Equilibrium Default

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June, 2010

Abstract

This paper studies the optimal financing of an investment project subject to the risk of default. A project needs outside funding from a lender, but the borrower can walk away at any moment and take some outside opportunity. The value of this opportunity is random and not observable by the lender. We show that the optimal dynamic contract may allow default along the equilibrium path. Focusing on the dynamics of default, debt and capital accumulation, we find that over the life of the project the probability of default declines, long-term debt falls and capital rises.

Introduction

This paper derives the optimal dynamic contract to finance an investment project when the borrower lacks the commitment to not default. In our model, the borrower has a project that needs outside funds to cover an initial fixed cost as well as a stream of capital investments. Lending is constrained by voluntary repayment: the borrower can, at any moment, walk away for some outside opportunity. The value of the outside opportunity is random and private information. This assumption implies that default actually occurs in equilibrium. Within this environment, we study constrained-efficient long-term contracts between the outside lender and the borrower. That is, we place no ad hoc assumptions on contracting, but impose the constraints arising from the entrepreneur's private information and lack of commitment.

Our model allows for a few different economic interpretations. The most obvious is the financing of a firm, where the lender is an entrepreneuer undertaking an investment project and the borrower is a bank seeking to finance this project, as in Albuquerque and

Hopenhayn (2004). Alternatively, one can apply the model to human capital investments and on-the-job training. The borrower in this case is a worker accumulating skills while working at a firm. The firm helps finance these investment but is concerned about retaining the worker. Finally, the model can be applied to an international context by interpreting the borrower as a sovereign government seeking to finance its government spending, including public investments, from foreign investors, as in Thomas and Worrall (1994).

We assume that outside opportunities are unattractive relative to the investment project, so that it is never efficient to take them. With commitment the outside opportunities are completely irrelevant. However, without commitment, the availability of the outside opportunities does constrain lending because the entrepreneur may take them if they are privately, but not socially, efficient.

When the value of the outside option is non random or is publicly observed, as in Albuquerque and Hopenhayn (2004), then it is never optimal to exercise it. Default never occurs along the equilibrium path. This property is also characteristic of other limited commitment models, such as in applications to consumption insurance and smoothing among individuals (Kocherlakota, 1996; Alvarez and Jermann, 2000), or international borrowing and lending by sovereign countries (Thomas and Worrall, 1994; Kehoe and Perri, 2002). While all these models capture the friction that ex post *potential* default introduces on ex ante lending, they are completely silent on *actual* default behavior.

In our model, in contrast, default occurs along the equilibrium path. Whenever the value of the outside opportunity is random, and its realization is not observed by the lender, we show that the optimal contract allows firms to exercise the outside option. The contract treats all firms identically, but over time some firms default and choose to exit the contract, while others choose to stay.

Our focus is on the dynamics of capital, debt and default. The optimal contract offered by the lender is forward looking, so it specifies and anticipates all these variables. We find that the probability of default is highest at the beginning of the lending relationship and declines over time. Capital rises over time as the entrepreneur pays back the long-term debt to the lender for initiating the project. The lender may play an increasing role in the financing of capital investments through short-term debt.

We show that this optimum can be implemented using risk adjusted short-term debt. In this implementation the lender decides investment and borrowing sequentially. The interest rate depends on the current level of debt and the level of capital.

Our environment is closest to Albuquerque and Hopenhayn (2004) model of firm finance. Their model features limited commitment, but all uncertainty is publicly observable by the lender. Clementi and Hopenhayn (2006), on the other hand, considers unobservable shocks

to revenue, but with no commitment problems. In both papers, if liquidation is assumed valuable enough to the lender, then exit may occur after a sufficiently bad history of shocks, but there is no default. For contrast, we assume in our model that the liquidation value to the lender is zero: the interaction of private information and limited commitment is then key for our results on default.

Our paper is also connected to a strand of literature that features default and studies the impact on lending, but limits a priori the set of available contracts. In a seminal paper, Eaton and Gersovitz (1981) study a model of sovereign debt imposing the restriction that debt contracts be short term and not state contingent. In the model the sovereign country faces idiosyncratic income shocks that it wishes to smooth and insure with foreigners who are risk neutral. The punishment for default is exclusion from financial markets. Arellano (2007) extend this framework by allowing income shocks to be persistent. This kind of model has also been applied to consumer credit by Chatterjee et al. (2005). In all these models, because debt is required to be non-continent, default does occur in equilibrium and interest rates on loans adjust to the varying default risk. Our paper differs from this literature in a few dimensions. First, we model the financing of an investment project, instead of the provision of consumption smoothing. Second, we do not restrict contracts to be noncontingent, but instead derive them from the assymmetry of information regarding outside options. Finally, and most importantly, our contracts are derived from a long-term relation, instead of a sequence of short term contracts. In particular, this implies that we allow for an unrestricted maturity structure of debt, instead of imposing that all debt be short term.[1]

The rest of the paper is organized as follows. In Section 2 we present the basic model economy and solve for the first best allocation. In Section 3 we briefly discuss the optimal contracting problem if the outside options were observable and contractable. Section 4 contains our main analysis for the model with unobservable outside options and default. Section 5 discusses a possible implementation of this optimal contract. Section 6 discusses relates the financially constrained path of investment in our model with familiar Tobin-q characterization of optimal investment. Section 7 contains our conclusions.

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[1]Arellano and Ramanarayanan (2006) considers a consumption smoothing model with noncontingent bonds, as in Eaton and Gersovitz (1981) but allowing for bonds of two maturities, one- and two-period bonds.

1   An Investment Project in Need of Funding

We study the efficient contract between an agent and a principal. Both are infinitely lived, risk neutral and discount flows at the same rate $r > 0$. The agent is needed to run the project, she can exclusively operate the technology. The principal can help finance it by enduring negative flows. In contrast the agent must consume non-negatively and starts with no financial wealth.[2]

The project requires an initial irreversible investment of I_0. This payment initiates capital at k_0 and opens access to the following stream of profits. At any point in time t, if $k(t)$ is the current stock of capital and $\dot{k}(t) = \frac{\partial}{\partial t} k(t)$ is its rate of change, flow returns are $R(k(t), \dot{k}(t))$ which capture profits net of investment costs.

We assume the return function $R(k, \dot{k})$ is concave, and strictly so in at least one of its arguments. When it is strictly concave in $\dot{k}$ we say there are adjustment costs in capital accumulation. If, on the other hand, it is linear in $\dot{k}$ there are no adjustment costs and we write $R(k, \dot{k}) = R(k, 0) - \dot{k}$. Finally, we impose a condition on the cross partial that ensures there is at most one steady-state level capital at the first-best.

Assumption 1. *The return function $R(k, \dot{k})$ is twice continuously differentiable and concave, with either: (a) $R(\cdot, \dot{k})$ strictly concave for all $\dot{k}$; or (b) $R(k, \cdot)$ strictly concave for all k. In addition, the cross-partial derivative satisfies $R_{21}(k, 0) \leq 0$ for all k.*

A simple example satisfying these assumptions is

$$R(k, \dot{k}) = \Phi(k) - \delta k - c(\dot{k}) - \dot{k},$$

where Φ is concave and represents revenues, Ψ is convex with $\Psi'(0) = 0$ and represents adjustment costs, and $\delta \geq 0$ is the rate of depreciation. In later sections we develop a few other examples.

The principal may commit to any contractual agreement, but the agent lacks such commitment. Before describing these contracts and the constraints that arise from this lack of commitment, it is useful to study the first-best benchmark allocation that maximizes the present value of profits without any such constraints.

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[2]Alternatively, we could assume the principal also consumes non-negatively but holds a large enough initial wealth position.

1.1 First best

Denote by $W(k)$ the value of the optimal accumulation problem given some initial capital k_0:

$$W(k_0) \equiv \sup_{\{k(t)\}} \int_0^\infty e^{-rt} R(k(t), \dot{k}(t))\, dt \qquad \text{s.t.} \quad k(0) = k_0.$$

Let $k^*(t)$ for $t \in [0, \infty)$ denote an optimal trajectory. Since the problem is convex, the value function W is increasing, continuously differentiable and concave. Its Bellman equation is given by

$$rW(k) = \sup_{\dot{k}} \left(R(k, \dot{k}) + W'(k)\dot{k} \right). \tag{1}$$

Consider the case with adjustment costs, so that $R(k, \cdot)$ strictly concave. Let $g(k)$ denote the policy function for capital accumulation $\dot{k}$ that maximizes (1). Then the optimal trajectory $k^*(t)$ solves the ordinary differential equation

$$\dot{k}^*(t) = g\big(k^*(t)\big) \quad \text{with} \quad k^*(0) = k_0.$$

There exists at most one finite steady state k_{ss} with $g(k_{ss}) = 0$. To see this, combine the first order condition $R_2(k_{ss}, 0) + W'(k_{ss}) = 0$, with the envelope condition, $rW'(k_{ss}) = R_1(k_{ss}, 0)$, to obtain the steady-state condition

$$\frac{1}{r} R_1(k_{ss}, 0) + R_2(k_{ss}, 0) = 0.$$

Assumption 1 then ensures that the left-hand side is locally decreasing, so there exists at most one value k_{ss} satisfying this equation. It also ensures the global stability of any such steady state: $\dot{k} = g(k) > 0$ for $k < k_{ss}$ and $\dot{k} = g(k) > 0$ for $k > k_{ss}$.

We assume that the project starts with capital below its steady-state level. If no steady state exists, so that $g(k) > 0$ for all k then we say that $k_{ss} = \infty$.

Assumption 2. $k_0 \le k_{ss}$.

Things are particularly simple without adjustment costs, when $R(k, \cdot)$ is linear. Integrating by parts gives

$$\int_0^\infty e^{-rt} R(k(t), \dot{k}(t))\, dt = k_0 + \int_0^\infty e^{-rt} (R(k(t), 0) - rk(t))\, dt.$$

Thus, the supremum is attained by a path that is discontinuous at $t = 0$ and sets capital $k^*(t)$ to the constant level k_{ss} that maximizes flow profits net of capital rental, $R(k, 0) - rk$. The value function is linear in initial capital: $W(k) = k_0 + \frac{1}{r} \max_k \{R(k, 0) - rk\}$.

In either case, the project is worthwhile investing if and only if $W(k_0) - I_0 > 0$. To get things going, we assume this is the case throughout.

Assumption 3. $W(k_0) - I_0 > 0$.

1.2 Outside option

The agent receives outside options which, if taken, leave a residual value of zero to the lender. (This could be modified and a positive liquidation value included.) The outside value s arrives with Poisson intensity λ and is distributed according to $F(s, k)$, which increases in first stochastic order with k. Notice that aside from the dependence on k there is no persistence in the process for the shocks. This simplifies the analysis, but is not essential.[3]

When the outside option is publicly observable, this problem has the same structure as Thomas-Worrall and has been studied in Albuquerque-Hopenhayn in a discrete time setting. Our main objective is to depart from the previous literature by assuming that this outside value is privately observed. For contrast, however, we will first consider the observable case.

To make things stark, we assume that the outside option is never good enough to warrant separation: within the support of $F(\cdot, k)$ it is never efficient to so, for all values of k.

Assumption 4. *If $F(s, k) \in (0, 1)$ then $s \leq W(k)$.*

As we show below, this implies that there will be no liquidation when the outside option is observable. Separations will be optimal, however, when outside options are privately observed by the agent, since avoiding them may be too costly. This emphasizes a novel aspects of our model: default, as inefficient separations, occurs in equilibrium.

1.3 Two Interpretations

We offer two economic interpretations of the model.

In the first, investment takes place in physical capital and the contract describes the lending relationship between an entrepreneur/firm/borrower and an outside lender. The value of the outside option may depend on capital to the extent that the entrepreneur is able to walk away with some of the firm's resources.

In the second, investment takes place in human capital and the contract describes the optimal employment relationship between a worker and her employer. The outside option may be interpreted as a new employment opportunity, the value of which may depend on capital to the extent that human capital is at least partly "general" and is not entirely "firm specific".

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[3]We also studied a version of the model with persistent shocks and obtained similar results.

2 Observable Outside Options: No Default

When the outside option is observable, the principal can always retain the agent by ensuring that she is better off within the relationship. This may require the contract to match outside options as they arise. Assumption 4 ensures that doing so is preferable to breaking the relationship. Thus, default, defined here as an inefficient separation, never occurs in equilibrium.

The problem can be solved recursively using the continuation utility of the borrower V as a state variable together with the stock k. Let $B(V, k)$ denote the value to the lender, which must satisfy the Hamilton-Jacobi-Bellman equation

$$rB(V, k) = \max_{\dot{k}, V(s), \dot{V}, c} \left(R(k, \dot{k}) - c + \lambda \int (B(V(s), k) - B(V, k)) F(ds, k) \right.$$
$$\left. + B_1(V, k)\dot{V} + B_2(V, k)\dot{k} \right) \quad (2)$$

subject to

$$c \geq 0, \quad (3)$$

$$rV = c + \lambda \int (V(s) - V) F(ds, k) + \dot{V}, \quad (4)$$

$$V(s) \geq s \qquad \forall s. \quad (5)$$

Here $V(s) - V$ represents the promised jumps in continuation utility when an outside offer s occurs.

For high enough values of V the first best is attainable as follows. Define $V_{\min}(k)$ to be the smallest value that can be delivered to the agent assuming k follows the optimal path $\dot{k} = g(k)$. This value satisfies the following Bellman equation with $c = 0$:

$$rV_{\min}(k) = \lambda \int_{V_{\min}(k)} (s - V_{\min}(k)) F(ds, k) + V'_{\min}(k) g(k). \quad (6)$$

For any value $V \geq V_{\min}(k)$ define the dividend schedule $d(V, k)$ as follows:

$$rV = d(V, k) + \lambda \int_V (s - V) F(ds, k). \quad (7)$$

This schedule will be decreasing in k and will reach zero precisely at the point where $V_{\min}(k(t))$ catches up to V.

Proposition 2.1. *For any (V, k) where $V \geq V_{\min}(k)$, $B(V, k) = W(k) - V$ so the corresponding investment policy coincides with the optimal one.*

Proof. Use the dividend schedule defined above for consumption. We show now that $B(V, k) = W(k) - V$ solves the Hamilton-Jacobi Bellman equation. Substituting on the right hand side of equation (2) gives

$$
\begin{aligned}
rB(V, k) &= \max_{k, c} R(k, \dot{k}) - c + \lambda \int_V (W(k) - s - (W(k) - V(V(s), k))) F(ds, k) - \dot{V} + W'(k)k \\
&= \max_{k} R(k, \dot{k}) - [d(V, k) + \lambda \int_V (s - V) F(ds, k)] + \dot{V} + W'(k)k \\
&= r(W(k) - V).
\end{aligned}
$$

This verifies that $B(V, k) = W(k) - V$ is a solution to the Hamilton-Jacobi equation. $\qquad \square$

Corollary 2.2. *The optimal contract is financially feasible if and only if $W(k_0) - I_0 \geq V_{\min}(k_0)$.*

Note that at this particular point with $V_0 = V_{\min}(k_0)$ the value of debt $B(V, k_0)$ is decreasing in V. This implies that the value to the lender is maximized at some $V < V_{\min}(k_0)$. Under the condition given in the above Corollary, competitive bidding of lenders would lead to an efficient investment policy while monopoly power would lead to underinvestment. If the condition of the corollary is not satisfied, competitive lending would also lead to $V_0 < V_{\min}(k_0)$, where $B(V_0, k_0) = I_0$. Underinvestment will occur until a sufficiently high outside option occurs that sets the continuation value of the borrower $V(t) \geq V_{\min}(k(t))$.

3 Unobservable Outside Options: Equilibrium Default

We turn now to the case where the outside value s is privately observed by the borrower. Assume for now that there is no recall with respect to this outside option, so it is lost if not taken when the arrival occurs. As before, we consider the a recursive representation for the lenders problem with state variables (V, k). The key difference to the case of observable outside option is that continuation values for the agent cannot be made contingent on s, for otherwise all agents would claim having the outside option that maximizes the continuation $V(\tilde{s})$. Letting $\dot{V}$ represent the rate of change of V, the agent's value evolves according to:

$$
rV = c + \lambda \int_V (s - V) F(ds, k) + \dot{V}. \tag{8}
$$

Setting $c = 0$ (which will be optimal until the first best is attainable) defines a function $\dot{V} = m(V, k) = rV - \lambda \int_V (s - V) F(ds, k)$ which is increasing in V and decreasing in k. Let $h(V, k) \equiv \lambda(1 - F(V, k))$ denote the hazard rate of separation, which is increasing in k and

decreasing in V. Note that a high current value of capital k increases the current hazard rate directly, and has a positive indirect effect on future hazard rates through a lower rate of change $\dot{V}$.

The value to the lender $B(V, k)$ satisfies the Hamilton-Jacobi-Bellman equation

$$rB(V, k) = \max_{\dot{k}, c} \left(R(k, \dot{k}) - c - h(V, k)B(V, k) + B_1(V, k)(m(V, k) + c) + B_2(V, k)\dot{k} \right). \quad (9)$$

The first-order condition for consumption is

$$-1 + B_1(V, k) \geq 0,$$

and $c = 0$ is the unique optimum if this inequality is strict; otherwise, consumption is indeterminate. Thus, without loss of generality we set $c = 0$. The first order conditions for capital accumulation

$$R_2(k, \dot{k}) + B_2(V, k) = 0. \quad (10)$$

This equation implicitly defines the policy function $\dot{k} = g(V, k)$.

It would be useful to know whether the value function is concave in V for given k. Although we cannot answer this question in general, we can relate the local curvature of V to the dynamics of V and k. Note that $m_1(V, k) = r + h(V, k)$. Using this and applying the envelope theorem to (9) the above is obtained. The value function $B(V, k)$ satisfies the following equation:

$$-h_1(V, k)B(V, k) + B_{11}(V, k)m(V, k) + B_{21}(V, k)\dot{k} = 0 \quad (11)$$

The first order condition implies that $B_{21}(V, k) = -R_{22}(V, k)\frac{\partial \dot{k}}{\partial V}$, which is nonnegative if $\dot{k}$ is increasing in V or if R is linear in $\dot{k}$, e.g. no adjustment cost. If in turn $\dot{V} \geq 0$ equation (11) implies that $B_{11} \leq 0$.

4 Two Cases Solved

This section considers two special cases. In the first, the distribution of outside options F is independent of capital k. This would be the case if installed physical capital cannot be taken by the borrower upon default. Or in the human-capital accumulation interpretation, if human capital may be entirely firm specific. This case provides a stark contrast between the situation where the outside option is publicly observable and the one where it is not. In the second case, returns and outside values are linear in capital k. For example, this

would be the case in the aggregate international economy context with an "AK" model of technology, which implies sustained growth, and if the country keeps only part of the capital after a default. The homogeneity assumption allows us to reduce the state variables of the contracting problem and gives a simple characterization of the optimum.

4.1   Outside Value Independent of Capital

We begin with the case where outside options is unaffected by capital, so that the distribution of outside options is given by some c.d.f. $F(s)$. This case provides the strongest contrast with the observable outside option case, since with observability the lack of commitment does not affect contracting. Either the project is implemented with first-best capital accumulation, or not at all, and this choice is fully efficient. This follows because $V_{\min}(k)$ is independent of k. If, in addition, there are no adjustment costs, then there are no transitional dynamics: capital is initiated immediately at its steady state level. With observable outside options inefficiencies arise only when the outside option depends on capital.

When the outside option is privately observed by the agent this is no longer the case. Due to the agent's lack of commitment inefficient separations or default then arise. This, in turn, affects the desired capital accumulation path as well as the decision to finance the project or not. Finally, even without adjustment costs, there are now nontrivial transitional dynamics towards the steady state. We conclude that, in contrast to the observable outside options case, it is no longer crucial that outside options depend on capital k for the lack of commitment to create losses in efficiency. Thus, assuming that capital does not affect outside options generates the sharpest contrast between the optimal contract with observable and unobservable outside options.

When F is independent of k, equation (8) specializes to

$$rV = c + \lambda \int_V (s - V)F(ds) + \dot{V}. \tag{12}$$

As in Albuquerque-Hopenhayn it is optimal to set $c = 0$ until V reaches $\bar{s}$. At that point a constant wage $w = r\bar{s}$ can be used and the first-best is attained. Setting $c = 0$ and integrating by parts gives

$$rV = \int_V h(s)ds + \dot{V}, \tag{13}$$

where $h(s) \equiv \lambda(1 - F(s))$. The lower bound $V_{\min}$ for V is obtained just like before, by setting $\dot{V} = 0$.

For any initial value $V_0 > V_{\min}$ the differential equation (13) generates an increasing path $V(t)$ that reaches $\bar{s}$ in finite time. This implies a decreasing path for the hazard rate of sep-

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aration, defined by $H(t) \equiv \lambda(1 - F(V(t)))$. Let $S(t) = e^{-\int_0^t H(\tau)d\tau}$ denote the corresponding survival function. The optimal accumulation path from (k_0, V_0) then maximizes

$$\int e^{-rt} S(t) R(k(t), \dot{k}(t)) \, dt.$$

This is a fairly standard accumulation problem with a risk adjusted discount factor. It can be shown that the optimal accumulation rule satisfies

$$(r + H(t))R_2(k, \dot{k}) = -R_1(k, \dot{k}) + \frac{d}{dt}R_2(k, \dot{k}) \tag{14}$$

Intuitively, delaying investment saves on the interest and hazard of loss of the corresponding adjustment cost. This is weighted against the loss of marginal revenue and changes in the adjustment cost. Note that in the special case with no adjustment cost where $R(k, \dot{k}) = f(k) - \dot{k}$ this condition reads $(r + h(t)) = f'(k)$, a standard investment equation relating the stock of capital to the risk adjusted interest. For that case, the stock of capital rises over time to the stationary optimal level as the risk of default decreases to zero.

We summarize this discussion in the next proposition.

Proposition 4.1. *Suppose the distribution of outside options does not depend on installed capital k. The optimal contract generates a declining path for the default probability. Without adjustment costs, capital $k(t)$ rises over time.*

For the case without adjustment costs, consider the comparative static exercise of an improvement in outside options, so that the new distribution $\tilde{F}$ first-order stochastically dominates the original F. Intuitively, this increases the risk of default and makes financing more costly. Indeed, it follows directly that $\tilde{h}(V) \geq h(V)$ and that $\tilde{m}(v) \leq m(v)$. For given v_0 this implies that hazard rates are higher at all points in time with distribution $\tilde{F}$. As a result, $\tilde{B}(v) \leq B(v)$. Thus, if the initial value of v_0 is set at a break even point for the lender, so that $B(v_0) - I_0 = 0$, then $\tilde{v}_0 \leq v_0$, which only reinforces the result. We summarize this result in the next proposition.

Proposition 4.2. *Suppose no adjustment costs and that $\tilde{F}$ first-order stochastically dominates F, so that $\tilde{F}(s) \leq F(s)$ for all s. Then the optimal contract for $\tilde{F}$ features higher default rates along the equilibrium path and lower investment.*

4.2 Homogeneous Returns and Outside Options

In this section we consider a specification where the outside value is affected by k in a linear fashion and where the revenue function R is homogeneous of degree one in $(k, \dot{k})$.

That is, the outside option has the form sk, where s has c.d.f. $F(ds)$. Note that with this specification, the hazard rate of separation $h(V,k) = \lambda(1 - F(\frac{V}{k}))$ is homogenous of degree zero in (V,k) and the function $m(V,k)$ is homogenous of degree one. The value and policy functions inherits this property.

Proposition 4.3. *The value function $B(V,k)$ and the policy function $g(V,k)$ are homogeneous of degree one in (V,k).*

We can exploit the homogeneity to rewrite the Bellman equation as a function of the ratio $v = V/k$:

$$rb(v) = \max_{\dot{k}}\{R(1,\dot{k}) - w - h(v,1)b(v) + b'(v)m(v,1) + (b(v) - b'(v)v)\dot{k}\}$$

where $b(v) = B(v,1)$. The optimal choice in the rate of growth $\dot{k} = g(v)$ solves:

$$R_2(1,\dot{k}) + b(v) - b'(v)v = 0. \tag{15}$$

$$R_{22}(1,\dot{k})\frac{\partial\dot{k}}{\partial v} = b''(v) \tag{16}$$

so $\frac{\partial\dot{k}}{\partial v}$ has the opposite sign of b''. Finally, the envelope condition for this problem is

$$-h_1(v,1)b(v) + b''(v)(m(v,1) - v\dot{k}) = 0. \tag{17}$$

Note that

$$\dot{v} = \frac{\dot{V}}{k} - \frac{V}{k}\frac{\dot{k}}{k} = \frac{1}{k}(m(v,1) - v\dot{k}) \tag{18}$$

so the envelope condition can be rewritten as:

$$-h_1(v,1)b(v) + b''(v)\dot{v} = 0, \tag{19}$$

showing that the sign of b'' is the opposite of $\dot{v}$. This proves the following.

Proposition 4.4. *If $B(v)$ is concave, both v and $\dot{k}$ increase over time and the hazard rate decreases.*

4.3   Simulations

This section presents numerical computations of the two special cases considered above. The first set of simulations corresponds to the case of no outside options and the second to the linear case.
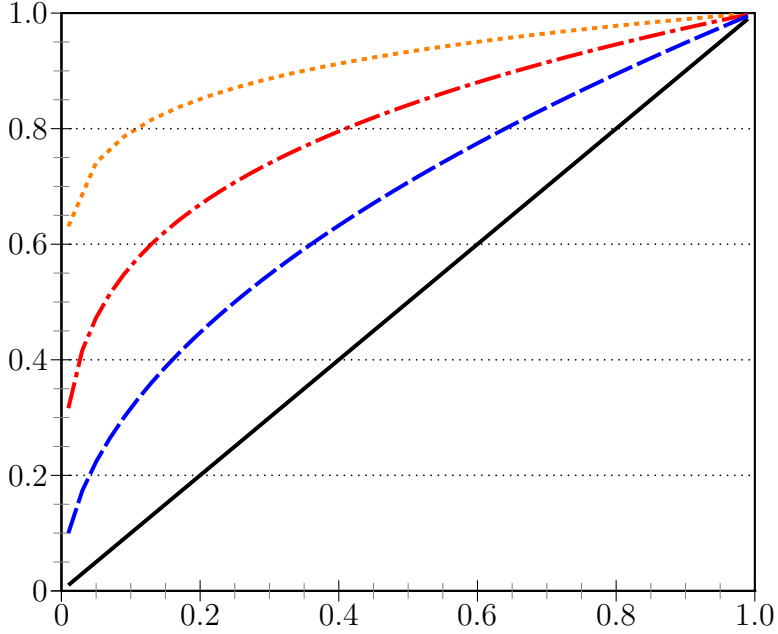
Figure 1: Distributions of outside options:

4.3.1 Outside option independent of k

The environment is as follows. The return function $R(k, \dot{k}) = f(k) - \delta k - \dot{k}$, where $f(k) = k^{0.5}$ and δ corresponds to the depreciation rate. Letting k^* denote the optimal steady state capital stock, the optimal value in absence of constraints is

$$W = \frac{f(k^*) - (r + \delta)k^*}{r},$$

where $f'(k^*) = r + \delta$. The interest rate used is $r = 5\%$ and depreciation rate $\delta = 5\%$. Outside values are given by a distribution function $F(s) = (s/\bar{s})^\gamma$, where $\bar{s}$ corresponds to the upper bound of the support that we chose equal to W, which equals 50 with the above parameter values. This is the highest possible value consistent with inefficient liquidation in the first best (or in the case where the contract is contingent on the outside option.) The alternative scenarios we consider differ in the parameter γ of this distribution taking values $\gamma \in \{0.1, 0.25, 0.5, 1\}$. Figure 1 below presents the c.d.f. corresponding to different values of γ, where lower values correspond to higher functions. For higher γ outside options are more attractive, so default and borrowing constraint should be more prevalent.

For illustrative purposes, Figure 2 shows the value functions $B(V)$ corresponding to $\gamma = 0.1$ and 0.25. The upper value function corresponds to $\gamma = 0.1$ and the lower one
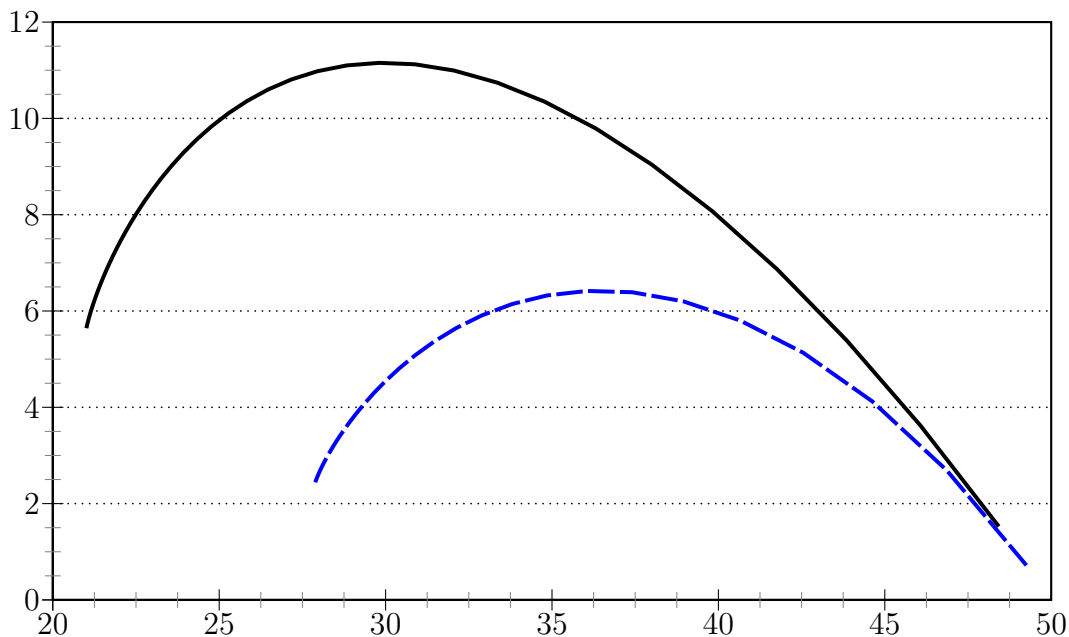
Figure 2: Value functions

to $\gamma = 0.25$. Each is plotted starting at the corresponding value $V_{\min}$. Note that there is a section over which B is increasing, so that Pareto improvements are obtained by increasing V. Competition would drive the initial value to the area where B is decreasing. The maximum debt is given at the peak and is larger in the lower γ scenario. In absence of own financing of the entrepreneur, the project will be financially feasible if and only if the initial investment does not exceed this maximum debt value.

Table 1 gives the evolution of the value of a firm starting in period zero at the point where debt is maximized:

It takes 7 periods (years) to reach the unconstrained steady state (because there are

t	V	B	$W = B + V$	h	Survival	k	risk adjusted r
0	36.5	6.4	42.9	7.6	1.00	8.1	12.6%
1	37.9	6.3	44.2	6.7	0.93	9.0	11.7%
2	39.5	6.1	45.6	5.7	0.87	10.1	10.7%
3	41.2	5.6	46.9	4.7	0.83	11.6	9.7%
4	43.2	4.8	48.0	3.6	0.79	13.5	8.6%
5	45.3	3.7	49.0	2.4	0.77	16.2	7.4%
6	47.6	2.1	49.7	1.2	0.76	19.9	6.2%
7	50.0	0.0	50.0	0	0.75	25.0	5.0%

Table 1: Evolution of Optimal Debt with $\gamma = 0.25$.

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| γ | V_0 | B_0 | W_0 | k_0 | Leverage | |
					excl. k	incl. k
0.1	30.0	11.2	41.1	11.1	27%	42.6%
0.25	36.5	6.4	42.9	8.1	15%	28.5%
0.5	40.3	4.0	44.3	6.1	9%	20.1%
1	43.2	2.4	45.6	4.5	5%	13.8%

Table 2: Summary statistics: maximum debt.

no adjustment costs, it would be reached immediately in the absence of informational constraints.) The value to the borrower rises as debt decreases, resulting in an increase in total surplus W. The hazard rate h decreases monotonically, starting from an initial value of 7.6%. There is a 75% probability of surviving to the unconstrained stage. The last column gives the adjusted interest rates that support these allocations and values.

Table 2 gives some summary values that are useful to compare across these scenarios. The upper section gives the starting values in each scenario at which debt value is maximized. Observing the third column labelled B_0 shows how stringent constraints to total debt become as γ increases, going from a maximum leverage of 27% for the case of $\gamma = 0.1$ to slightly above 5% when $\gamma = 1$. Correspondingly, risk adjusted interest rates are higher in the latter scenarios and the initial capital financed k_0 much lower.

In order to compare properties of the contract, the Table 3 provides the evolution in the cases $\gamma = 0.1$, 0.25 and 0.5, starting from the same level of debt (equal to the maximum value for $\gamma = 0.5$). One way of motivating this exercise is if all these projects had the same initial cost I_0 equal to this initial debt B_0. As γ increases, the initial risk of default increases considerably thus decreasing the initial capital k_0, decreasing the total survival probability and increasing the length of time required to reach the unconstrained state. Correspondingly, the total initial surplus W_0 decreases. This decrease is not that dramatic since as γ increases, default is typically associated with higher outside options.

γ	V_0	W_0	k_0	Risk adjusted r_0	Years to efficient	Survival
0.1	45.6	49.6	21.0	5.9%	1.9	99%
0.25	44.8	48.8	15.5	7.7%	2.2	97%
0.5	40.3	44.3	6.1	15.2%	4.7	77%

Table 3: Summary statistics: starting from maximum B_0 for $\gamma = 0.5$.
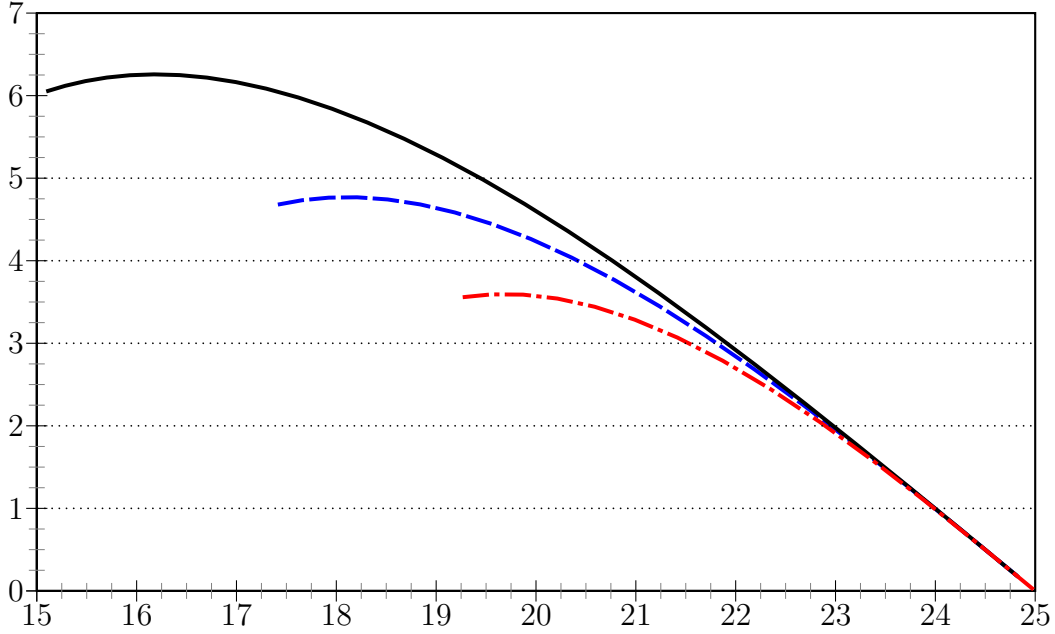
Figure 3: Value functions

4.3.2 Linear case

We now provide some results for the linear case where as described the value function is homogenous of degree one in (V, k). As discussed above, it is convenient to define $v = V/k$ as state variable. This economy grows at a constant rate in the first best, giving total value wk. The technology for production is linear Ak, and the adjustment cost quadratic: $C(\frac{\dot{k}}{k}) = \frac{1}{2}(\frac{\dot{k}}{k})^2$. Outside opportunity is of the for sk, where s is distributed on $[0, \bar{s}]$, where following similar arguments as above $\bar{s}$ is chosen equal to w. The distribution for s is identical as in the previous case, with values of $\gamma = 0.25$, 0.5 and 1. The statistics we present parallel those discussed above.

The value functions for the three scenarios are presented in Figure 3, where as before higher value functions $B(v)$ correspond to lower γ and is associated with higher maximum debt.

Table 4 gives the paths starting at time zero from the point of maximum debt for $\gamma = 1$. It takes over 10 years to reach the unconstrained point, which happens with only 30% probability. Initial hazard rates are very high, reflecting good outside opportunities and correspondingly risk adjusted interest rates are very high and capital accumulation very low (0.5% as opposed to 2.5% in the unconstrained case.). The following table provides summary

t	V	B	h	Survival	k growth	risk adjusted r
0	19.7	3.6	21.3	1	0.4%	26.3%
1	20.1	3.6	19.8	0.81	0.6%	24.8%
2	20.5	3.5	18.2	0.67	0.9%	23.2%
3	20.9	3.3	16.4	0.57	1.2%	21.4%
4	21.4	3.1	14.5	0.48	1.5%	19.5%
5	21.9	2.8	12.5	0.42	1.7%	17.5%
6	22.4	2.4	10.4	0.38	2.0%	15.4%
7	22.9	2.0	8.2	0.34	2.2%	13.2%
8	23.5	1.4	5.9	0.32	2.3%	10.9%
9	24.1	0.9	3.5	0.31	2.4%	8.5%
10	24.7	0.3	1.1	0.30	2.5%	6.1%
10.4	25.0	0.0	0.0	0.30	2.5%	5.0%

Table 4: Evolution of Optimal Debt with $\gamma = 1$.

γ	V_0	B_0	W_0	k growth	Risk adjusted r	Leverage B_0/W_0
0.25	16.2	6.	3 22.4	0.6%	15.3%	27.9%
0.5	18.1	4.8	22.9	0.5%	19.9%	20.9%
1	19.7	3.6	23.3	0.4%	26.3%	15.4%

Table 5: Summary statistics: maximum debt.

statistics to compare across different γ's.

As in the case analyzed in the previous section, higher γ considerably decreases maximum leverage, from 27.9% when $\gamma = 0.25$ to 15.4% for $\gamma = 1$. Table 4 provides a comparison across scenarios starting all from a point where the value of debt is 3.6, the upper bound for the $\gamma = 1$ case. As before, higher γ's are associated to much lower total survcival and increased time to the unconstrained state, a much higher initial adjusted interest rate and corresponding much lower growth rate (0.4% when $\gamma = 1$ as opposed to 2.2% when $\gamma = 0.25$). As before and for the same reasons, the differences in default separation rates do not translate into large differences in initial total value W_0.

γ	V_0	W_0	k growth	Risk adjusted r_0	Years to efficient	Survival
0.25	21.2	24.8	2.2%	9.0%	6.5	88%
0.5	21.0	24.6	1.9%	13.3%	7.0	74%
1	19.7	23.3	0.4%	26.3%	10.4	30%

Table 6: Summary statistics: starting from maximum B_0 for $\gamma = 0.5$.

5 Implementation with Risk Adjusted Loans

In this section we ask whether the optimal contract defined above can be implemented by a risk adjusted interest schedule. To make the analysis more transparent assume the function $B(V, k)$ is concave in V for all k and the optimal contract operates in a region where B is decreasing in V. As a consequence, we may invert for each k this function expressing the utility of the borrower as a function $V(B, k)$ of debt B and assets k.

Consider the following capital accumulation problem for the borrower. Every period, as function of the outstanding debt B and k, the agent owes a flow payment $r(B, k)$ to the lender. The borrower has flow income $R(k, \dot{k})$ and consumes c. The law of motion for debt is therefore

$$\dot{B} = c + r(B, k) - R(k, \dot{k})$$

The agent's Hamilton-Jacobi Bellman equation is

$$rV(B, k) = \max_{k, c} \left(c + V_2(B, k)\dot{k} + V_1(B, k)(c + r(B, k) - R(k, \dot{k})) + \int_{V(B, k)} h(s, k)\, ds \right). \quad (20)$$

At an optimum, the first-order condition for consumption requires

$$1 + V_1(B, k) \leq 0$$

if this condition holds with strict inequality $c = 0$ is the unique optimum consumption and the lender pays $R(k, \dot{k})$ so that $\dot{B} = r(B, k) - R(B, k)$. The first order condition for the agent's choice of $\dot{k}$ is

$$V_2(B, k) - V_1(B, k)R_2(k, \dot{k}) = 0 \quad (21)$$

If V were indeed the inverse function of B, then

$$\frac{V_2(B, k)}{V_1(B, k)} = B_2(V(B, k), k) \quad (22)$$

so the choice of $\dot{k}$ is the same. We will now determine the function $r(B, k)$ that will generate the same path for B as $B(V(t), k(t))$. First note that

$$\dot{B} = B_2(V, k)\dot{k} + B_1(V, k)V = r(B, k) - R(k, \dot{k}) \quad (23)$$

Moreover, the value to the lender must satisfy the following:

$$rB = R(k, \dot{k}) - h(V, k)B + \dot{B} \quad (24)$$

which after substitution of $\dot{B}$ gives

$$r(B,k) = \big(r + h(V(B,k),k)\big)B \tag{25}$$

Note that this payment as a ratio of B is the risk adjusted interest rate $r + h$.

Proposition 5.1. *With $r(B,k)$ given by equation (25), the solution to the borrower's problem (20) yields a trajectory $k(t)$ that coincides with the solution to the lender's problem (9). The function $V(\cdot,k)$ is the inverse of $B(\cdot,K)$, for all k.*

6 A Modified Tobin Q Investment Equation

This section derives an equation that draws a connection with Tobin's q. We begin by defining q as the marginal value of debt $B(V,k)$, that is $q = B_2(V,k)$ (equivalently, it is the marginal value of total surplus $B(V,k) + V$). This leads to a simple relationship between q and investment. We then consider defining q as the marginal effect on the value of the firm, given by total expected discounted cash flows excluding the agent's outside options. The relationship is then more subtle. We show that under this definition the relationship continues to hold but is no longer as straightforward.

Start with the Hamilton-Jacobi-Bellman equation

$$rB(V,k) = \max_{\dot{k},c}\{R(k,\dot{k}) - c - h(V,k)B(V,k) + B_1(V,k)(m(V,k) - c) + B_2(V,k)\dot{k}\}.$$

As usual, we can set consumption to zero $c = 0$. The investment policy function $\dot{k} = g(V,k)$ is determined by the first order condition

$$R_2(k,\dot{k}) + B_2(V,k) = 0. \tag{26}$$

This equation can be inerpreted as

$$R_2(k,\dot{k}) + B_2(V,k) = 0. \tag{27}$$

So that investment $\dot{k}$ is determined by the current value of q.

Below we discuss the evolution equation for q. Before doing so, consider the linear homogeneous case, where $R(k,\dot{k})$ and $F(k,s)$ are homogeneous of degree 1 and 0, respectively. Then $R_2(V,k)$ is homogeneous of degree zero so that $\dot{k}/k$ is then a function of q only. In addition, in this case $B(V,k)$ is homogeneous of degree one. This does not imply that average and marginal q coincide, as in the standard theory developed by Hayashi (1982). However,

it does imply that marginal q can be expressed as an increasing function of $B(V,k)/k$, since both are monotone functions of V/k over the relevant range. Thus, the investment rate $\dot{k}/k$ can be expressed as a function of "average Q" $W(V,k)/k$.

To develop the law of motion for q, differentiate the Hamilton-Jacobi-Bellman equation with respect to k to obtain

$$(r + h(V,k))B_2(V,k) = R_1(k,\dot{k}) - h_2(V,k)B(V,k) + \frac{dB_2}{dt}(V,k) + B_1(V,k)m_2(V,k) \quad (28)$$

where $\frac{dB_2(V,k)}{dt}$ is convenient notation to represent the derivative of the composed function $B_2(V(t),k(t))$ with respect to time t. That is:

$$\frac{dB_2}{dt}(V,k) = B_{21}(V,k)\dot{V} + B_{22}(V,k)\dot{k} = B_{21}(V,k)m(V,k) + B_{22}(V,k)g(V,k).$$

Rewriting equation (28) as

$$(r + h(V,k))q = R_1(k,\dot{k}) + \dot{q} - h_2(V,k)B(V,k) + B_1(V,k)m_2(V,k).$$

gives an equation for the evolution of q. Interpreting equation (31) as $R_2(k,\dot{k}) + q = 0$ determines investment $\dot{k}$ as a function of q.

To really match the notion of q, we need to assess the marginal effect that investment has on expected discounted cash flows. Let $W(V,k) = B(V,k) + E(V,k)$ denote this value, where $E(V,k)$ denotes the equity component of V. That is, the value without taking into account the outside option. The equity value $E(V,k)$ solves

$$rE(V,k) = -h(V,k)E(V,k) + E_1(V,k)m(V,k) + E_2(V,k)\dot{k}. \quad (29)$$

where $\dot{k} = g(V,k)$ and g is the optimal policy function obtained from the Hamilton-Jacobi-Bellman equation for $B(V,k)$.

Differentiating equation (29) with respect to k gives

$$(r + h(V,k))E_2(V,k) = -h_2(V,k)E(V,k) + \frac{dE_2}{dt}(V,k) + E_1(V,k)m_2(V,k) + E_2(V,k)g_k(V,k) \quad (30)$$

Adding $(R_2(k,\dot{k}) + B_2(V,k))g_k(V,k) = 0$ to the right-hand side of equation (28), and then

adding the resulting expression to equation (30) and rearranging delivers:

$$(r + h(V, k) - g_k(V, k))W_2(V, k)$$
$$= R_1(k, \dot{k}) + R_2(k, \dot{k})g_k(V, k) - h_2(V, k)W(V, k) + \frac{dW_2}{dt}(V, k) + W_1(V, k)m_2(V, k)$$

and letting $q = W_2(V, k)$ gives

$$(r + h(V, k) - g_k(k, \dot{k}))q = R_1(k, \dot{k}) + R_2(k, \dot{k})g_k(V, k) - h_2(V, k)W(V, k) + \dot{q} + W_1(V, k)m_2(V, k). \tag{31}$$

In the special case where the outside value is independent of k we obtain the simple expression

$$(r + h(V))q = R_1(k, \dot{k}) + \dot{q},$$

which is a standard risk-adjusted q equation. In contrast, when k affects the outside value, equation (31) incorporates the effect that higher k has on immediate and future separation.

In the linear homogeneous case, where $R(k, \dot{k})$ and $F(k, s)$ are homogeneous of degree 1 and 0, respectively, we have that the value functions $W(V, k)$, $B(V, k)$ and $E(V, k)$ are all homogeneous of degree one. Although this does not imply that average and marginal q coincide, as in the standard theory developed by Hayashi (1982), it does imply that marginal B_2 can be expressed as an increasing function of $W(V, k)/k$, since both are monotone functions of V/k over the relevant range. Thus, the investment rate $\dot{k}/k$ can be expressed as a function of $W(V, k)/k$, that is, as a function of "average Q".

Although in the homogeneous case we find that the investment rate can be expressed as a function of "average Q" alone, it is no longer the case, as in the standard neoclassical theory, that the shape of this relation is dictated exclusively by the investment technology. In particular, the local slope is not dictated by the local convexity of $R(1, \cdot)$, that is, the degree of adjustment costs. Instead, the relationship is now mediated by the mapping between $W(V, k)/k$ to $B_2(V, k)$, which is determined by the solution of the entire problem, requiring a full specification of $R(k, \dot{k})$ and $F(k, s)$, not even just local properties of these functions. We conclude that identifying the adjustment cost technology from the shape of the relationship between $W(V, k)/k$ and the investment rate $\dot{k}/k$ is much less immediate than in the neoclassical case.

7   Conclusions

In this paper we studied the optimal financing of a project subject to the constraint that borrowers default with positive probability. Our model bridges two strands of literature: the limited commitment model where the risk of default is present but never takes place, and the incomplete contract literature where default takes place because contracts are assumed to be noncontingent.

By allowing for default, we are able to study the dynamics of default in an optimal long-term relationship. Our results seem broadly consistent with stylized facts on the patterns for debt, equity and default of new firms. Our model is highly stylized but seems easily extendable. In particular, future work could allow the shocks to the outside option to be persistent, include shocks to the project's returns and explore other possible contract implementations.

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