} denote the continuation value of unemployed workers searching for a high-type job, W_{02} the continuation value of an employed worker, and define the rent of finding a job as $R_{02} \equiv W_{02} - U_{02}$. The indifferent constraint of unemployed workers read

$$\begin{aligned} rU &= z + p_{02}.R_{02} \\ &= const \end{aligned}$$

which defines a unique relationship between R_{02} and p_{02}, which we write $p_{02}(R_{02})$. (Note that when $U_1 = U_2$, which is always the case in equilibrium, this relationship is equal to $p_{01}(R_1)$ defined above). Write

$$rU - z = p_{02}(R_{02})R_{02}$$

Taking the elasticity with respect to R_{02} gives

$$\varepsilon_{p_{02},R_{02}} = -1$$

We still have that

$$\begin{aligned} M_2 &= J_2 + R_{02} \\ &= J_2 + W_{02} - U \end{aligned}$$

Now

$$(r + \delta + s)(J_2 + W_{02}) = y_2 + (\delta + s)U$$

which gives

$$M_2 = \frac{y_2 + (s + \delta)U}{r + \delta + s}$$

Again, the key firm decisions concern the number of vacancies to be opened and the rent to be paid to each worker. In other words the firm problem is

$$Max_{R_2, v_2(0)} = -\frac{v_2(0)}{c} + v_2(0)q(R_{02})[M_2 - R_{02}]$$

By proceeding in exactly the same way as above we get that the first order conditions are given by

The three first order condition are

$$\begin{aligned} R_{02} &= \beta(M_2 - U) \\ \frac{v_2(0)}{c} &= (1-\beta)(M_2 - U)q(R_{02}) \\ M_2 &= \frac{y_2 + (r+\delta)U}{r+\delta+s} \end{aligned}$$

The value of a type 2 firm searching for unemployed workers is

$$\begin{aligned} \Pi_{02} &= \frac{1}{r+\delta}\left\{[(M_2\_U)(1-\beta)q(R_{02})]^2 c - \frac{[(M_2-U)(1-\beta)q(R_{02})]^2 c^2}{2c}\right\} \\ &= \frac{1}{r+\delta}\frac{[(M_2-U)(1-\beta)q(R_{02})]^2 c}{2} \end{aligned}$$

8 General Equilibrium

In order to close the model, a set of equilibrium conditions has to be satisfied. The first regards high-productivity firms and unemployed workers. Productive firms choose optimally what type of workers to search for. In any equilibrium, at least some high-type firms search for employed workers (if not, q_2 would be infinite for any $R_{12} > 0$, ensuring an efficient firm infinite profit). Let τ denote the fraction of the firms that search for employed workers. We require that

$$\begin{aligned} \Pi_{12} &\geq \Pi_{02} \text{ for all } \tau \\ \Pi_{12} &= \Pi_{02} \text{ if } \tau < 1 \end{aligned}$$

The free entry of firms will ensure that the number of firms f adjusts so that there are zero profit *ex ante*, which means that

$$\alpha Max[\Pi_1; 0] + (1-\alpha)Max[\Pi_{12}; 0] = K$$

Definition 3 *The general equilibrium is given by a vector of value function and job finding rates* $\{U^*, p_{01}, p_{12}, \tilde{p}_2^*\}$ *and a vector of market quantities* $\{u_1, \tilde{u}_2, u_2, n_1, n_2, \tau, f, g\}$ *satisfying*

- optimal vacancy and rent posting by firm 1 and 2 in different submarkets
- optimal search for the unemployed and the employed workers
- free entry of firms
- indifference across operating submarkets of firms and workers

When deriving the equilibrium of the model, two different scenarios will be consider, with and without an active third market.

8.1 Equilibrium with All Submarkets

Suppose first that the third market is operative. The determination of $\{U^*, p_{01}, p_{12}, \tilde{p}_2^*\}$ is given by this system

$$
\begin{aligned}
\alpha \Pi_1 + (1-\alpha)\Pi_{12} &= k \quad \text{Free Entry of firm} \\
\Pi_{12} &= \Pi_{02} \quad \text{Firm Indifference across submarkets} \\
rU_1 &= r\tilde{U}_2 \quad \text{Unemployed indifference across submarkets} \\
& \quad\ rU^* \quad \text{Value of Unemployment}
\end{aligned}
$$

Or, written out,

$$
\begin{gathered}
\alpha \frac{[(M_1 - U)(1-\beta)p_{01}]^2 c}{2} + (1-\alpha)\frac{[(M_2 - M_1)(1-\beta)p_{12}]^2 c}{2} = K \\
\frac{[(M_2 - U)(1-\beta)p_{12}]^2}{2} = \frac{[(M_2 - M_1)(1-\beta)p_{02}^{-\frac{\beta}{1-\beta}}]^2}{2} \\
p_{02}(M_2 - U) = p_{01}(M_1 - U) \\
rU = z + p_{01}.\beta(M_1 - U)
\end{gathered}
$$

where

$$
\begin{aligned}
M_1 &= \frac{y_1 + rU_1 + p_{12}\beta(M_2 - M_1)}{r + \delta + s} \\
M_2 &= \frac{y_2 + (r+\delta)U}{r + \delta + s}
\end{aligned}
$$

This system can the be solved for $\{p_{12}, p_{02}, p_{01}, U\}$. Let n_0, n_1 and n_2 denote respectively the measure of workers unemployed, employed in type 1 firm and in type 2 firm. Furthermore, let k_{01} and k_{02} denote the fraction of unemployed workers searching for type 1 and type 2 firm respectively. The following balance flow conditions apply

$$
\begin{aligned}
p_{02}k_{02}n_0 + p_{12}k_{01}n_1 &= (\delta + s)n_2 \\
p_{01}k_{01}n_0 &= (\delta + s + p_{12})n_1 \\
n_0 + n_1 + n_2 &= 1 \\
k_{01} + k_{02} &= 1
\end{aligned}
$$

To close the model and characterize the equilibrium, we utilize the aggregate consistency condition that all firms searching in the same submarket offers the same wage. Hence

$$
\begin{aligned}
p_{01} &= \left[\frac{(1-\alpha)fv_1(o)}{k_{01}n_0}\right]^{1-\beta} \\
p_{12} &= \left[\frac{\tau\alpha f v_2(1)}{n_1}\right]^{1-\beta} \\
p_{02} &= \left[\frac{(1-\tau)\alpha f v_2(0)}{k_{02}n_0}\right]^{1-\beta}
\end{aligned}
$$

The appendix reports the routine for solving numerically the model.

Proposition 4 *The equilibrium satisfies the following property:* $W_{12} > W_{02} > W_{01}$.

Proof: In all submarket, the wage is set so that the indifference curve of the workers and the iso profit constraint of the firms are tangent. It is easy to show that the at any given point in $W, \theta-$ space, the indifference curve of employed workers is steeper than the indifference curve of unemployed workers, reflecting that unemployed workers are more eager to find jobs. Furthermore, as a type two-firms in equilibrium is indifferent as to which submarket to enter, it follows that (W_{02}, θ_{02}) and (W_{12}, θ_{12}) are at the same iso-profit curve. From a revealed-preference type of argument it follows easily that $W_{12} > W_{02}$. Furthermore, the iso profit-curve of a high-type firm at any given point in the W, θ- space is flatter than that of the low-type firm, reflecting that high-type firms are more eager to speeding up the hiring process than are low-productivity firms. Since unemployed workers are indifferent as to what firm to apply to, a simple-revealed preferences type of argument delivers that $W_{02} > W_{01}$.

Corollary 5 *Unemployed workers have no incentives to join the submarket for employed workers.*

Proof: Suppose the unemployed workers obtained $U' \geq U^*$ by searching in the employed-search submarket. Since the indifference curve of the low-type worker is flatter in the (W, θ) space than that of the high-type workers, it follows from a revaled-preference type of argument that a type-2 firm could increase its profits by offering a lower wage, and hence obtain a higher profit than when offering W_{02}, a contradiction.

We can also show the following result

Proposition 6 *There exists a value* α^* *such that for any* $\alpha \geq \alpha^*$, $\tau \geq 1$, *while for all* $\alpha \in (0, \alpha^*)$, $0 < \tau < 1$.

Sketch of proof: For any number $\varepsilon > 0$. Consider a firm that sets $w = y_1 + \varepsilon$. As $\alpha \to 1$, the arrival rate of workers to this firms goes to infinity, independently of which wage $w \in (y_1, y_2)$ the other high-type firms choose. Thus profits go to infinity. If a high-type firm searches for unemployed workers, the arrival rate of workers to the firm will be bounded, and hence also profit. The claim thus follows. By a similar argument, it also follows that at least some high-type firms searches for employed workers as long as $\alpha > 0$.

8.2 Pure Job Ladder Equilibrium

Suppose now that $\alpha > \alpha^*$ so that the equilibrium describe above does not hold because the definition required τ, the proportion of firms hiring directly from the emplyed to be strictly less than one. We obtained thus a pure job ladder equilibrium since high productivity firms hire only from the employment pool. The determination of $\{U^*, p_{01}, p_{12}, \}$ alongside the distribution of employment across states $\{n_1, n_2, n_0\}$ and the size of entry f is given by the following 7 equation system

$$
\begin{aligned}
\alpha \Pi_1 + (1-\alpha)\Pi_{12} &= k \\
rU^* &= z + \beta p_{01} S_1 \\
p_{01} &= \left[\frac{(1-\alpha) f v_1}{n_0} \right]^{1-\beta} \\
p_{12} &= \left[\frac{\alpha f v}{n_1} \right]^{1-\beta} \\
p_{01} n_0 &= (\delta + s + p_{12}) n_1 \\
p_2 n_1 &= (\delta + s) n_2 \\
u_1 + n_1 + n_2 &= 1
\end{aligned}
$$

The routine for solving and charatcerizing the equilibrium between the pure job ladder and the general specification is outlined in the appendix.

9 Basic Calibration and Comparative Static

Table 1 and 2 report the basic parameter values for our calibration. The calibration is based on quarterly statistics and the pure interest rate is 1 percent. The productivity level in low type firm is set to a baseline reference value of $y_1 = 1$, while the premium for the high type is 10percent. The flow value of unemployment z is 0.6, a value far the replacement rate observed in real life labour markets. The matching function is Cobb Douglas with an elasticity β equal to 0.5. The parameter of the search cost is 0.15, while the entry cost k is 5, a value roughly equal to five times times the output produced by a low productivity job.The sum of the separation s and the firm death rate is 0.06. The proportion of low productivity firms is rather high at 0.93.The rest of the parameters are reported in 1

The baseline equilibrium features an unemployment rate equals to 5.8 percent and a job finding probability equal to 1, in line with the basic quarterly statistics in the United States labour market. Unemployment flows are 5.7 percent, consistent with quarterly job creation rate in the US manufacturing sector compiled by Davis and Haltiwanger. Job to job mobility is slightly below 5 percent. In Table 1 most of the unemployed workers search for low productivity firms, as indicated by $k_{01} = 0.96$. Similarly, high productivity firms search mainly among the employed sector, as indicated by the fraction of firms hiring from the employment pool ($\tau = 0.97$) The equilibrium allocation is described in the central part of Table 1. The job finding rate for unemployed workers p_{01} is the largest among the various job finding rates, but the bulk of workers in the labor market is employed in type 2 firms. Indeed, type 2 firms absorb 75 percent of the total workforce. As a result, the submarket 02, albeit significant, represents a fringe of the entire economy.

As we mentioned above, the labor market features unemployment flows and job to job flows that are comparable in absolute magnitude, and the *job ladder* mechanism is clearly present in the simulated economy. Workers start out in low productivity firms and eventually graduate to high type jobs through on the job search. Eventually, firm and match specific shocks at rate δ and s induce another round of job ladder. The bottom part of the Table 1 features also an important relationship between firm size and firm wages, where the latter are measured in terms of PDV wages. Clearly, high type firms are larger in size and pay higher wage.

The idea of the baseline simulation from Table 1 to Table i2 is to show that an increase in the share of low productivity firms α lead the economy to move toward a pure job ladder equilibrium. Indeed, the only parameter that changes between Tables and 1 and 2 is α. Recall that in the baseline specification of Table 1 the equilibirum value of τ is very close to one and as a result the submarket 02 is very small. A small increase in α, similarly to that experienced from Table 1 to Table i2 leads to an equilibrium value of $\tau > 1$, a value that is not consistent with all three submarkets being operative. In other words, as α is increased with respect to the value assigned in Table 1, the economy moves to a pure job ladder equilibrium. In moving from Table 1 to Table 2 α increases from 0.93 to 0.94, suggesting that α^* in our numerical example is inside this small interval. The economy described in Table 2 does look very similar to that described in Table 1, even though two only submarkets are operative. Note also that the equilibrium value of unemployment U^* does slightly fall as α increases. This is not surprising since in a pure job ladder equilibrium the number of low productivity firms is higher.

9.1 Comparative Static

Figures 1 and 2 describe the equilibrium of the model following an increase in the proportion of low productivity firms, α. The parameters used in the simulations are identical to those of Table 1, with the only exception of α that ranges from close to zero to close to 1. In each panel in Figure 1 and 2, the horizontal axis ranges from 0 to 1. The first panel shows that as α reaches α^*, or a share of low productivity firms sufficiently higherm the submarket 02 shut down and the economy moves to a pure job ladder equilibrium where firms search only for employed workers. Such features is accounted for by a value of $\tau = 1$. In the second panel of 1 we report the share k_{01}, or the share of unemployed workers searching for low productivity firms. As α reaches α^* such proportion becomes very close to 1.

The fourth panel of 1 reports the comparative static with respect to the value of unemployment rU as the economy increases the share α. The value of unemployment clearly falls monotonically. This is probably the most important and clear result following the increase in α. An economy with a larger proportion of low productivity firm is an economy that brings lower utility to non employed workers. This simple result is true regardless of the type of equilibrium in which the economy settles, as demonstrated by the monotonic fall in rU across all ranges of α. Similar results hold for the joint income and the wage obtained by workers, which decline monotonically as the low productivity share α increases. The last two panels on Figure 1 shows the effects effects on profits following an increase in the proportion of low productivity firms. Profits in both type 1 and type 2 firm increase. Consider first low productivity firms. The increase in α, by reducing the the share of high productivity firms in the economy, increases the employment and profit opportunity in low productivity firms. It is a simple competition effect due to the fact that there are fewer firms with superior technology. The effect on high productivity firms is similar, since existing firms face lower competition from firms of similar technology. The latter effect is milder in the pure job ladder equilibrium.

The effects of α on the number of firms is rather non linear. As long as α is lower than α^*, an increase in the proportion of low type firms reduce the number of entrants. As the economy switches to the pure job ladder equilibrium, the number of firms increase dramatically.

Figure 2 focuses on the aggregate labour market. To understand the overall effect it is important to first look at n_2, employment in high productivity firms, displayed in panel 5 of Figure 2 . Following an increase in α, there are fewer high productivity firms, and thus aggregate employment in these of firms fall. This monotonic falls is the counterpart of the fall in the value functions displayed in Figure 1. The market composition in terms of employment changes in favor of low productivity firms, and as a result employment n_1 increases substantially. If one looks at employment n_1 as a sort of first step toward employment in good firms, the increase in this employment is akin to an increase in "bad employment". The fall in overall unemployment should also not be surprising, especially if we look at the sum of n_0 and n_1 as the pool of workers that are waiting to move to high employment n_2.

Turning back to job finding rate, the first three panels of Table 2 show that all job finding rates fall as the proportion of low productivity firms falls.

The increase in the share of low productivity firms induce an increase of both vacant firms

$V_1(0)$ and searching workers into the submarket 01, the latter being obtained by the product $k_{01}n_0$. The simulation shows that the effects from searching worker effects dominates despite the fall in in n_0 and thus the job finding rate p_{01} falls. The effect on p_{02} is similar. As α increases, both high productivity firms and searching workers falls, but the effects obtained by the reduction in searching firm is stronger. The reduction in p_{12} is simpler, since in such submarket there is an increase of searching workers and a reduction in vacant firms $V_2(0)$. Finally, there is an increase in job to job flows, as the increase in α implies that the only channel to reach employment in high productivity firms is through a passage through employment n_1

Figures 3 and 4 focus on the comparative statics following an increase in the productivity of high type firms y_2. The lowest value of y_2 in the simulation is 1.08, a value lower than that displayed in Table 2. The simulation clearly shows that as long as y_2 is less than 1.09, the economy settles in a pure job ladder equilibrium and the market 02 shuts down. The first two panels in Figure 3 show also that as the high productivity premium increases, the economy moves out of a pure job ladder equilibrium toward an equilibrium in which the market 02 is operative. Not surprisibgly, the equilibrium value of unemployment increases monotonically, as well as the joint income and the PDV wages in both low and high productivity firms. The increase in the value of low productivity jobs is linked to the expectation of a capital gain associated to a future move toward a high productivity job that, as a result of the larger y_2 has higher value. The effects of y_2 on profits depends entirely on the type of equilibrium in which the economy settles. For values of $y > 1.09$, all three submarkets are operative and the larger productivity y_2 increase profits in high type firms and lower profits in low productivity firms. The opposite happens when the pure job ladder equilibrium prevails. In a pure job ladder equilibrium the increase in y_2 increases the demand for employed workers n_1 with obvious positive impact on firm 1 profits. With convex hiring costs it is possible that such increase in demand leads to an overall increase in costs, depsite the larger produtctivity y_2.

Figure 4 focuses on aggregate quantities. The clear and simpler effect is the increase in aggregate employment in high productivity firms. This result is the counterpart of the increase in the value functions described in the panel of 3.Employment in low productivity firms falls, since more and more workers move to better jobs. The increase in the job finding rate p_{12} is consistent with such change. Things are more complicated when we look at the unemployment level. To understand this effect one has first to realize that the fall in n_1 employment is quantitatively very sizeable, as displayed in the fifth panel in Figure 4. In the pure job ladder equilibrium low productivity firms are now smaller and reduce the demand for unemployed workers. As a result the job finding rate p_{01} falls and and n_0 increases. As the economy moves to the equilibrium in which all submarket opens up, high productivity firms hire directly from the unemployed and induce a reduction in the supply of searching workers for low productivity jobs. In other words there is a jump in p_{02} and p_{01} increases, since unemployed are a more scarce resource in the 01 submarket. As the high productivity premium increases further, the overall effect on unemployment is ambiguous and non monotonic.

Figures 5 6 report the comparative static following an increase in the entry costs. Results

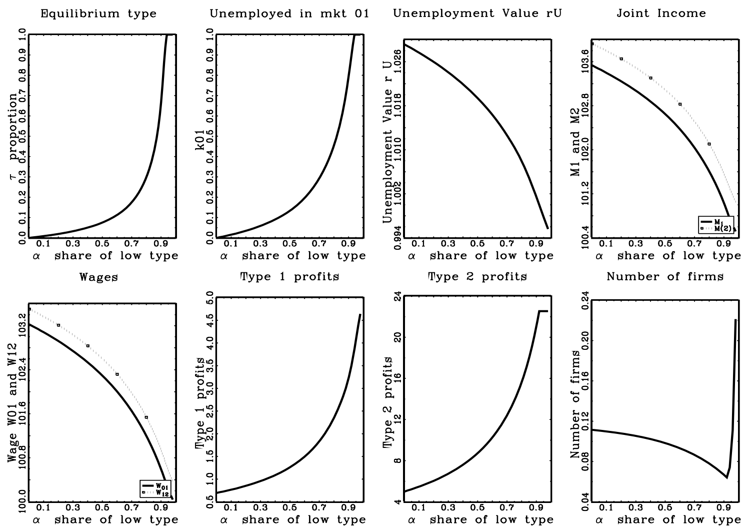

Figure 1: Increase in α, the proportion of low productivity firms. Value Functions

are straightforward. The value of unemployment falls monotonically, as do all PDV wages. An economy with larger costs is an economy with more frictions and barriers to entry and does yield lower utility. The number of firms fall while profits for incumbent firms increase. The latter ìresult is not surprising, since free entry implies that expected profits match the entry costs. All job finding rates fall as does fall employment in high productivity firms. There is as a consequence an increase in employment and unemployment at lower productivity level.

10 n types of firms

The model easily generalize to n types of firms with productivities $z < y_1 < y_2, ..., < y_n$. Let market ij denote the market for workers currently employed in firms of type i searching for jobs in firms of type j. We thus require that $j > i$. Define M_i as the joint NPV income of a worker and a firm on level i, so that

$$(r + s + \delta)M_i = y_i + \max_{j>i} p_j R_j + (s + \delta)U$$

It can be convenient to write $z = y_0$ and $U = M_0$. Suppose market ij operates, and let $R_{ij} = W_j - M_i$ denote the rents when going from level i to level j. Then it is easy to show

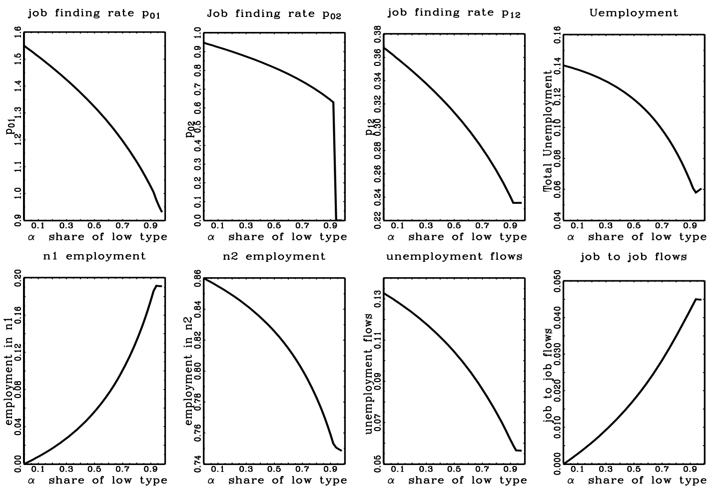

Figure 2: Increase in α, the proportion of low productivity firms. Aggregate Quantities

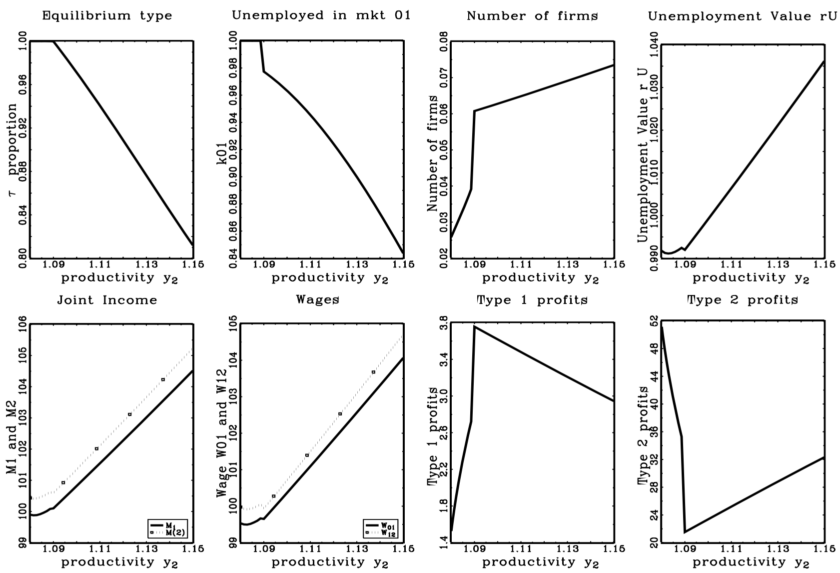

Figure 3: Increase in productivity y_2 of high type firms. Value Functions

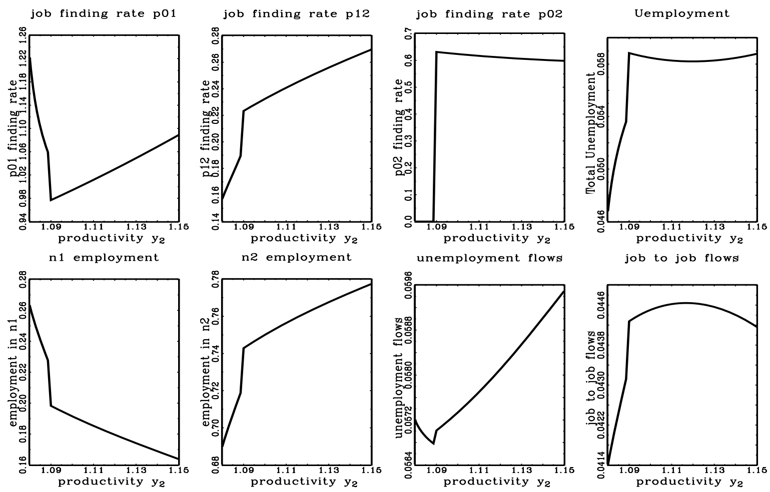

Figure 4: Increase in productivity y_2 of high type firms. Aggregate Quantities

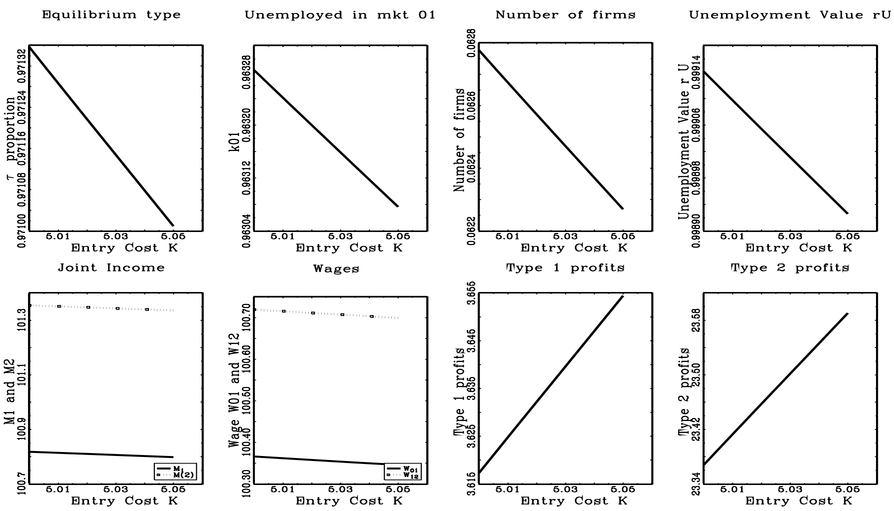

Figure 5: Increase in entry cost k. Value Functions

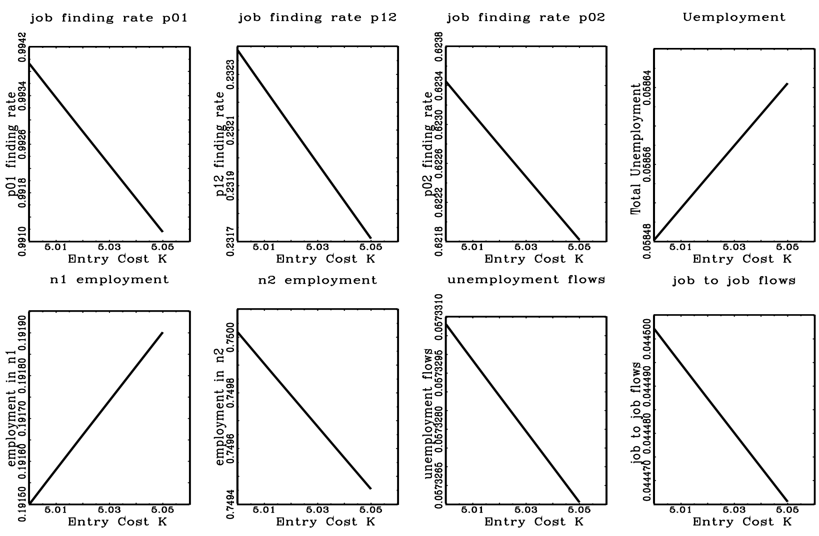

Figure 6: Increase in entry cost k. Aggregate quantities

that $R_{ij} = \beta(M_j - M_i)$.The equilibrium asset value equations thus generalize to

$$\begin{aligned} R_{ij} &= \beta(M_j - M_i) \\ \frac{v_{ij}}{c} &= (1-\beta)(M_j - M_i)q(Rij) \\ \Pi_{ij} &= \frac{[(1-\beta)(M_j - M_i)q(Rij)]^2 c}{r+s+\delta} \end{aligned}$$

Let N_i denote the measure of workers in type i firms, τ_{ij} the fraction of type j firms searching for type i workers, and κ_{ij} the fraction of "type" i workers searching for type j firms. Clearly

$$\begin{aligned} \sum_{i=0}^{n} N_i &= 1 \\ \sum_{i=0}^{j-1} \tau_{ij} &= 1 \text{ for all } j \\ \sum_{j=i+1}^{n} \kappa_{ij} &= 1 \text{ for all } i \end{aligned}$$

The labor market tightness in each submarket is denoted by θ_{ij}. It follows that

$$\theta_{ij} = f\frac{\tau_{ij} v_{ij}}{\kappa_{ij} N_i}$$

Finally, the flow equations read

$$\sum_{i=0}^{j-1} N_i p_{ij} \kappa_{ij} = [s + \delta + \sum_{k=j+1} p_{jk}\kappa_{jk}] N_j$$

In equilibrium we require that submarket ij is open if and only if

$$\Pi_{ij} = \max_{k<j} \Pi_{kj}$$

Finally, free entry of firms implies that

$$E\Pi = K$$

We are now able to show the following result, which we refer to as a maximum separation result

Proposition 7 *In any n-firm equilibrium the following holds*

a) Workers employed in a firm of type k always search for jobs with strictly higher wages than workers employed in firms of type $l < k$. Firms of type k always offer a strictly higher wage than firms of type l if $k > l$.

b) Let I_k denote the set of worker types searching for firms of type k. Consider I_k and I_l, $k > l$. Then

- *All elements in I_k are greater than or equal to all elements in I_l.*
- *I_k and I_l have at most one common element.*

It follows that the market, to the largest extent possible, separates workers and firms so that the low-type workers search for the low-type firms. Note the similarity with the non-assortative matching results in the search litterature (Shimer and Smith (2001), Eeckout and Kirkcher (2008). If the production technology is linear in the productivities of the worker and the firm, it is optimal that the high-type firms match with the low-type workers and vice versa. Similarly, in our model it is optimal that the workers in a firm with a high current productivity search for vacancies with high productivity, and vice versa.

To understand this result, recall that if vacancies are filled quickly that requires long worker queues, and the flip-side of the coin is that workers find jobs slowly. It is therefore optimal that the most "patient" workers, i.e., the workers with the highest current wage, search for the most "impatient" firms, the firms with the highest productivity. It is also trivial to exstend the efficiency result above to the n-firm case.

Let us then move to the continous case. Let the support of (active) firms be denoted by $[y_{\min}, y_{\max}]$. A firm's type continues to be its productivity, a worker's type the productivity of the firm she is working for. Let the type of the unemployed workers be denoted by z. Define the set I_j as for the discrete case. Clearly, the unemployed workers cannot appy only to the lowest type of firms, since the number of jobs in this firm has mass zero.

From a revealed preference argument, it follows that firms with different productivities advertize different wages. It also follows that the sets I_y and $I_{y'}$ have at most one element in common. Furthermore, if chosing from a continuous and monotone set of $(p, W(p))$ combinations, a high-type worker is always more willing to trade off p for W than is a lower-type worker. This leads us to the following conjecture:

Conjecture 8 *In equilibrium, a worker of type y_i searches for one firm type y_j only, where $y_j = f(y_i)$. Furthermore f is continuous and strictly increasing in y_j, and*

$$\begin{aligned} \lim_{y_j \to y_{\max}^-} f(y) &= y_{\max} \\ \lim_{y_j \to y_{\min}^+} f(y) &= y_z \end{aligned}$$

The next issue is then how to characterize $f(y)$. Let $(W(y), p(y))$ denote equilibrium values. It follows that $p = p(W)$. Let $\widetilde{V}(y_i) \equiv \max V(W, p(W), y_i)$ denote the profitability of a vacancy. Similarly, let $\widetilde{M}(y_j) \equiv M(p(W), y_j)$. The equilibrium is then has to satsify the following envelope conditions

$$\begin{aligned} \widetilde{V'}(y_j) &\equiv \frac{\partial V(W(y_j), p(y_i), y_j)}{\partial y_j} \\ \widetilde{S}(y_j) &\equiv \frac{S(p(W), y_j)}{\partial y_j} \end{aligned}$$

11 Endogenizing productivity differences

Suppose the firms can choose between (y_1, K_1) and (y_2, K_2), $y_1 < y_2$, $K_1 < K_2$. Furthermore, suppose that with no on-the-job search, the parameters are such that all firms choose the lowest investments.

Conjecture 9 *Suppose we allow for on-the-job search. Then the resulting equilibrium is a pure job-ladder equilibrium, determined by the zero profit conditions*

$$
\begin{aligned}
\Pi_1 &= K_1 \\
\Pi_{12} &= K_2
\end{aligned}
$$

Scetch of proof: First note that the 2-market cannot be empty. Suppose it is. Then a firm that opens up can fill its vacancies infinitely quickly, and thus as long as $y_2 > y_1$ can obtain unbound profit. Furthermore, by assumption it will not be profitable for firm 2 to search for unemployed workers, since in this market they will be dominated by type-1 firms.

Suppose the firms could choose investment-output combinations from a menu defined by the function $y = F(K)$, where $F(K)$ is increasing and concave and bounded above by $\overline{y}$. Then we conjecture that the equilibrium is a pure jobladder equilibrium with infinitely many steps.

12 APPENDIX:

13 Computation of the General Equilibrium

To solve the model one needs to set following 10 parameters: r, s, δ, y_1 and y_2, z, β c, k, α. In addition, the matching function we use is cobb douglas with share parameter β and with constant A.

The procedure to compute the equilibrium is as follows. First, the procedure tries to solve for the model with three submarkets. If this fails the procedure switches to the pure job ladder equilibrium. The solution is basically computed in four steps. The first steps (step i) solves for the asset equations in the general model, the second steps (step ii) computes τ, the proportion of good firms that hire directly from the unemployement pool and the final steps solves for the stock. Step three (step iii) is reached only if the proportion of firms that hires directly from the unemployed is less than one. In case this proportion τ is greater than one, the procedure goes to the step four (step iv) and solves for the pure job ladder equilibrium.

13.1 Step i): Solving for the Asset values in the general model

The procedure starts from assigning an arbitrary initial guess value of $M_1 = M_1^{'}$ and $rU = rU^{'}$. Given the initial guess, one can compute recursively $M_2^{'}, p_{01}^{'}, p_{02}^{'}, p_{12}^{'}$

$$
\begin{aligned}
M_2' &= \frac{y_2 + (r+\delta)U'}{r+\delta+s}; \qquad \text{using } M_2 = \frac{y_2 + (r+\delta)U}{r+\delta+s} \\
p_{12}' &= \frac{(r+\delta+s)M_1' - y_1 + rU'}{\beta(M_2' - M_1')} \qquad \text{using } M_1 = \frac{y_1 + (s+\delta)U + p_{12}\beta(M_2 - M_1)}{r+\delta+s} \\
p_{01}' &= \frac{rU' - z}{\beta(M_1' - U')} \qquad \text{using } rU = z + \beta p_{01}(M_1 - U) \\
p_{02}' &= \frac{rU' - z}{\beta(M_2' - U')} \qquad \text{using } rU = z + \beta p_{02}(M_2 - U)
\end{aligned}
$$

Given these values we define the function $d(M_1', U')$ as the difference in profits across high type firms so that

$$
\begin{aligned}
d(\Pi) &= \Pi_{12}() - \Pi_{02}() \\
d &= \frac{[(M_2' - U')(1-\beta)p_{12}']^2}{2} - \frac{[(M_2' - M_1')(1-\beta)p_{02}'^{-\frac{\beta}{1-\beta}}]^2}{2}
\end{aligned}
$$

For given value of U', the procedure updates the value of M_1' so that

$$
M_1'' = M_1' - \lambda d(\Pi)
$$

where $\lambda > 0$ is an adjustment parameter. In other, words we reduce the value M_1'' as long as $d()$ is positive. Given M_1'' and holding fixed U' update $M_2''', p_{01}'', p_{02}'', p_{12}''$ using M_2'' and proceed further until

$$
d(\Pi) \simeq 0
$$

Given M'' expected profits at entry are

$$
dE\Pi = \alpha\Pi_{01} + \Pi_{12} - k
$$

and update the value of U' so that

$$
U'' = U' + \lambda_1 dE\Pi
$$

Given U'', update the asset values and redo the procedure for finding $d(\Pi) \simeq 0$, and calculating U'''. The equilibrium in the first step is obtained for a couple M_1^* and U^* so that

$$
\begin{aligned}
d(\Pi) &\simeq 0 \\
dE\Pi(\Pi) &\simeq 0
\end{aligned}
$$

13.2 Step ii): Obtaining the fraction of firms τ that hire directly from the employed

The first step of the model has solved for M_1, rU $M, p_{01}, p_{12}, p_{02}$. The rest of the equations are obtained from

$$
\begin{aligned}
(p_{01})^{\frac{1}{1-\beta}} &= \frac{(1-\alpha)fv_1(0)}{k_{01}n_0} \\
(p_{12})^{\frac{1}{1-\beta}} &= \frac{\tau\alpha f v_2(1)}{n_1}^{1-\beta} \\
(p_{02})^{\frac{1}{1-\beta}} &= \frac{(1-\tau)\alpha f v_2(0)}{(1-k_{01})n_0}
\end{aligned}
$$

and the flows conditions

$$
\begin{aligned}
p_{02}k_{02}n_0 + p_{12}k_{01}n_1 &= (\delta+s)n_2 \\
p_{01}k_{01}n_0 &= (\delta+s+p_{12})n_1 \\
n_0+n_1+n_2 &= 1 \\
k_{01}+k_{02} &= 1
\end{aligned}
$$

Since $\frac{n_1}{k_{01}n_0} = \frac{p_{01}}{\delta+s+p_{12}}$ dividing the equation for $\theta_{01} = (p_{01})^{\frac{1}{1-\beta}}$ and $\theta_{12} = (p_{12})^{\frac{1}{1-\beta}}$one obtains immediately and expression for τ as

$$
\tau^* = \frac{\theta_{12}}{\theta_{01}} \frac{\alpha v_1(0)}{(1-\alpha)v_2(1)} \frac{p_{01}}{\delta+s+p_{12}}
$$

where $v_i = c(M_i - U)_i q(p_i)$ $i = 1, 2$. If $\tau^* < 1$ the equilibrium with all submarket is consistent and steps iii can be completed. Conversely, if $\tau^* > 1$ the routine solves for the pure job ladder equilibrium.

13.3 Step iii): Obtaining stocks in the general model

Assume $f = f'$ and $k_{01} = k'_{01}$ and obtain recursively

$$
\begin{aligned}
n'_0 &= \frac{\delta+s}{\delta+s+p_{01}k'_{01}+p_{12}(1-k'_{01})} \\
n'_1 &= \frac{p_{01}k_{01}n_0}{\delta+s+p_{12}} \\
n'_2 &= 1-n'_0-n'_1
\end{aligned}
$$

Given these values obtain the function dk as

$$
dk = (1-k_{01})\theta(p_{02})n_0 - (1-\tau)(1-\alpha)f'v_2(0)
$$

and update

$$
k'' = k' + \lambda dk
$$

Continue the procedure as long as k'' is such that

$$
dk \simeq 0
$$

With the completion of step iii the general equilibrium is fully solved.

Given k'' obtain the function df

$$df = f - \frac{n_1 \theta_{12}(p_{12})}{\tau * (1-\alpha) * v_2(1)}$$

and update the value of f' so that

$$f'' = f' - \lambda_1 df$$

Given f'' , update the stocks and redo the procedure for finding $d(k) \simeq 0$, and calculating k'''. The equilibrium in the first step is obtained for a couple f^* and k^* so that

$$\begin{aligned} d(f) &\simeq 0 \\ dE\Pi(k) &\simeq 0 \end{aligned}$$

14 Step iv. Solve for the pure job ladder equilibrium

The step iv is reached only if the routine finds a value of $\tau > 1$ in step ii. The procedure starts from an arbitrary initial guess value of $M_1 = M_1^{'}$ and $rU = rU^{'}$. Given the initial guess, it computes recursively $M_2^{'}, p_{01}^{'}, p_{02}^{'}, p_{12}^{'}$

$$\begin{aligned} M_2^{'} &= \frac{y_2 + (r+\delta)U^{'}}{r+\delta+s}; \qquad \text{using } M_2 = \frac{y_2 + (r+\delta)U}{r+\delta+s} \\ p_{12}^{'} &= \frac{(r+\delta+s)M_1^{'} - y_1 + rU^{'}}{\beta(M_2^{'} - M_1^{'})} \qquad \text{using } M_1 = \frac{y_1 + (s+\delta)U + p_{12}\beta(M_2 - M_1)}{r+\delta+s} \\ p_{01}^{'} &= \frac{rU' - z}{\beta(M_1^{'} - U')} \qquad \text{using } rU = z + \beta p_{01}(M_1 - U) \\ n_0^{'} &= \frac{\delta+s}{\delta+s+p_{01}^{'}} \\ n_1^{'} &= \frac{p_{01}^{'}(\delta+s)}{(\delta+s+p_{01}^{'})(\delta+s+p_{12}^{'})} \\ n_2^{'} &= 1 - n_1^{'} - u_1^{'} \\ f' &= \frac{n_1^{'}\theta_2(p_{12}^{'})}{(1-\alpha)v_2(1)} \end{aligned}$$

15 Welfare

Let κ denote the fraction of the unemployed workers that search for low-type firms, and $1-\kappa$ the fraction of the unemployed workers that search for high-type firms. Let x denote the inflow of new firms

The welfare function is

$$\begin{aligned} W &= \int_0^\infty [N_1 y_1 + (N_{12}+N_{02})y_2 + uz \\ &\quad -\alpha f\frac{v_1^2}{2c} - \tau(1-\alpha)f\frac{v_{12}^2}{2c} - (1-\tau)(1-\alpha)f\frac{v_{02}^2}{2c} - xK]e^{-rt}dt \end{aligned}$$

The law of motions are defined as

$$\begin{aligned} \dot{N}_1 &= x(\kappa u, \alpha f v_1) - (s+\delta)N_1 - x(N_1, (1-\alpha)\tau f v_{12}) \\ \dot{N}_{12} &= x(N_1, (1-\alpha)\tau f v_{12}) - (s+\delta)N_{12} \\ \dot{N}_{02} &= x((1-\kappa)u, (1-\alpha)(1-\tau)f v_{02}) - (s+\delta)N_{02} \\ \dot{u} &= (N_1+N_{12}+N_{02})(s+\delta) - x(\kappa u, \alpha f v_1) - x((1-\kappa)u, (1-\alpha)(1-\tau)f v_{02}) \\ \dot{f} &= x - \delta f \end{aligned}$$

The initial conditions take care of the requirement that $N_1+N_{12}+N_{02}+u=1$. The controls are x, v_1, v_{12}, v_{02}, κ, and τ. All fractions have to be between zero and 1, this will be discussed later. The current-value Hamiltonian reads

$$\begin{aligned} H &= N_1 y_1 + (N_{13}+N_{02})y_2 + uz - \alpha f\frac{v_1^2}{2c} - \tau(1-\alpha)f\frac{v_{12}^2}{2c} - (1-\tau)(1-\alpha)f\frac{v_{02}^2}{2c} - xK \\ &\quad +\lambda_1[x(\kappa u, \alpha f v_1) - (s+\delta)N_1 - x(N_1, (1-\alpha)\tau f v_{12})] \\ &\quad +\lambda_{12}[x(N_1, (1-\alpha)\tau f v_{12}) - (s+\delta)N_{12}] \\ &\quad +\lambda_{02}[x((1-\kappa)u, (1-\alpha)(1-\tau)f v_{02}) - (s+\delta)N_{02}] \\ &\quad +\lambda_u[(N_1+N_{12}+N_{02})(s+\delta) - x(\kappa u, \alpha f v_1) - x((1-\kappa)u, (1-\alpha)(1-\tau)f v_{02})] \\ &\quad +\lambda_f[x-\delta f] \end{aligned}$$

The controls are chosen so as to maximize H. Note that $x(u,v) = Au^\beta v^{(1-\beta)}$ it follows that $x_v = (1-\beta)Au^\beta v^{-\beta} = (1-\beta)q(\theta)$. We thus get the following first order conditions for vacancy creation:

$$\begin{aligned} \frac{v_1}{c} &= (1-\beta)q_1[\lambda_1 - \lambda_u] \\ \frac{v_{12}}{c} &= (1-\beta)q_{12}[\lambda_{12} - \lambda_1] \\ \frac{v_{02}}{c} &= (1-\beta)q_1[\lambda_{02} - \lambda_u] \end{aligned}$$

The first order conditions for the other controls read

$$\begin{aligned} \lambda_f &= K && (13) \\ p_u(\lambda_1 - \lambda_u) &\geq p_{u2}(\lambda_{02} - \lambda_u) \text{ with equality if } \kappa < 1 && (14) \\ q_{12}(\lambda_{12} - \lambda_1) &\geq q_{02}(\lambda_{02} - \lambda_u) \text{ with equality if } \tau < 1 && (15) \end{aligned}$$

(note that it is never optimal to set κ or τ equal to zero). Here we have used that $x_u = \beta p$. Note that if $\tau < 1$, the last condition implies that $v_{12} = v_{02}$. Finally, the value functions for the adjungated variables are given by (in steady state)

$$
\begin{aligned}
(r+s+\delta)\lambda_1 &= y_1 + \beta p_{01}(\lambda_{21}-\lambda_1) - (s+\delta)\lambda_u && (16)\\
(r+s+\delta)\lambda_{12} &= y_2 - (s+\delta)\lambda_u && (17)\\
(r+s+\delta)\lambda_{02} &= y_2 - (s+\delta)\lambda_u && (18)\\
r\lambda_u &= z + \kappa p_{01}(\lambda_1-\lambda_u) + (1-\kappa)p_{02}(\lambda_{02}-\lambda_u) = z + p_{01}(\lambda_1-\lambda_u) && (19)\\
(r+\delta)\lambda_f &= \alpha[(1-\beta)v_1 q_1(\lambda_1-\lambda_u) - \frac{v_1^2}{2c}] \\
&\quad +(1-\alpha)[\tau((1-\beta)v_{12}q_{12}(\lambda_{12}-\lambda_1) - \frac{v_{12}^2}{2c}) + (1-\tau)((1-\beta)v_{02}q_{02}(\lambda_{02}-\lambda_u) - \frac{v_{02}^2}{2c}) \\
&= \alpha[(1-\beta)v_1 q_1(\lambda_1-\lambda_u) - \frac{v_1^2}{2c}] + (1-\alpha)[(1-\beta)v_{12}q_{12}(\lambda_{12}-\lambda_1) - \frac{v_{12}^2}{2c}] && (20)
\end{aligned}
$$

(where the second last equality follows from the first order condition for κ). Note that

$$
\begin{aligned}
\lambda_1 - \lambda_u &= \frac{y_1 - r\lambda_u + \beta p_{01}(\lambda_{21}-\lambda_1)}{r+s+\delta} && (21)\\
\lambda_{12} - \lambda_1 &= \frac{y_2 - y_1}{r+s+\delta+\beta p_{01}} && (22)\\
\lambda_{02} - \lambda_u &= \frac{y_2 - r\lambda_u}{r+s+\delta} && (23)
\end{aligned}
$$

It follows that the first order conditions of the planner is exactly equal to the market solution. More than that, the maximization problem for the controls is exactly equal to the maximization problem of the firm. Thus, the planner's solution and the decentralized solution is the same.

16 Proof of claim related to α^*

We want to show the following claim: For a given number of firms f, there exists a unique α^* with the following property: If $\alpha > \alpha^*$ there exists a pure job ladder. If $\alpha < \alpha^*$ some high-type firms search for unemployed workers. We start by assuming that the number of vacancies per firm is constant.

Consider first the case where $\alpha \to 1$. Note that λ_{12} is limited above. We want to show that $\lim_{\alpha\to 1} q_{12} = \infty$. Suppose not, and suppose instead that q is bounded by $\overline{q}$. Since λ_{12} is limited above by $\overline{\lambda} = y_2/(r+s+\delta)$ it follows that v is limited above by $\frac{\overline{\lambda}\overline{q}}{c}$.

Let $\overline{N_1}$ denote the value of N_1 in the limit as $\alpha \to 1$. Clearly $\overline{N_1} > 0$ and $rU > z$. It follows that

$$\begin{aligned}
\lim_{\alpha\to 1} q_{12} &= \lim_{\alpha\to 1} A[\frac{(1-\alpha)fv}{N_1}]^{-\beta} \\
&\leq \lim_{\alpha\to 1} A[\frac{(1-\alpha)f\overline{v}}{N_1}]^{-\beta} \\
&= \infty
\end{aligned}$$

Hence q cannot be limited above. But then it follows that the profit of searching for employed workers goes to infinity as α goes to zero.

Consider then the profitability of a high-type firm searching for unemployed workers. Since λ_{02} is bounded above by $\overline{\lambda} = y_2/(r+s+\delta)$, the profit can only goes to infinity if q_{12} does. Suppose it does. Then workers applying to this job has a job finding rate of $p = 0$ and thus receives $rU = z$. However, the workers would then prefer to search for the low-type firm and we cannot be in equlibrium. It follows that it is more profitable to search for employed than for unemployed workers if α is sufficiently close to 1.

Suppose then $a \to 0$. It follows that $N_1 \to 0$. We want to show that the proportion of high-type firms searching for employed workers goes to 0. Suppose not, and suppose the share is bounded below by $\tau^{\min} > 0$. Suppose that in the limit, $v_{12} > 0$. It follows that

$$\begin{aligned}
\lim_{\alpha\to 0} q_{12} &= \lim_{\alpha\to 1} A[\frac{(1-\alpha)f\tau v_{12}}{N_1}]^{-\beta} \\
&\leq \lim_{\alpha\to 1} A[\frac{(1-\alpha)f\tau^{\min} v_{12}}{N_1}]^{-\beta} = 0
\end{aligned}$$

Note also that $v_{12} = 0$ if and only if $q_{12}\lambda_{12} = 0$. Thus both if $v_{12} = 0$ in the limit and when it is not the assumption that $\tau^{\min} > 0$ is inconsistent with (15).

Finally we want to show that for any $\alpha > 0$, $\tau > 0$. Suppose not. Then there exists a $\alpha > 0$ such that $\tau = 0$. If (15) is satisfied we must have that $v_{12} < \infty$. But then it follows that $q_{12} = \infty$, hence (15) cannot be satisfied. Again we have derived a contradiction.

Finally we want to show that there exists a unique α^* as described above. That there exists a α^* such that (15) is satisfied with equality for $\tau = 1$ follows from continuity and the results just laid out. What is left is to show that this α^* is unique. To this end it is sufficient to show that if (15) is satisfied with equality for $\tau = 1$, then an decrease in α implies that the right-hand side of (15) is striclty greater than the left-hand side for $\tau = 1$.

Suppose first that v is constant in all submarkets. In what follows we will work with α_2 rather than α, the fraction of high-type firms. We want to show that an increase in α_2 for a given f at $\tau = 1$ implies that searching for unemployed workers become strictly more profitable than searching for employed workers. (from 15

$$q_{12}(\lambda_{12} - \lambda_1) < q_{02}(\lambda_{02} - \lambda_u) \tag{24}$$

for α_2' marginally greater than α_2^*.

Suppose λ_u decreases. Then p_{01} decreases. From (23) it follows that $\lambda_{02} - \lambda_u$ increases. From (14) it follows that q_{02} increases. Thus the right-hand side of 24 increases. An increase in α_2 increases p_2, and from (22) it follows that $(\lambda_{12} - \lambda_1)$ decreases and mq_{12} decreases. Thus the left-hand side of 24 decreases. Hence we are done in this case.

Suppose then that λ_u is increasing in α_2 (which indeed seems likely). From (18) and (17) it follows that λ_{12} and λ_{02} are identical. In what follows we rescale the model by setting $z = 0$. Clearly this can be done without loss of generality, as the maxmization problem is unchanged if all flows z, y_1, y_2 are reduced equally much. It follows that we can write

$$\lambda_u = \frac{p_{01}}{r + p_{01}} \lambda_1$$

Thus, from (16)

$$\lambda_u \left(1 - \frac{s+\delta}{r+s+\delta}\right) \frac{r + p_{01}}{p_{01}} = \frac{y_1 + p_{01}(\lambda_{12} - \lambda_1)}{r+s+\delta}$$

Taking elasticities wrt α_2 gives

$$el\lambda_u + X < elp_{01} + el(\lambda_{12} - \lambda_1)$$

where $X = el\frac{r+p_{01}}{p_{01}} > 0$. It follows that

$$el\lambda_u < elp_{01} + el(\lambda_{12} - \lambda_1)$$

From (14) and (16) it follows that $r\lambda_u = p_{02}(\lambda_{02} - \lambda_u)$. Taking elasticities and using the above equation give

$$elp_{02} + el(\lambda_{02} - \lambda_1) < elp_{01} + el(\lambda_{12} - \lambda_1) - el(\lambda_{02} - \lambda_0)$$

or

$$elp_{02} < elp_{01} + el(\lambda_{12} - \lambda_1) - el(\lambda_{02} - \lambda_0) \tag{25}$$

From (18) and (17) it follows that λ_{12} and λ_{02} are identical. Furthermore, as $\frac{\delta\lambda_1}{\delta\alpha_2} > \frac{\delta\lambda_u}{\delta\alpha_2}$ and $\lambda_{12} - \lambda_u < \lambda_{02} - \lambda_u$ it follows that $0 > el(\lambda_{02} - \lambda_0) > el(\lambda_{12} - \lambda_1)$. From (25) it thus follows that $elp_{02} < elp_{01}$ and thus that $elq_{02} > elq_{12}$. Furthermore, since $el(\lambda_{02} - \lambda_0) > el(\lambda_{12} - \lambda_1)$ this implies that (24) is satisfied.

17 References

Bartelsman, Eric J. and Mark Doms (2000). Understanding productivity: Lessons from longitudinal

microdata. *Journal of Economic Literature* 38, no. 3: 569: 594

Bartelsman, Eric; Scarpetta, Stefano and Schivardi, Fabiano, (2005) "Comparative Analysis of Firm Demographics and Survival: Micro-level Evidence for the OECD countries " *Industrial and Corporate Change*, Vol. 14, pp. 365-391, 2005.

Baily, Martin Neil, Charles Hulten and David Campbell. (1992) “Productivity Dynamics in Manufacturing Establishments,” *Brookings Papers on Economic Activity*: Microeconomics, 1992, 187-249.

Belzil, Christian, (2000). "Job Creation and Job Destruction, Worker Reallocation, and Wages," *Journal of Labor Economics*, University of Chicago Press, vol. 18(2), pages 183-203,

Bertola and Cabalero, (1994); Cross Sectional Efficiency and Labour Hoarding in a Matching Model of Unemployment, *Review of Economic Studies*, Vol 61(3) 435-457

Bertola, G. and Garibaldi, P. (2001), "Wages and the Size of Firms in Dynamic Matching Models." *Review of Economic Dynamics* 4 (2001): 335-368

Davis, Steven J., and John Haltiwanger (1999) "Gross Job Flows," in Ashenfelter and Card.(eds.), *Handbook of Labor Economics*, North-Holland, 1999, 2711-2805.

Delacroix, Alan and Shi, Shouyong (2006), "Directed Search On the Job and the Wage Ladder", *International Economic Review* 2006, vol 47, 651-699

Hopenhyn, Hugo. (1992) "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," *Econometrica*, 60(5), 1127-50.

Hopenhyn, Hugo, and Richard Rogerson. (1993) "Job Turnover and Policy Evaluation: A General Equilibrium Approach," *Journal of Political Economy*, 101(5), 915-38.

Melitz (2003) The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity *Econometrica* 71

Nobuhiro Kiyotaki & Ricardo Lagos, (2006). "A model of job and worker flows," Staff Report 358, *Federal Reserve Bank of Minneapolis*

Foster Lucia; Haltiwanger, John; and Chad Syverson (2007). "Reallocation, Firm Turnover and Efficiency: Selection on Productivity or Profitability?" *University of Maryland.*

Klette TJ and S Kortum (2004), " Innovating Firms and Aggregate Innovation", *Journal of Political Economy*, 112, 986–1018

Lane, Julia; Haltiwanger, John and Spletzer, James (1999) "Productivity Differences Across Employers: The Role of Employer Size, Age, and Human Capital," *American Economic Review*, 94-98.

Restuccia Diego , and Richard Rogerson (2007): Policy distortions and aggregate productivity with heterogenous plants. *NBER Working Paper* 13018

Moen, Espen (1997) "Competitive Search Equilibrium". *Journal of Political Economy* 1997, vol 105, 385- 411

Moen Espen, and Åsa Rosen (2004) "Does poaching distort training?" *Review of Economic Studies*, 2004 vol. 71, 1143-1162.

Lentz, Rasmus, and Mortensen, Dale, (2005) "Productivity Growth and Worker Reallocation" *International Economic Review*, Vol. 46, No. 3, pp. 731-749,

Lentz, Rasmus, and Mortensen, Dale (2006), "An Empirical Model of Growth through Product Innovation". Unpublished paper.

Lentz, Rasmus, and Mortensen, Dale (2007), "Labor Market Friction, Firm Hetrogeneity, and Productivity". Unpublished paper.

Mortensen Dale., and Pissarides, Christopher Albert 1994, Job Creation and Job Destruction in the Theory of Unemployment, *Review of Economic Studies* Vol 61(3) 397-415

Oi, Walter and Idosn, Todd (1999) Firm Size and Wages, in *Handobook of Labor Economics*, Vol 3. O. Ashenfleter and D. Card eds, 2165-2214, Elsevier Science, Amsterdam

Pissarides (1994), Search Unemployment with On-the-job-search, *Review of Economic Studies*, Vol 61(3) 457-494

Shi (2006)

Shimer (2006), On-the-job Search and Strategic Bargaining, *European Economic Review*, 50(4): 811-830.

Yashiv, Eran (2000a) The Determinants of Equilibrium Unemployment, *American Economic Review*, Vol. 90 No. 5, 1297-1322

Yashiv, Eran (2000b) Hiring as Investment Behavior, *Review of Economic Dynamics*, Vol. 3, 486-522.

Table 1: Baseline Calibration with three submarkets

Parameter	Notation	Value
Pure Discount Rate	r	0.010
Separation Rate	s	0.040
Firm Bankruptcy Rate	δ	0.020
Bargaining Share	β	0.500
entry cost	k	5.000
low type proportion	α	0.9300
high type productivity	y_1	1.000
low type productivity	y_2	1.100
unemployed income	z	0.550
search cost parameter	c	0.150
matching function parameter	A	7.000
matching function elasticity	β	0.500
Equilibrium Values		
Joint Income 1	M_1	100.8178
Joint Income 2	M_2	101.3549
unemployment flow value	rU	0.9991
unempl. job finding rate in low type	p_{01}	0.9939
on the job finding rate	p_{12}	0.2324
unempl. job finding rate directly to high type	p_{02}	0.6234
Equilibrium Quantities		
Unemployment	n_0	0.0585
Employment in Low productivity type	n_1	0.1915
Employment in High productivity type	n_2	0.7500
Proportion of unemployed in submkt 01	k_{01}	0.9633
Number of Firms	f	0.0628
Proportion of high type firms in submarket 12	τ	0.9714
Worker Flows		
Unemployment Flows	$n_0 * (p_{01} + p_{02})$	0.0573
Job to Job Flows	$n_1 * p_{12}$	0.0445
Firm Size, PDV Wages and Profits		
Profits in submarket 01	Π_{01}	3.6174
Profits in submarket 02	Π_{02}	23.3687
Profits in submarket 12	Π_{12}	23.3687
Firm Size in submarket 01	N_{01}	0.1031
Firm Size in submarket 02	N_{02}	2.0363
Firm Size in submarket 12	N_{12}	5.4628
Wages in submarket 01	W_{01}	100.3659
Wages in submarket 02	W_{02}	100.7196
Wages in submarket 12	W_{12}	102.0753
Source: Authors' calculation		

Table 2: Baseline Calibration with two submarkets

Parameter	Notation	Value
Pure Discount Rate	r	0.010
Separation Rate	s	0.040
Firm Bankruptcy Rate	δ	0.020
Bargaining Share	β	0.500
entry cost	k	5.000
low type proportion	α	0.9400
high type productivity	y_1	1.000
low type productivity	y_2	1.100
unemployed income	z	0.550
search cost parameter	c	0.150
matching function parameter	A	7.000
matching function elasticity	β	0.500
Equilibrium Values		
Joint Income 1	M_1	100.7446
Joint Income 2	M_2	101.2824
unemployment flow value	rU	0.9983
unempl. job finding rate in low type	p_{01}	0.9797
on the job finding rate	p_{12}	0.2319
Equilibrium Quantities		
Unemployment	n_0	0.0577
Employment in Low productivity type	n_1	0.1937
Employment in High productivity type	n_2	0.7486
Number of Firms	f	0.0713
Worker Flows		
Unemployment Flows	$u * (p_{01} + p_{02})$	0.0565
Job to Job Flows	$n_1 * p_{12}$	0.0449
Firm Size, PDV Wages and Profits		
Profits in submarket 01	Π_{01}	3.8171
Profits in submarket 12	Π_{12}	23.5327
Firm Size in submarket 01	N_{01}	0.1222
Firm Size in submarket 12	N_{12}	6.2383
Wages in submarket 01	W_{01}	100.2870
Wages in submarket 12	W_{12}	100.6362
Source: Authors' calculation		