# Dynamic Contracts when Agent's Quality is Unknown*

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January 2012

**Abstract**

We solve a long-term contracting problem with symmetric uncertainty about the agent's quality, and a hidden action of the agent. As information about quality accumulates, incentives become easier to provide because the agent has less room to manipulate the principal's beliefs. This result is opposite to that in the literature on "career concerns" in which incentives via short-term contracts get harder to provide as the agent's quality is revealed over time.

Keywords: Principal-agent model, optimal contract, learning, private information, reputation, career

JEL Codes: D82, D83, E24, J41

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*Previous versions circulated under the title "Dynamic Incentive Contracts under Parameter Uncertainty". We thank Heski Bar-Isaac, Pieter Gautier, Robert Kohn, Albert Marcet, Ennio Stacchetti as well as seminar participants at the IMT Institute in Lucca, the Tinbergen Institute, University of Cergy-Pontoise, University of Essex, Stockholm Institute for International Economic Studies (IIES), and the Minnesota Macro Workshop for comments and suggestions. We acknowledge the support of the Kauffman Foundation and of the Barcelona GSE .

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# 1 Introduction

In an agency problem the agent may have not just a hidden action, but also an unknown quality. Many relationships between firms and workers, between shareholders and CEOs, or between lenders and borrowers, are of this kind. Yet, little is known about the optimal design of multi-period contracts in such situations. For example, the question remains open as to whether quality uncertainty encourages effort.

When risk-neutral principals and agents deal in spot markets and quality is fixed over time, Holmström (1999) provides a clear answer: Quality uncertainty is good for incentives because it creates a reputational concern. Gibbons and Murphy (1992) confirm this result for one-period incentive contracts and risk averse agents.

We find that the opposite holds true under full commitment: Quality uncertainty harms incentives. Our conclusion differs from Holmström's because markets reward perceived talent whereas contracts are designed to extract effort; once committed to the relationship, it is never in the interest of the principal to discourage the agent by punishing him for having a low productivity. This creates an incentive for the agent to manipulate the principal's beliefs about quality through his choice of effort.

An agent that has provided less effort than expected knows that output would have been higher had he taken the recommended action. Such private information drives a persistent wedge between the principal's and the agent's posteriors, with shirkers remaining more optimistic about quality. This motivates the manipulation an agent might undertake: By inducing the principal to underestimate his productivity, a shirker anticipates that he will benefit from overestimated inferences about his effort in future periods and thus higher rewards. Hence, of two agents with identical performance histories, the shirker will enjoy a higher expected future utility.

The benefits of manipulating the principal's belief downward is reminiscent of the "ratchet effect" discussed in Laffont and Tirole (1988).[1] In order to prevent such *belief manipulation*, contracts under quality uncertainty must link pay more tightly to performance, which lowers the welfare of the risk-averse agent. Since true quality is constant in our model, belief manipulation is more effective

[1]In contrast to Laffont and Tirole (1988) our model features no adverse selection from the outset. Instead, we model a pure moral hazard problem where the principal and agent share the same prior. Asymmetric information can arise only off the equilibrium path through the persistent influence of past actions on posteriors.

early on when posteriors put higher weight on new information, and the sensitivity of pay to performance declines over time.

We use a first-order approach to characterize the optimal contract. We focus on the necessary conditions for recommended effort to be incentive compatible and derive sufficient conditions under which the agent's problem is globally concave. Then we solve the contract in closed form when the agent has exponential utility.[2]

We cast our model in continuous time so as to use the optimization techniques originally introduced by Schättler and Sung (1993). Their methodology has already been extended by Williams (2011) to an environment with persistent private information. There are several differences between our paper and that of Williams; he assumes that the agent observes his productivity and that it evolves stochastically, whereas we keep productivity fixed but neither the agent nor the principal know its actual value. Second, Williams assumes that the initial ability is common knowledge. It therefore remains commonly known when the agent reports truthfully and the equilibrium features no learning. In contrast, we have a common learning process along the equilibrium path. Modeling it requires that we introduce contract duration as an additional state. Third, we use a proof strategy that does not rely on the stochastic maximum principle. Instead, we follow the approach proposed by Cvitanić *et al.* (2009) and use a variational argument to derive the first order conditions.

Our paper seems to be the first to study commitment in a repeated agency problem when the agent's quality is unknown and constant, and where the principal makes transfers to the risk-averse agent in each period. A few papers have analyzed the interactions between quality and moral hazard but under different assumptions about the structure of payments or the timing of actions. Giat *et al.* (2010) add initial private information to Holmström and Milgrom (1987) so that there is a single transfer at the end of the contracting horizon. Conversely, Hopenhayn and Jarque (2007) analyze persistent unknown quality when the effort decision occurs solely in the first period. Adrian and Westerfield (2009) assume that principal and agent disagree about the resolution of uncertainty and

[2]Establishing incentive compatibility when private information is fully persistent entails the following technical issue: As the duration of the relationship increases, the state space becomes unbounded because the entire history of actions matters for evaluating the agent's options off the equilibrium path. A recursive approach to the problem quickly becomes intractable since, as originally explained by Fernandes and Phelan (2000), it takes the beliefs of the agent and of the principal as separate states. The first order approach bypasses this difficulty by focusing on the equilibrium path. Then the challenge consists in deriving sufficient conditions. To the best of our knowledge, the only proof in discrete time is by Kapicka (2006) and is rather specific to the reporting problem analyzed in his paper. One remedy is to numerically check the incentive compatibility of the contract, as in Abraham and Pavoni (2008).

let the two parties agree to disagree. DeMarzo and Sannikov (2008) study a problem which is similar in structure to ours but consider that agents are risk neutral and impose a stationary Markov process on the agent's quality.

The paper proceeds as follows. Section 2 lays out the model. The agent's necessary and sufficient conditions are derived in Section 3. Section 4 displays the contract under exponential utility that is optimal for the principal. It characterizes the set of parameters and initial beliefs under which the agent's first-order conditions represent a global optimum. Section 5 discusses the properties of the optimal contract and equilibrium wage schedule. Section 6 contrasts our full-commitment contract with the no-commitment model of Holmström (1999). Section 7 sums up our findings whereas the proofs of the main Propositions and Corollaries are in Appendix A. We relegate the proofs of some tangential claims to Appendix B.

# 2 The Problem

*Production process.*— Let $\{B_t\}_{t\geq 0}$ be a standard Brownian Motion on a probability space $(\Omega, \mathcal{F}, P)$. The *cumulative output* $Y_t$ of a match of duration $t$ is observed by both parties and satisfies the stochastic integral equation

$$Y_t = \int_0^t (\eta + a_s)ds + \int_0^t \sigma dB_s \ . \tag{1}$$

The *time-invariant* productivity is denoted by $\eta$ whereas $a_t \in [0,1]$ is the effort provided by the agent. The agent's action shifts average output without affecting its volatility.

*Learning.*— No one knows $\eta$ at the outset and common priors are normal with mean $m_0$ and precision $h_0$. Posteriors over $\eta$ depend on $Y_t$ and on cumulative effort $A_t \triangleq \int_0^t a_s ds$. Conditional on $(Y_t, A_t, t)$, they are also normal with mean

$$\hat{\eta}(Y_t - A_t, t) \triangleq E_t\left[\eta | \, Y_t, A_t\right] = \frac{h_0 m_0 + \sigma^{-2}\left(Y_t - A_t\right)}{h_t} \ , \tag{2}$$

and with precision

$$h_t \triangleq h_0 + \sigma^{-2} t \ . \tag{3}$$

Focusing on normal priors over the mean of a normally distributed process enables us to summarize all the statistically significant information with three variables: cumulative output $Y$, cumulative effort $A$ and elapsed time $t$. Especially useful for the characterization of optimal contracts is the fact that beliefs depend on the history of $a$ through $A$ alone. Hence it is sufficient to keep track of cumulative effort instead of the whole effort path.[3]

*Preferences.*—The agent is risk averse and cannot borrow and lend. For all $t \geq 0$ and any given event $\omega \in \Omega$, we define a wage function $w : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$. The agent preferences as of time $0$ read

$$\mathcal{U}_0 \triangleq \int_0^\infty e^{-\rho t} U\left(w_t\left(\omega\right), a_t\right) dt\ , \tag{4}$$

with $\rho > 0$. Our specification of wages is quite general since they can depend on the entire past and present $\{Y_s; 0 \leq s \leq t\}$ of the output process.

The principal is risk neutral and seeks to maximize output net of wages. His inter-temporal preferences are

$$\pi_0 \triangleq \int_0^\infty e^{-\rho t}\left(dY_t - w_t\left(\omega\right)\right) dt\ , \tag{5}$$

where we have imposed a common discount rate for the agent and principal.

*Long-term contract.*—We assume that the parties are able to commit to a long-term contract which can depend on realized history in an arbitrary way. We follow the usual practice of adding *recommended effort* $a^*$ to the contract definition. Accordingly, since a given output path is a random element of the space $\Omega$, a contract is a mapping $(w, a^*) : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R} \times [0,1]$ that associates at each time $t$ a wage-effort pair to any output path. The mapping must be measurable based on information that the principal has, and so, can depend on past output but not on past effort. Otherwise contracts remain general since they can depend on the entire sample path $\{Y_s; 0 \leq s \leq t\}$ of the output process.[4]

*Beliefs.*—The principal assumes that the agent always takes his equilibrium action $a_t^*$. His beliefs are governed by (2) in which $A = A^*$ and by (3). By contrast, the agent's beliefs incorporate the actual level of effort $a$ which only he knows. Thus his beliefs are governed by (2) in which $A$ and not $A^*$ enters. Let $\mathcal{F}_t^a \triangleq \sigma\left(Y_s, a_s; 0 \leq s \leq t\right)$ denote the filtration generated by $(Y, a)$ and $\mathbb{F}^a \triangleq \{\mathcal{F}_t^a\}_{t\geq 0}$ the $P-$augmentation of this natural filtration. Denote by $Z_t$ the cumulative surprise of someone who believes that $Y_t$ was accompanied by the effort sequence

[3]This is why most of the literature on career concerns, Holmström's (1999) model included, focuses on the additive normal case. Dewatripont et al. (1999) discuss in their Remark on page 186 the complications that arise when more general production functions are considered.

[4]Given the diffusion property of the output process, one should think of $\Omega = C\left([0,T];\mathbb{R}\right)$ as the space of continuous functions $\omega : [0,T] \rightarrow \mathbb{R}$ and of the process defined in (6) $Z_t\left(\omega\right) = \omega\left(t\right)$, $0 \leq t \leq T$, as the coordinate mapping process with Wiener measure $P$ on $\left(\Omega, \mathcal{F}_t^Y\right)$. Accordingly a contract is a mapping $(w, a^*) : \mathbb{R}^+ \times C\left([0,T];\mathbb{R}\right) \rightarrow \mathbb{R} \times [0,1]$.

$\{a_s; 0 \le s \le t\}$. The filtering theorem of Fujisaki et al. (1972) implies that the *innovation process*

$$dZ_t \triangleq \frac{1}{\sigma}\left[dY_t - (\hat{\eta}(Y_t - A_t, t) + a_t)dt\right] \tag{6}$$

is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}^a, P)$.[5] Moreover, $\hat{\eta}$ is a $P-$martingale with decreasing variance[6]

$$d\hat{\eta}(Y_t - A_t, t) = \frac{\sigma^{-1}}{h_t} dZ_t \ . \tag{7}$$

The agent is restricted to the class of control processes $\mathcal{A} \triangleq \{a : \mathbb{R}^+ \times \Omega \to [0, 1]\}$ that are $\mathbb{F}^a-$predictable.[7] Given that the principal does not observe actual effort, the information available to him is restricted to the filtration $\mathcal{F}_t^Y \triangleq \sigma(Y_s; 0 \le s \le t)$ generated by $Y$ whose augmentation we denote by $\mathbb{F}^Y \triangleq \left\{\mathcal{F}_t^Y\right\}_{t\ge 0}$. An effort path is an equilibrium path when recommended and actual effort coincide, i.e., if $a_t = a_t^*$ for all $(t, \omega)$.

# 3 Incentive Compatible Contracts

This section focuses on the agent's problem. We derive the necessary conditions for a given action to be optimal and then establish a restriction under which they are also sufficient. We impose a terminal date $T$ on the contracting horizon. Until then, both principal and agent are *fully committed* to the relationship. The agent's continuation value at time $t$ reads

$$v_t \triangleq \max_{a\in\mathcal{A}} E\left[\int_t^T e^{-\rho(s-t)} U\left(w(\overline{Y}_s), a_s\right) ds + e^{-\rho(T-t)} W\left(\overline{Y}_T\right) \middle| \mathcal{F}_t^a\right], \tag{8}$$

where the output path is denoted by $\overline{Y}_t \triangleq \{Y_s; 0 \le s \le t\}$ and $W(\cdot)$ is the terminal utility which depends on output history.[8] The agent computes his continuation

[5]As shown in Section 10.2. of Kallianpur (1980), the linearity of the filtering problem implies that the filtrations generated by the output and innovation processes coincide. More formally, for $\mathcal{F}_t^Z \triangleq \sigma(Z_s; 0 \le s \le t)$, we have $\mathcal{F}_t^a = \mathcal{F}_t^Z$.

[6]Equation (7) follows directly from Ito's lemma. Let $X_t \triangleq Y_t - A_t$ denote cumulative output net of cumulative effort so that

$$d\hat{\eta}(X_t, t) = \frac{\partial \hat{\eta}(X_t, t)}{\partial t} dt + \frac{\partial \hat{\eta}(X_t, t)}{\partial X_t} dX_t = -\frac{\sigma^{-2}}{h_t}\hat{\eta}(X_t, t) + \frac{\sigma^{-2}}{h_t}\left(\hat{\eta}(X_t, t) + \sigma dZ_t\right) = \frac{\sigma^{-1}}{h_t} dZ_t \ .$$

[7]A mapping is predictable when it is $\mathcal{P}-$measurable, with $\mathcal{P}$ denoting the $\sigma$-algebra of predictable subsets of the product space $\mathbb{R}^+ \times \Omega$, i.e. the smallest $\sigma$-algebra on $\mathbb{R}^+ \times \Omega$ making measurable all left-continuous and adapted processes.

[8]Since we shall let $T \to \infty$, we have assumed a tractable form for $W$. It is straightforward to let $W$ also depend on cumulative effort $A$. Then one would have to adjust the stochastic process

value by taking a conditional expectation under the filtration $\mathcal{F}_t^a$ which varies with the level of cumulative effort. The principal, on the other hand, does not observe actual actions. Thus he needs to keep track of continuation values for any potential level of cumulative effort. We shall simplify the problem by adopting a first order approach: We focus on the continuation value along the equilibrium path and then establish conditions under which our solution is indeed globally optimal.

## 3.1 Necessary conditions

The optimization problem $(8)$ cannot be analyzed with standard methods because the objective function depends on the process $w_t$ which is *non-Markovian*. We instead use a martingale approach. Faced with a contract $(w, a^*)$, the agent controls the distribution of wages through his choice of effort. Under this interpretation, the agent chooses the probability measure over realizations of $w_t$. This approach renders our optimization problem treatable with optimal control techniques because the Radon–Nikodym derivative associated with any effort path is a Markovian process.

The idea of treating distributions as controls in order to solve principal-agent models goes back to Mirrlees (1974). The learning process complicates our problem as past efforts affect not only current wages but also future expectations. We show in the Appendix how this difficulty can be handled through an extension of the proof by Cvitanić et *al.* (2009) which leads to the necessary condition stated below.

**Proposition 1** *There exists a unique decomposition for the agent's continuation value*

$$\begin{aligned} dv_t &= [\rho v_t - U(w_t, a_t)]\, dt + \gamma_t \sigma dZ_t\,, &(9)\\ v_T &= W(Y_T)\,, &(10)\end{aligned}$$

*where $\gamma$ is a square integrable predictable process. The necessary condition for $a_t^*$ to be an optimal control reads*

$$\left[\gamma_t + E_t\left[-\int_t^T e^{-\rho(s-t)}\gamma_s \frac{\sigma^{-2}}{h_s} ds\right] + U_a(w_t, a_t^*)\right](a - a_t^*) \leq 0\,, \tag{11}$$

*for all $a \in [0,1]$.*

$p$ defined in equation $(13)$ by replacing $W(\overline{Y}_T)$ with $W_A(\overline{Y}_T, A_T)$. One can verify that our results hold with few or no changes under this more general definition. The specification of the terminal utility would matter if we were to focus on repeated contracts, with $W$ capturing the agent's outside option and the ability of the principal to reward him at the end of the relationship. We do not consider such generalizations because this paper focuses on the limit situation where both parties are forever committed. Then, as long as standard transversality conditions hold, the specification of the terminal utility is immaterial to the analysis.

An increase in current effort has two effects: it raises the promised value along the equilibrium path and increases cumulative effort. The first effect is proportional to the process $\gamma$ which measures the sensitivity of the agent's value to output surprises. The second effect is captured by the expectation term in $(11)$. This term vanishes when $\eta$ is known, since then $\sigma^{-2}/h_s = 0$ for all $s \geq t$. As a special case of our model, we then get the necessary condition in Sannikov (2008) which says that an optimal control must maximize the expected change in continuation value $\gamma$ minus the marginal cost of effort $U_a(\cdot)$.

Quality uncertainty leads to the addition of the expected future sensitivities weighted by their precision ratios because they capture the marginal impact of current effort on expected earnings. To see this, observe that $\partial\hat{\eta}(Y_s - A_s, a)/\partial a_t = -\sigma^{-2}/h_s$ for all $s \geq t$. Hence a marginal increase in $a_t$ lowers date-$s$ posteriors about $\eta$ by the amount $\sigma^{-2}/h_s$. The impact in utils follows multiplying these marginal output surprises by the expected sensitivity $\gamma_s$ of the promised value.

Analytically, (11) is more convenient when re-written as follows:

$$\left[\frac{\sigma^{-2}}{h_t}p_t + \gamma_t + U_a\left(w_t, a_t^*\right)\right]\left(a - a_t^*\right) \leq 0\ ,\ \text{for all } a \in [0,1]\ , \tag{12}$$

where

$$p_t \triangleq h_t E\left[-\int_t^T e^{-\rho(s-t)}\gamma_s \frac{1}{h_s} ds \middle| \mathcal{F}_t^a\right] \tag{13}$$

is a stochastic process capturing the value of private information.

The reformulated necessary condition $(12)$ involves two stochastic variables, $\gamma_t$ and $p_t$. This is a usual result for dynamic contracts with private information.[9] First, we recover the now standard technique of using the promised value to encode past history. A related interpretation can be inferred for $p$ noticing that the incentive constraint implied by $(12)$ is

$$\gamma_t \geq -U_a\left(w_t, a_t\right) - \frac{\sigma^{-2}}{h_t}p_t\ . \tag{14}$$

Given that the agent is risk averse, it is reasonable to conjecture that the principal will minimize the volatility parameter $\gamma$. Hence, as long as $a_t^* > 0$ for all $t$,

[9] This feature was originally noticed by Werning (2001) considering principal-agent problems with hidden savings. He proved that one has to introduce both continuation value and expected marginal utility from consumption. A general approach has been recently proposed by Pavan et al. (2010). They establish an envelope formula for the derivative of an agent's equilibrium payoff. When applied to adverse selection problems with Markovian types, the envelope formula leads to the definition of an additional recursive variable.

To the best of our knowledge, Williams (2008) was the first to introduce two separate stochastic processes in order to solve dynamic incentive problems in continuous time. He also explains how one of them can be dispensed with when the utility function is exponential. We show in Section 4 that a similar simplification holds in our set-up.

the necessary condition $(12)$ will hold with equality almost everywhere along the equilibrium path. We show below that this indeed holds true when the agent has exponential utility. We therefore replace $\gamma_t$ by the expression implied for it when $(12)$ binds and, as shown in Appendix B.1., obtain the following solution:

$$p_t = E\left[\int_t^T e^{-\rho(s-t)} U_a\left(w_s, a_s\right) ds \middle| \mathcal{F}_t^a\right] < 0\,. \tag{15}$$

Since the stochastic variable $p_t$ is negative, it follows from $(12)$ that, for any recommended level of effort $a_t^*$ and any given wage $w_t$, the volatility of the agent's promised value $\gamma_t$ has to be higher with quality uncertainty than without. The implication does not hinge on any particular specification of the utility function. It only requires that the incentive constraint $(12)$ binds everywhere along the equilibrium path, thereby illustrating the generality of our main finding: An uncertain environment makes it harder to motivate the agent and so leads to greater exposure to risk.

*Intuition behind* $p$.— The second state variable $p$ is equal to the expected discounted marginal cost of future efforts. Multiplying it by the ratio $\sigma^{-2}/h_t$ yields the marginal effect of cumulative effort on the continuation value. The intuition for this result can be laid out considering *mimicking strategies*. Fix $\overline{Y}_t$ and lower cumulative effort by $\delta > 0$. Then define a strategy enabling the agent to reproduce the payoffs of an agent with the reference level $A_t^*$ of past effort. Let $a_t^*$ denote the optimal effort at time $t$ of the reference policy with cumulative effort $A_t^*$. By providing $a_t^\delta = a_t^* - \delta\sigma^{-2}/h_t$,[10] the agent with cumulative effort $A_t^* - \delta$ ensures that cumulative output will have the same drift as along the reference path

$$\hat{\eta}(Y_t - \left(A_t^* - \delta\right), t) + a_t^\delta = \frac{h_0 m_0 + \sigma^{-2}\left(A_t^* - \delta\right)}{h_t} + a_t^* - \frac{\sigma^{-2}}{h_t}\delta = \hat{\eta}(Y_t - A_t^*, t) + a_t^*\,.$$

Assume now that a similar strategy is employed afterwards, so that $a_s^\delta = a_s^* - \left(\sigma^{-2}/h_t\right)\delta$ for all $s \geq t$. Cumulative effort will be $A_s^\delta = A_s^* - \left[1 + \left(\sigma^{-2}/h_t\right)(s-t)\right]\delta$ leading to the following output drift

$$\begin{aligned}
\hat{\eta}(Y_s - A_s^\delta, s) + a_s^\delta &= \frac{h_0 m_0 + \sigma^{-2}\left(A_s^* - \left[1 + \left(\sigma^{-2}/h_t\right)(s-t)\right]\delta\right)}{h_s} + a_s^* - \frac{\sigma^{-2}}{h_t}\delta \\
&= \hat{\eta}(Y_s - A_s^*, s) + a_s^* - \frac{\sigma^{-2}}{h_t h_s}\left[\underbrace{\left(h_t + \sigma^{-2}(s-t)\right)}_{=h_s} - h_s\right] = \hat{\eta}(Y_s - A_s^*, s) + a_s^*\,.
\end{aligned}$$

[10]Such strategies are not feasible when the reference control is at the lower bound, i.e., when $a_t^* = 0$. One should therefore interpret our discussion of mimicking strategies as an heuristic one. The rigorous interpretation being that of the expectation term $E\left[-\int_t^T \gamma_s \frac{\sigma^{-2}}{h_s} ds \middle| \mathcal{F}_t^a\right]$ laid-out in the paragraph above.

As desired, the mimicking strategy reproduces the distribution of $Y_s$ for all $s \geq t$ and the product $-\left(\sigma^{-2}/h_t\right) p_t$ measures its expected discounted return in utils.[11] It is positive because it took the agent with cumulative effort $A_t^*$ more work to produce $Y_t$, implying that his productivity is likely to be lower. Returns decrease as $t$ increases because the influence of output on beliefs is lower when $\eta$ is known more precisely. This suggests that incentives become easier to provide over time, a result that we will discuss at length in Section 5.

## 3.2 Sufficient conditions

First-order conditions rely on the premise that the agent's objective is globally concave. Unfortunately, principal-agent problems do not always fulfill such a requirement. In our case, establishing concavity is complicated by the persistence of private information: Excluding one shot deviations does not necessary rule out multiple deviations because any departure from recommended effort drives a permanent wedge between the beliefs of the agent and that of the principal. Thus we have to verify the sufficiency of our necessary conditions. Only then can we be sure that the agent finds it indeed optimal to provide recommended effort when assigned the wage function satisfying the local incentive constraint $(12)$.

How to establish incentive compatibility for discrete time contracts with persistent information remains an open question.[12] By contrast, when the model is cast in continuous time, the sufficiency of the necessary conditions and thus the incentive compatibility of the effort path can be established directly.[13] In our case, sufficiency holds when the requirements stated in the following proposition are fulfilled.

**Proposition 2** *A control $a$ is incentive compatible if* $(11)$ *and*

$$-2U_{aa}\left(w_t, a_t\right) \geq e^{\rho t} \xi_t \sigma^2 h_t \tag{16}$$

[11]The correction term $\sigma^{-2}/h_t$ required to mimic the output distribution remains constant over time because of two countervailing mechanisms. One the one hand, as $h_s$ increases, the impact of past deviations on posteriors decreases over time. On the other hand, the mimicking strategy involves repeated deviations so that the gap between $A_s^*$ and $A_s^\delta$ widens over time. When the output distribution is normal, these two opposite forces offset each other.

[12]The difficulties arising in discrete time settings are thoroughly discussed by Abraham and Pavoni (2008). To circumvent them, they propose a numerical procedure verifying *ex-post* the implementability of contracts with hidden effort and savings. See also Kocherlakota (2008) for a discussion of the problem and an analytical example.

[13]When the optimization problem can be cast as an optimal control problem, the sufficiency of the necessary conditions follow from the concavity of the agent's Hamiltonian. This general mathematical result is summarized in Theorem 3.5.2 of Yong and Zhou (1999), and has already been used in principal-agent settings by Schättler and Sung (1993) and more recently by Williams (2008). The concavity requirement derived in Williams (2008) tend to be violated by his principal-agent problem. Corollary 2 below shows that this is not the case in our model because sufficiency is not anymore an issue when parameter precision $h_t$ goes to infinity.

*are true for almost all* $t$*, where* $\xi$ *is the predictable process defined uniquely by*

$$E\left[-\int_0^T e^{-\rho s}\gamma_s \frac{\sigma^{-2}}{h_s} ds \middle| \mathcal{F}_t^a\right] - E\left[-\int_0^T e^{-\rho s}\gamma_s \frac{\sigma^{-2}}{h_s} ds \middle| \mathcal{F}_0^a\right] = \int_0^t \xi_s \sigma dZ_s \,, \qquad (17)$$

*for all* $t \in [0, T]$.

According to $(15)$, the process $\xi_t$ is the random fluctuation in the discounted sum of marginal utilities as evaluated from time $0$. Proposition 2 imposes stronger restrictions on $\xi_t$ than required so that a control might violate them and nevertheless be incentive compatible. Moreover, $(16)$ and $(17)$ are stated in terms of $(w_t, \gamma_t)$ which are endogenous, implying that they have to be verified ex-post for any given contract. In some cases, however, one can translate $(16)$ and $(17)$ into a requirement on the parameters of the model. Indeed, when the agent's utility function is exponential, as in $(20)$, we shall show that the conditions of Proposition 2 are fulfilled if $(26)$ holds.

Finally, observe that letting the horizon $T$ go to infinity allows us to discard the terminal condition $(10)$ as long as the transversality condition $\lim_{T\to\infty} e^{-\rho t} W\left(\overline{Y}_T\right)$ is satisfied. Then we can replace the Backward Stochastic Differential Equation[14] $(9)$ by a Stochastic Differential Equation (SDE hereafter) and express the law of motion of the stochastic process $p$ as follows.

**Corollary 1** *In the infinite horizon case,* $p_t$ *(defined in (15)) satisfies*

$$dp_t = \left[p_t\left(\rho + \frac{\sigma^{-2}}{h_t}\right) + \gamma_t\right] dt + \vartheta_t \sigma dZ_t \,, \qquad (18)$$

*with*

$$\vartheta_t \triangleq e^{\rho t}\sigma^2 h_t \xi_t$$

*and* $\xi_t$ *being defined in* $(17)$.

Some general results about local incentive constraints and their sufficiency have been established in this section. We have derived qualitative results on the interaction between quality uncertainty and incentive compatibility. It is difficult to make further progress without being more specific about the agent's preferences. This is why we hereafter restrict our attention to a particular class of utility function.

[14]A Backward Stochastic Differential Equation is a Stochastic Differential Equation on which a terminal condition has been imposed. In our case, we assumed that the agent's value $v_t$ equals $W\left(\overline{Y}_t\right)$ at the end of the contracting horizon, i.e., when $t = T$.

# 4 Optimal Contract under Exponential Utility

We now explain how one can solve for the principal's problem and derive the optimal contract in closed form when attention is restricted to commitment over an infinite horizon and exponential utility functions. The main idea is to simplify the optimization program by eliminating two states: The first one is a component of the sufficient statistics for beliefs, $\hat{\eta}$; and the second one is the value of private information, $p$. We now describe how each of these is dealt with.

*Eliminating $\hat{\eta}$ from the list of states.*— According to $(5)$ the principal's problem has an infinite horizon, so that his objective reads

$$\begin{aligned} J_t &\triangleq E\left[\int_t^\infty e^{-\rho(s-t)}\left(\hat{\eta}(Y_s - A_s^*, s) + a_s - w_s\right) ds \middle| \mathcal{F}_t^Y\right] \\ &= \frac{\hat{\eta}(Y_t - A_t^*, t)}{\rho} + E\left[\int_t^\infty e^{-\rho s}\left(a_s - w_s\right) ds \middle| \mathcal{F}_t^Y\right] . \end{aligned}$$

The equality follows because the principal is risk neutral and beliefs are a martingale. This implies that the posterior mean $\hat{\eta}$ can be dispensed with as a state, leaving only precision as a belief state. Furthermore, since $h_t$ is deterministic, we may index precision by $t$. The fact that the expected value of $\eta$ is immaterial to the principal's objective shows that incentives are optimally designed to reward effort and not ability.

## 4.1 Incentives providing contract

We first characterize contracts when the incentive constraint holds almost everywhere in the future. We prove below that this is indeed optimal if precision exceeds a given threshold. As explained before, we can omit the posterior mean $\hat{\eta}$ and recast the principal's optimization problem as[15]

$$j_t \triangleq \max_{\{a,w,\gamma,\vartheta\}} E\left[\int_t^\infty e^{-\rho(s-t)}\left(a_s - w_s\right) ds \middle| \mathcal{F}_t^Y\right] ,$$

subject to the two promise-keeping constraints $(9)$ and $(18)$ and to the incentive constraint $(14)$. Since the state variables $v$ and $p$ are Markovian, we are justified in using a Hamilton-Jacobi-Bellman (HJB) equation to characterize the principal's value function.[16] If we had to keep all three states $(t, v, p)$, the HJB equation would

[15]Appendix B.2 shows that the HJB equations $(19)$ defined below can be extended to include $\hat{\eta}$ and would still be satisfied.

[16]We use a strong formulation for the principal's problem even though we have used a weak formulation to solve for the agent's problem. This change of solution method is usual for principal-agent models. Yet, as discussed in Cvitanic *et al.* (2009), it may lead to measurability issues if the optimal action directly depends on the Brownian motion. In our case, however, $a^*$ turns out to be constant over time so that measurability of the optimal control will not be problematic.

read

$$\rho j_t = \max_{\{a,w,\vartheta\}} \left\{ \begin{array}{c} a - w + \frac{\partial j}{\partial t} + \frac{\partial j}{\partial v}\left(\rho v - U\left(w,a\right)\right) + \frac{\partial j}{\partial p}\left(\rho p - U_a\left(w,a\right)\right) \\ + \frac{\sigma^2}{2}\left[\frac{\partial^2 j}{\partial v^2}\gamma\left(t,p,w,a\right)^2 + \frac{\partial^2 j}{\partial p^2}\vartheta^2 + 2\frac{\partial^2 j}{\partial v p}\gamma\left(t,p,w,a\right)\vartheta\right] \end{array} \right\} . \qquad (19)$$

Instead of solving this general HJB equations, we now impose a parametric assumption on the utility function in order to reduce the dimensionality of the state space.

*Eliminating $p$ from the list of states.—* We restrict attention to the following utility function[17]

$$U(w,a) = -\exp(-\theta\left(w - \lambda a\right)), \text{ with } \lambda \in (0,1)\ ,\ \theta > 0\ , \qquad (20)$$

and $a \in [0,1]$. Imposing $\lambda < 1$ ensures that the first-best action is $a = 1$ because the marginal utility of an additional unit of output exceeds the marginal cost of effort regardless of $\eta$.[18] The utility rules out agents with limited liability because it is defined even for negative consumption, which occurs with positive probability in equilibrium.

When $U(a,w)$ is given by $(20)$, the problem greatly simplifies because $U_a\left(w,a\right) = \theta\lambda U\left(w,a\right)$. Then $(8)$ and $(15)$ imply that, whenever the incentive constraint binds for almost all $s \geq t$, $p_t = \theta\lambda v_t$. The proportionality of $v$ and $p$ means that keeping track of one of the two states is sufficient. This further reduces the dimensionality of the problem and allows us to rewrite the HJB equation $(19)$ as

$$\rho j_t = \max_{\{a,w\}}\left\{a - w + \frac{\partial j}{\partial t} + \frac{\partial j}{\partial v}\left(\rho v - U\left(w,a\right)\right) + \left(\frac{\sigma^2}{2}\right)\frac{\partial^2 j}{\partial v^2}\gamma\left(t,v,w,a\right)^2\right\} . \qquad (21)$$

The first order conditions for $a$ and $w$ cannot be jointly satisfied. More precisely, the one for wages is always tight while the one for effort remains constrained, which leads to the following claim.

**Claim 1** *If the Incentive Constraint binds for almost all $s \geq t$, recommended effort is set equal to its upper-bound $a_t^* = 1$.*

Fixing the agent's action to its first-best level allows us to solve for the value function by guess-and-verify.

[17]Even though the full characterization of the contract will hold only for utilities of the form $(20)$, the optimality conditions derived in Section 3 are true independently of this parametric restriction. One of its implications is that there is no wealth effect on leisure because $U_w\left(\cdot\right)/U_a\left(\cdot\right) = -\lambda^{-1}$ is equal to a constant that does not depend on $w$.

[18]Accordingly, one could interpret our model as resulting from a situation where the agent is able to divert cash flows $1 - a$ at the rate $\lambda$. As in DeMarzo and Sannikov (2009), setting $\lambda$ below one ensures that cash diversion entails linear losses. Our problems differ because DeMarzo and Sannikov (2009) focus on risk neutral agents whereas we introduce risk aversion by taking a concave transformation of the agent's income net of his opportunity cost $\lambda a$.

**Proposition 3** *Assume that: (i) $U(w,a)$ is as specified in $(20)$; and (ii) the Incentive Constraint $(14)$ binds for almost all $s \geq t$. Then the recommended effort is set equal to the first-best level $a_t^* = 1$ and the principal's value function is of the form*

$$\rho j(t,v) = f(t) + \frac{\ln(-\rho v)}{\theta} \, . \tag{22}$$

*The function $f(t)$ is the unique solution of the first order ODE*

$$f'(t) = \rho f(t) - \rho\left(1 - \lambda + \frac{\ln(-k_t/\rho)}{\theta}\right) + \frac{\theta(\sigma\lambda)^2}{2}\left[\left(\frac{\sigma^{-2}}{h_t}\right)^2 - k_t^2\right] , \tag{23}$$

*with boundary condition $\lim_{t\to\infty} f'(t) = 0$ and $k_t$ being given by the negative root of the quadratic equation*

$$k_t^2(\sigma\lambda\theta)^2 - k_t\left[1 + \frac{(\lambda\theta)^2}{h_t}\right] - \rho = 0 \, . \tag{24}$$

Let us compare the expression of $j(t,v)$ to its counterpart if $a$ was contractible. Observing the action allows the principal to elicit full effort, $a = 1$, while perfectly insuring the agent. The cost of delivering value $v$ through a constant income stream is equal to $-\ln(-\rho v)/\theta$. The principal must add $\lambda$ to the baseline remuneration so as to compensate the agent for the effort cost. Accordingly, first-best wages read $w^{FB}(v) = \lambda - \ln(-\rho v)/\theta$ and the principal's value function is given by $\rho j^{FB}(t,v) = 1 - \lambda + \ln(-\rho v)/\theta$. Comparing $j(t,v)$ to $j^{FB}(t,v)$ it is apparent that $1 - \lambda - f(t)$ measures the per-period loss due to both hidden effort and quality uncertainty.[19] The following corollary shows that this loss decreases over time as quality uncertainty becomes less of a concern.

**Corollary 2** *The function $f(t)$ is increasing over time and has a finite limit which we denote $F$. The principal's expected profit as a function of the value $v$ promised to the agent is therefore increasing in belief precision $h$.*

[19] It can be shown that $1 - \lambda - f(t)$ is always positive, as one should expect since the value function cannot exceed its first best level. First one uses Corollary 3 to conclude that $1-\lambda-f(t) < 1 - \lambda - F$ where $F \triangleq \lim_{t\to\infty} f(t)$. The expression of $F$ immediately follows from $(23)$

$$F = 1 - \lambda + \frac{\ln(-K/\rho)}{\theta} + \frac{\theta(\sigma\lambda K)^2}{2\rho} \, , \tag{25}$$

where $K \triangleq \lim_{t\to\infty} k(t)$. It is easily shown that $\ln(-K/\rho) < 0$. One still has to establish that its absolute value is higher than that of the fourth term on the RHS of $(25)$. A Taylor approximation around $1$ yields $\ln(-K/\rho) < -(\rho + K)/\rho$. Reinserting this inequality into $(25)$ and using the definition of $K$, one can finally prove that $F$ is indeed less than $1 - \lambda$.

We still have to check whether the contract is incentive compatible. Applying the conditions in Proposition 2 to our particular model yields the following requirement.

**Corollary 3** *First best effort is incentive compatible (i.e., meets conditions* $(11)$ *and* $(16)$*) when*

$$\rho\sigma^2 > \frac{1}{h_t} + 2\left(\lambda\theta\right)^2 \frac{1}{h_t^2}\,. \tag{26}$$

*Since precision* $h_t$ *is increasing with time, the condition holds at all subsequent dates* $s \geq t$*.*

The sufficient condition $(26)$ is more likely to hold when: Both parties are impatient, output noise is high, the marginal cost of effort $\lambda$ is low, the coefficient of absolute risk aversion $\theta$ is small, or parameter precision $h_0$ is high. Indeed, $(26)$ always holds in the limit case, $h_0 = \infty$, where quality is know because multiple deviations are not anymore relevant.

We shall henceforth assume that our parameters satisfy $(26)$. The condition is sufficient and not necessary, however, and our comparative statics results hold independently of it, showing that they are robust over a wider region of the parameter space.

## 4.2 Optimal contract

The derivation of the value function $j\left(t, v\right)$ was based on the premise that the incentive constraint always binds. Obviously, the principal has the option to perfectly insure the agent while recommending zero effort. As explained in the discussion of first-best contracts following Proposition 3, implementing such a policy has a cost of $-\ln\left(-\rho v\right)/\theta$. By contrast, its return is $0$ because the agent does not exert any effort. This suggests that the principal would rather insure the agent when $f\left(t\right)$ is negative and offers him an incentives providing contract when $f\left(t\right)$ is positive. However, this conclusion is misleading because it is based on a comparison between contracts that recommend full effort or no effort at *every point* in the future. It may instead be optimal to insure the agent for a certain length of time and then to provide him with incentives to exert effort. In other words, the principal has the valuable option to delay incentives provision.

Due to the absence of wealth effect, the option does not depend on the current belief about $\eta$ but is instead deterministic. The marginal gains from delaying incentives are equal to $f'\left(t\right)$ while the costs due to discounting are given by $-\rho f\left(t\right)$. Hence, when $\psi(t) \triangleq \rho f\left(t\right) - f'\left(t\right) < 0$ the principal perfectly insures the agent. Conversely, when $\psi(t) \geq 0$ he offers the incentives providing contract described in the previous sub-section. Given that $\psi(t)$ is increasing over time,[20]

[20]See proof of Proposition 4.

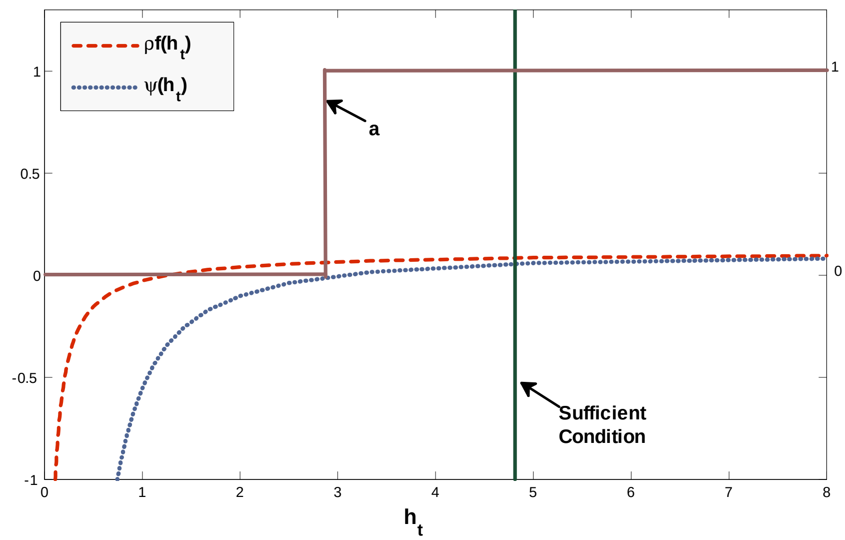


Figure 1: RECOMMENDED EFFORT AS A FUNCTION OF PARAMETER PRECISION $h_t$.

there is at most one precision level above which incentives provision is optimal, as illustrated in Figure 1 for the parameter values in Table 1. If quality uncertainty is high enough at the beginning of the relationship, the effort path starts at $0$ and switches to $1$ exactly at the time where $\psi(t)$ crosses the zero axis. Depending on the parameter constellation, it may also happen that $f(t)$, and consequently $\psi(t)$, remain negative at all $t$. in such cases, it is always optimal to perfectly insure the agent, as stated in the following proposition.

**Proposition 4** *Let $F$ denote $\lim_{t\to\infty} f(t)$, if: (i) $F > 0$, $a_t$ is a step function: There exists a unique precision $\tilde{h}$ such that recommended effort is $a_t^* = 0$ whenever $h(t) < \tilde{h}$, and $a_t^* = 1$ otherwise. The principal's value function $j^*$ is given by*

$$\rho j^*(t,v) = \begin{cases} e^{-\rho(\tau-t)} f(\tau) + \frac{\ln(-\rho v)}{\theta}; & \text{when } h(t) < h(\tau) = \tilde{h} \\ f(t) + \frac{\ln(-\rho v)}{\theta}; & \text{when } h(t) \geq \tilde{h} \end{cases};$$

*(ii) $F \leq 0$ : Recommended effort $a_t^* = 0$ for all $t$ and the principal's value function reads*

$$\rho j^*(v) = \frac{\ln(-\rho v)}{\theta}.$$

This proposition completes our description of the optimal contract. Its properties and implications for wage dynamics are explored in the next section.

# 5 Characterization of the Optimal Contract

*Wage dynamics.*— Optimal wages under incentives provision are given by[21]

$$w_t(v) = -\frac{\ln(k_t v)}{\theta} + \lambda\,. \tag{27}$$

The variable $k_t$ and consequently the mapping associating wages to the continuation value $v$ are decreasing functions of time.[22] As beliefs become more precise, the agent's ability to manipulate them weakens. The principal can extract more effort with the same risk exposure and so trades lower wages in exchange of better insurance.

**Corollary 4** *For any given promised value $v$, the optimal wage $w_t^*(v)$ is a decreasing function of beliefs precision and thus time.*

Corollary 4 does not directly apply to income dynamics because the promised value $v$ evolves over time. To obtain its law of motion, we reinsert the optimal volatility

$$\gamma_t\left(v\right) = \Gamma_t v \triangleq \lambda\theta\left(k_t - \frac{\sigma^{-2}}{h_t}\right)v\,, \tag{28}$$

derived in the proof of Proposition 3 into the SDE $(9)$ to obtain

$$dv_t = v_t\left[\left(\rho + k_t\right)dt + \Gamma_t\sigma dZ_t\right]\,. \tag{29}$$

The sign of the deterministic trend $v_t\left(\rho + k_t\right)$ is ambiguous. It indicates how earnings are allocated over time: When it is positive, wages are back loaded so that expected wages are above their current level. Conversely, when the trend is negative, payments are front loaded. Given that $k_t$ decreases over time and $v_t$ is negative, the principal resorts more intensively to back loading early in the relationship. Payments are deferred because incentives can be provided at a cheaper cost in the future through higher income stabilization.

Williams (2011) proves qualitatively similar results in a reporting problem with persistent income shocks: Efficiency losses due to private information increase with the persistence of the endowment and, parallel to our result that the principal back loads payments more when $h_t$ is lower, Williams also finds that persistence of shocks leads to a tendency to back load payments that is absent in reporting problems with i.i.d. shocks.

Deriving the law of motion of wages allows us to analytically identify the income stabilization and back loading channels. According to equation $(27)$, wages satisfy the following SDE

$$dw_t = -\left(\frac{1}{\theta}\right)\left[\left(\frac{1}{k_t}\right)dk_t + d\ln(-v_t)\right]\,. \tag{30}$$

[21] See proof of Proposition 3.
[22] See the proof of Corollary 3.

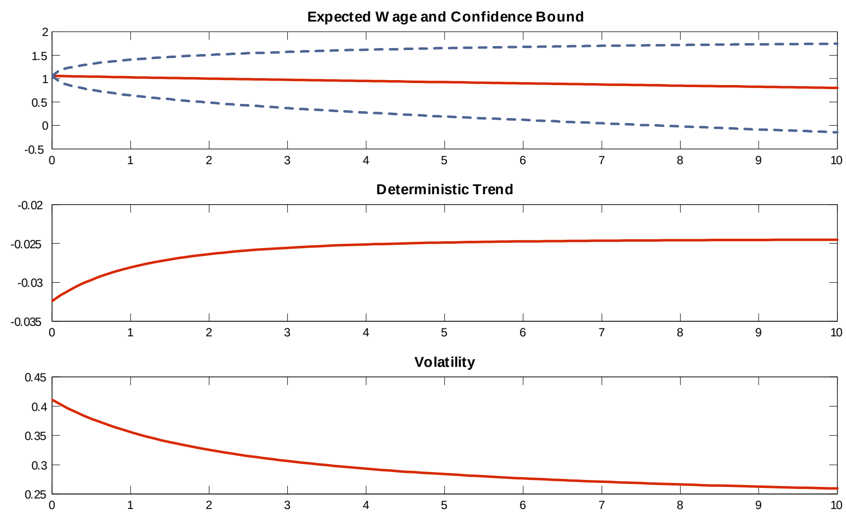


Figure 2: WAGE DYNAMICS AS A FUNCTION OF CONTRACT DURATION.

Reinserting (29) into (30) and applying Ito's lemma to the logarithmic transformation of $v$ yields the following "reduced form" for wage growth

$$dw_t = \frac{1}{\theta}\left(\underbrace{-\frac{dk_t/dt}{k_t}}_{\text{Insurance}} + \underbrace{\frac{(\theta\lambda)^2}{2}\left(\frac{\sigma^{-1}}{h_t}\right)^2}_{\text{Back Loading}} \underbrace{-\frac{(\sigma\theta\lambda)^2}{2}k_t^2}_{\text{Immiserization}}\right)dt + \frac{\Gamma_t}{\theta}\sigma dZ_t\,. \tag{31}$$

The SDE neatly sums up the three mechanisms that drive income dynamics: (i) For a constant promised value, wages decrease over time due to better insurance; (ii) Back loading weakens over time, raising current income; (iii) Wages are driven downwards by the agent's immiserization. Of the three channels, only the first two are due to learning. Immiserization, by contrast, remains relevant when belief precision is infinite. It follows from the *inverse Euler equation* that can be established in the infinite-precision limit using Ito's lemma

$$dU_w\left(w_t, a_t\right)^{-1} = -\frac{\lambda\sigma}{v}dZ_t\,,\ \text{when}\ \sigma^{-2}/h_t = 0\,.$$

Under (20), $U_w\left(w_t, a_t\right)^{-1} = \exp\left(\theta\left[w-\lambda\right]\right)/\theta$ is convex in $w$, hence the immiserization. However, if we had solved the problem using preferences for which the inverse marginal utility of income is concave,[23] the inverse Euler equation would

[23]An example of such utility function could be $U\left(w,a\right) = c(a)w^{1-\phi}/\left(1-\phi\right)$ with $\phi < 1$ and $c'(a) < 0$.

imply that wages exhibit a positive trend. Immiserization is therefore specific to the exponential specification of the utility function.

The trend and volatility terms in $(31)$ are both deterministic. We plot them in the second and third panels of Figure 2. The assumed parameter values are shown in Table 1. They are used as baseline numbers for all the simulations reported in the paper. The value $h_0 = 4.81$ is the smallest precision that satisfies the second-order condition $(26)$ given the assumed values of the other parameters.

| $\rho$ | $\sigma^2$ | $\theta$ | $\lambda$ | $h_0$ |
|---|---|---|---|---|
| 0.5 | 0.5 | 1 | 0.7 | 4.81 |

TABLE 1: BASELINE PARAMETERS

The middle panel of Figure 2 shows that the trend is increasing over time. Hence, quality uncertainty reinforces the immiserization process because the back loading channel is dominated by the income stabilization channel. This is not always true: Other parameter constellations yield decreasing or even hump-shaped profiles for the deterministic trend.

The top panel of Figure 2 plots the mean wage and the one-standard-deviation bands for the parameter values in Table 1. The stochastic term $\sigma dZ$ is the output surprise defined in $(6)$, which means that the solution $w_t$ to the SDE $(31)$ is a normally distributed random variable. The distribution of wages at date $t$ is the frequency distribution of wages among age-$t$ workers with abilities randomly drawn from $\eta \sim N\left(0, h_0^{-1}\right)$. By normality, the bands are equidistant from the mean, hence, symmetric. Furthermore $k_t$ has a strictly negative limit

$$\lim_{t\to\infty} |k_t| = |K| = \frac{1}{2}\left(\sqrt{\left(\frac{1}{(\sigma\lambda\theta)^2}\right)^2 + \frac{4\rho}{(\sigma\lambda\theta)^2}} - \frac{1}{(\sigma\lambda\theta)^2}\right) > 0\,;$$

implying that the volatility of the wage increments does not die off as

$$\left|\frac{\Gamma_t}{\theta}\sigma\right| = \left|\lambda\sigma\left(k_t - \frac{\sigma^{-2}}{h_t}\right)\right| \to \lambda\sigma\,|K| > 0\,.$$

Since these increments are independent, the cross-sectional variance of wages converges to infinity. We sum up our findings in the corollary below, whereas Figure 2 illustrates them.

**Corollary 5** *The volatility of the wage increments is decreasing to a positive limit so that the cross-sectional variance of wages grows without bound. Provided that the sufficient condition* $(16)$ *is satisfied, wages exhibit a negative trend.*

*Surplus as a function of prior precision.*— Instead of focusing on wage dynamics within a given match, we can use the model to compare the surplus associated

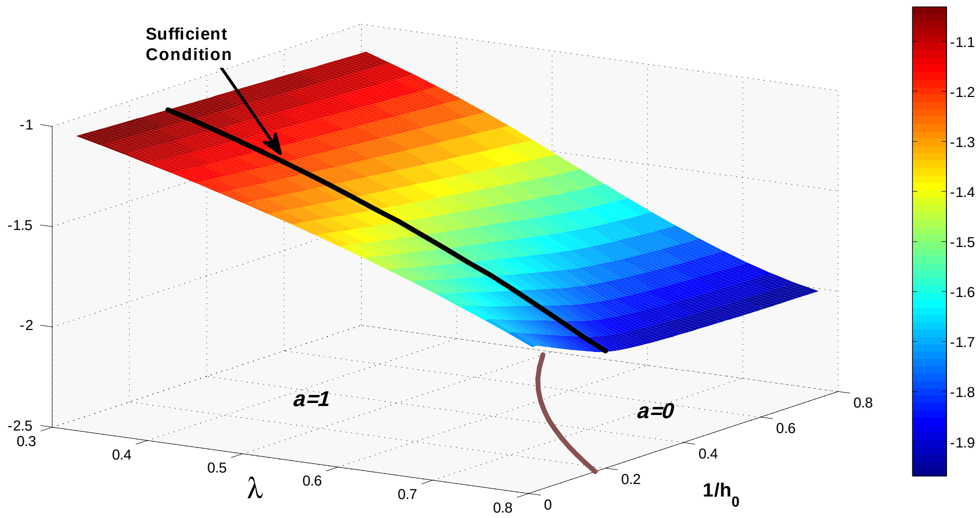


Figure 3: AGENT'S VALUE AS A FUNCTION OF $1/h_0$ AND $\lambda$.

to commitment across different environments. As stated in Corollary 2, the surplus is higher when priors are more accurate. The intuition for this result directly follows from Corollary 4: An increase in the precision with which the productivity of the match is known enables the principal to further stabilize income. As contracts get closer to the second best, the principal can deliver the promised value $v$ at a lower expected cost.

Figure 3 plots the agent's value as a function of the prior variance $1/h_0$ and of the marginal cost of effort parameter $\lambda$, holding the principal's value constant at zero. The other parameters are as given in Table 1. We report on the horizontal plane a line that separates the regions where recommended effort is zero or one.

We also include in Figure 3 a solid black line labeled "sufficient condition" which identifies the maximal prior variance $1/h_0$ and $\lambda$ above which incentive compatibility holds surely. In particular, $(26)$ (which involves both $\lambda$ and $h$) holds to the left of the line. For the parameter values used in the plot, $(26)$ reads $1/4 > h_0^{-1} + 2\left(\lambda h_0^{-1}\right)^2$, and so the maximal $\lambda$ as a function of $h_0$ is given by

$$\lambda = \sqrt{\frac{h_0}{2}\left(\frac{h_0}{4} - 1\right)}. \tag{32}$$

The RHS of this equation is positive only if $h_0 \geq 4$. In other words, $(26)$ can be met only if $h_0^{-1} < 25\%$, and then more easily if $\lambda$ is low enough. However, the RHS of $(32)$ exceeds unity once $h_0^{-1} \leq 0.1830$. Then $(26)$ holds for all $\lambda \in (0, 1)$.

As expected, the agent's value is decreasing in the prior variance $1/h_0$. Figure 3 also illustrates how an increase in $\lambda$ lowers the surplus. This is what one should expect because the higher $\lambda$, the more costly it is to provide effort. Hence an increase in $\lambda$ intensifies the severity of the moral hazard problem, making it more costly for the principal to deliver a given utility.

# 6 Commitment vs. Spot Market

We now wish to relate our model to the literature on reputations that typically adopts the interpretation that $\eta$ is general ability. We focus on the canonical model of Holmström (1999; "H" hereafter) which assumes spot-market wages that may reflect the worker's history but cannot reflect current output.

In both ours and Holmström's model, the principal is risk neutral. The agents' utility functions, however, differ because Holmström assumes that agents are risk-neutral. To make our analysis of commitment comparable to his analysis, we shall derive the spot-market equilibria of H in our environment, i.e., for the case where the agent has lifetime utility $(4)$ and period utility $(20)$.

Holmström imposes zero expected profits for the principal after every history and at each date. In our model, the principal has full commitment and his profits will not be zero at an arbitrary date. To compare our solution to H, it is natural to impose zero expected lifetime profits on the principal at the outset. Thus we shall assume that at date zero, the agent gets all the rents from the relationship.

We first show that the equilibrium behavior of spot-market wages and effort under risk aversion is essentially the same as in H: Reputational concerns are the only reason why the agent exerts any effort, and when information about $\eta$ accumulates and as these concerns disappear, his effort converges to zero, just as in the risk-neutral case. Of itself this is not surprising. Rather, the result is useful because it enables us to isolate the role that full commitment plays in generating economic outcomes for the parties to the contract.

Employers cannot commit to paying wages that depend on performance, and competition among employers bids wages up to expected output. Denoting as before equilibrium actions by a star, expected productivity reads

$$w_t^S = \hat{\eta}\left(Y_t - A_t^*, t\right) + a_t^* \ , \tag{33}$$

where we have added an $S$ superscript for spot wages.

Equilibrium action entails a deterministic sequence for $a_t^*$ because the utility function does not display any wealth effect. Effort is sustained by the market's imprecise knowledge of $\eta$ and the agent's attempts to raise the market's expectation. With our utility function and a spot market, the sequence $a_t$ eventually reaches zero and remains there.

**Proposition 5** *Assume that* $(i)$ $U(w, a)$ *is as specified in* $(20)$*; and* $(ii)$ *Wages are set on the spot market, i.e. (33). Then*

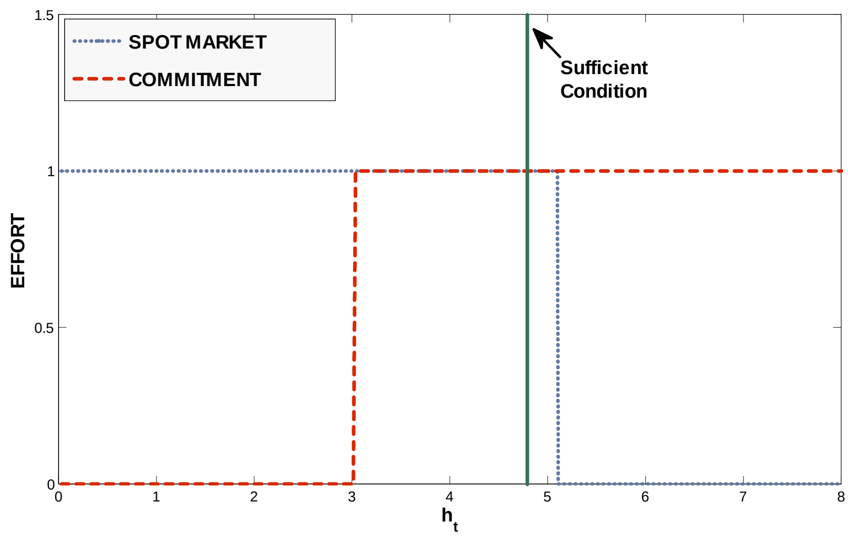


Figure 4: EFFORT AS A FUNCTION OF PARAMETER PRECISION $h_t$.

$(a)$ *The equilibrium effort path* $a_t^*$ *is deterministic;*
$(b)$ *There exists a precision* $\bar{h}$ *such that* $a_t^* = 0$ *whenever* $h(t) > \bar{h}$.

Figure $4$ reports the effort path as a function of $h_t$ when wages are set on the spot market along with its counterpart in the commitment scenario. It illustrates how uncertainty about general ability affects incentives in opposite directions. Spot markets elicit higher effort when beliefs are less precise because reputations have not yet been established. By contrast, under commitment, incentives are more costly to provide when precision is low. This is why the two effort paths are almost mirror images of each other: It switches from one to zero in the spot market and from zero to one under commitment.[24] Their profile are not smooth because the marginal cost of effort is *decreasing* in consumption, full effort can always be sustained through a less than proportional increase in wages whenever interior effort, $a^* \in (0, 1)$, is incentive compatible. Such a deviation is Pareto optimal and so dominates any equilibrium path with intermediate action.

Figure $4$ does not accurately represent the distribution of lifetime gains that full commitment offers. That would be the distribution of the random variable $\mathcal{U}_0$ defined in $(4)$, which we report in Figure $5$.[25] While wages themselves are normally distributed, utilities are nonlinear and bounded above. This is why the

[24]Full effort in the spot market is incentive compatible for $h_t < \bar{h}$ since the function $R_t$ defined in the proof of Proposition 5 is decreasing in $h_t$.

[25]The distribution of lifetime utilities is obtained through Monte Carlo simulations. We simulate 10000 sample paths and compute the resulting kernel densities. We verify the accuracy of

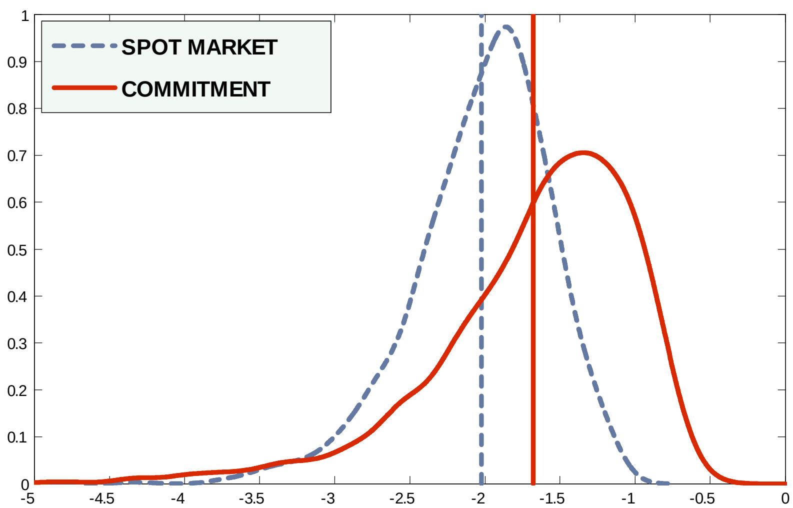


Figure 5: DISTRIBUTION OF LIFETIME UTILITIES.

distributions of $\mathcal{U}_0$ are skewed to the left with their means, as represented by the vertical lines, to the left of the modes. Figure 5 illustrates that commitment results in a noticeably higher expected lifetime utility $E_0[\mathcal{U}_0]$: Long-term contracts raise the agent's utility by $17.1\%$, a gain that is equivalent to a compensating variation of $26.4\%$ in wages across first-best allocations.[26] Even though utilities derived from contracts exhibit more dispersion, they dominate from a stochastic point of view. In other words, not only the average worker but most workers do benefit from contracting.

*$\eta$ as a match-specific ability.*— If, instead of denoting general ability, $\eta$ were match specific, then neither the optimal contract nor the Pareto frontier would change under full commitment. By contrast, spot-markets would work poorly. The agent now has no reputational concern; implying that effort would remain constant at zero. The wage would equal $E_t[\eta]$ at *all* dates. The value of commitment is then even larger than in the case where ability is transferable.

*Participation constraints.*— We have described two separate economies, each

the procedure by comparing the simulated and theoretical average utilities. The approximation error turns out to be around $10^{-3}$ in relative difference.

[26]The welfare gain is obtained dividing the difference between the two expected utilities $E_0[\mathcal{U}_0]$ by the expected utility when wages are set on the spot market. To obtain the compensating variation, we first derive the wage such that $U(w,1)/r = E_0[\mathcal{U}_0]$, which yields $w^{Com}$ under commitment and $w^{Spot}$ under spot market. The compensating variation follows taking the difference between the two wages and dividing it by $w^{Spot}$.

with its own wage setting protocol. Our commitment solution is for a contract that would yield the principal zero expected profit at the outset, but after some histories his expected profit will fall below zero. Similarly, the agent's continuation value may fall below the spot market solution. An extension would add participation constraints as Rudanko (2010) and Lustig *et al.* (2007) have done for multi-agent environments without learning. In partial equilibrium settings without learning there are more papers with limited commitment. Closely related to ours is the principal-agent model of Sannikov (2008) which, under some adjustments to the parametric form of the utility function, is encompassed in our framework as $h_0 \to \infty$, i.e., when posteriors have converged to the true value of $\eta$. More precisely, Sannikov considers a utility function that is (i) defined over the positive real line; (ii) bounded from below; and (iii) separable in income and effort. By contrast, our utility function $(20)$ is not bounded from below and, as a result, we do not have a low retirement point. Observe, however, that our characterization of the agent's necessary condition $(11)$ does not depend on the parametric assumption $(20)$ and so coincides with Sannikov's when $h_t = \infty$.

# 7 Conclusion

We have solved a contracting problem with quality uncertainty and explained why it worsens the incentive insurance trade-off. We developed an approach that works for any utility function when the quality and noise are normally distributed. We found that the agent faces two opposite effects when considering a downward deviation from recommended effort. On the one hand, he will be punished by a lower promised value because of the decrease in observable output. On the other hand, he will benefit from higher expectations than the principal about the unknown productivity of the match. This second channel that we label belief manipulation is specific to problems under quality uncertainty. The extent to which it influences incentive provisions depends on the remaining length of the relationship. This is why it is not relevant in markets based on spot agreements.

Although the prospect of belief manipulation reduces the gains from commitment, our simulation shows that it does not eliminate them altogether. We found, in particular, that quality uncertainty makes it harder to reward effort under full commitment, in direct contrast to its tendency to stimulate effort in spot markets. Spot and full commitment settings are both highly stylized depictions of how markets operate in reality. Thus a promising task would be to combine both environments in a model with limited commitment so as to evaluate how the two incentive channels interact.

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# Appendix A: Proofs of propositions and corollaries

**Proof. Proposition 1:** Consider the Brownian motion $Z^0$ under some probability space with probability measure $Q$, and $\mathbb{F}^{Z^0} \triangleq \left\{\mathcal{F}_t^{Z^0}\right\}_{0\le t\le T}$ the suitably augmented filtration generated by $Z^0$. Let

$$Y_t = \int_0^t \sigma dZ_s^0 \; ,$$

so that $Y_t$ is also a Brownian motion under $Q$. Given that expected output is linear in cumulative output,[27] the exponential local martingale

$$\Lambda_{t,\tau}^a \triangleq \exp\left(\int_t^\tau \left(\frac{\hat\eta(Y_s - A_s, s) + a_s}{\sigma}\right) dZ_s^0 - \frac{1}{2}\int_t^\tau \left|\frac{\hat\eta(Y_s - A_s, s) + a_s}{\sigma}\right|^2 ds\right) \; , \; t \le \tau \le T \; ,$$

is a martingale, i.e. $E_t\left[\Lambda_{t,T}^a\right] = 1$. Hence Girsanov theorem holds and ensures that

$$Z_t^a \triangleq Z_t^0 - \int_0^t \left(\frac{\hat\eta(Y_s - A_s, s) + a_s}{\sigma}\right) ds$$

is a Brownian motion under the new probability measure $dP^a/dP \triangleq \Lambda_{0,T}^a$. Given that both measures are equivalent, the triple $(Y, Z^a, Q^a)$ is a weak solution of the SDE

$$Y_t = \int_0^t \left(\hat\eta(Y_s - A_s, s) + a_s\right) ds + \int_0^t \sigma dZ_s^a \; .$$

Adopting a weak formulation allows us to view the choice of control $a$ as determining the probability measure $Q^a$. In order to define the agent's optimization problem, let $R^a(t)$ denote the reward from time $t$ onwards so that

$$R^a(t) \triangleq e^{\rho t}\left[\int_t^T U\left(s, \overline{Y}_s, a_s\right) ds + W\left(T, \overline{Y}_T\right)\right] \; ,$$

where $U\left(s, \overline{Y}_s, a_s\right) \triangleq e^{-\rho s} U\left(w\left(\overline{Y}_s\right), a_s\right)$ and $W\left(T, \overline{Y}_T\right) \triangleq e^{-\rho T} W\left(\overline{Y}_T\right)$ are utilities at time $t$ discounted from time $0$. The agent's objective is to find an admissible control process that maximizes the expected reward $E^a\left[R^a(0)\right]$ over all admissible controls $a \in \mathcal{A}$. In other words, the agent solves the following problem

$$v_t = \sup_{a\in\mathcal{A}} V^a(t) \triangleq \sup_{a\in\mathcal{A}} E_t^a\left[R^a(t)\right] \; , \text{ for all } 0 \le t \le T \; .$$

[27] More formally, the martingale property holds true because

$$\left|\hat\eta(Y_t - A_t, t) + a_t\right| \le K\left(1 + \left\|Z^0\right\|_t\right) \; , \text{ for all } t \in [0, T] \; ,$$

with $K = \frac{\sigma^{-1}}{h_0} + 1$ and $\left\|Z^0\right\|_t \triangleq \max_{0\le s\le t}\left|Z^0(s)\right|$.

The objective function can be recast as

$$V^a(t) = E_t^a\left[R^a(t)\right] = E_t\left[\Lambda_{t,T}^a R^a(t)\right] \ , \tag{34}$$

where the operator $E^a[\cdot]$ and $E[\cdot]$ are expectation under the probability measure $Q^a$ and $Q$, respectively. One can see from $(34)$ that varying $a$ is indeed equivalent to changing the probability measure. The key advantage of the weak formulation is that, under the reference measure $Q$, the output process does not depend on $a$. Hence, we can treat it as fixed which enables us to solve our problem in spite of its non-Markovian structure.

Our derivation of the necessary conditions builds on the variational argument in Cvitanić *et al.* (2009). Define the control perturbation

$$a^\varepsilon \triangleq a + \varepsilon \Delta a \ ,$$

such that there exists an $\varepsilon_0 > 0$ for which any $\varepsilon \in [0, \varepsilon_0)$ satisfy $|a^\varepsilon|^4$, $\left|U^{a^\varepsilon}\right|^4$, $\left|U_a^{a^\varepsilon}\right|^4$, $\left|\Lambda_{t,\tau}^{a^\varepsilon}\right|^4$, $\left(\mathcal{U}_{t,\tau}^{a^\varepsilon}\right)^2$ and $\left(\partial_a \mathcal{U}_{t,\tau}^{a^\varepsilon}\right)^2$ being uniformly integrable in $L^1(Q)$ where

$$\mathcal{U}_{t,\tau}^a \triangleq \int_t^\tau U\left(s, \overline{Y}_s, a_s\right) ds \ .$$

We introduce the following shorthand notations for "variations"

$$\begin{aligned}
\nabla \mathcal{U}_{t,\tau}^a &\triangleq \int_t^\tau U_a\left(s, \overline{Y}_s, a_s\right) \Delta a_s ds \ , && (35)\\
\nabla A_t &\triangleq \int_0^t \Delta a_s ds \ , && (36)\\
\nabla \Lambda_{t,\tau}^a &\triangleq \Lambda_{t,\tau}^a \left(\frac{1}{\sigma}\right)\left[\int_t^\tau \left(-\frac{\sigma^{-2}}{h_s}\nabla A_s + \Delta a_s\right) dZ_s^0 - \int_t^\tau \left(\hat{\eta}_s + a_s\right)\left(-\frac{\sigma^{-2}}{h_s}\nabla A_s + \Delta a_s\right) ds\right] \\
&= \Lambda_{t,\tau}^a \left(\frac{1}{\sigma}\right) \int_t^\tau \left(-\frac{\sigma^{-2}}{h_s}\nabla A_s + \Delta a_s\right) dZ_s^a \ . && (37)
\end{aligned}$$

**Step 1:** We first characterize the variations of the agent's objective with respect to $\varepsilon$

$$\begin{aligned}
\frac{V^{a^\varepsilon}(t) - V^a(t)}{\varepsilon} &= E\left[\Lambda_{t,T}^{a^\varepsilon} R^{a^\varepsilon}(t) - \Lambda_{t,T}^a R^a(t)\right] \\
&= E\left[\left(\frac{\Lambda_{t,T}^{a^\varepsilon} - \Lambda_{t,T}^a}{\varepsilon}\right) R^{a^\varepsilon}(t) + \Lambda_{t,T}^a \left(\frac{R^{a^\varepsilon}(t) - R^a(t)}{\varepsilon}\right)\right] \\
&= E\left[\nabla \Lambda_{t,T}^{a^\varepsilon} R^{a^\varepsilon}(t) + \Lambda_{t,T}^a \left(\frac{R^{a^\varepsilon}(t) - R^a(t)}{\varepsilon}\right)\right] \ .
\end{aligned}$$

To obtain the limit of the first term as $\varepsilon$ goes to zero, observe that

$$\nabla\Lambda_{t,T}^{a^{\varepsilon}} R^{a^{\varepsilon}}(t) - \nabla\Lambda_{t,T}^{a} R^{a}(t) = \left[\nabla\Lambda_{t,T}^{a^{\varepsilon}} - \nabla\Lambda_{t,T}\right] R^{a}(t) + \nabla\Lambda_{t,T}^{a^{\varepsilon}}\left[R^{a^{\varepsilon}}(t) - R^{a}(t)\right] .$$

As shown in Cvitanić *et al.* (2009), for any $\varepsilon \in [0, \varepsilon_0)$, this expression is integrable uniformly with respect to $\varepsilon$ and so

$$\lim_{\varepsilon\to 0} E\left[\nabla\Lambda_{t,T}^{a^{\varepsilon}} R^{a^{\varepsilon}}(t)\right] = E\left[\nabla\Lambda_{t,T}^{a} R^{a}(t)\right] .$$

The limit of the second term reads

$$\lim_{\varepsilon\to 0} \frac{R^{a^{\varepsilon}}(t) - R^{a}(t)}{\varepsilon} = e^{\rho t}\nabla\mathcal{U}_{t,T}^{a} .$$

Due to the uniform integrability of $\Lambda_{t,T}^{a}\left(R^{a^{\varepsilon}}(t) - R^{a}(t)\right)/\varepsilon$, the expectation is also well defined. Combining the two expressions above, we finally obtain

$$\lim_{\varepsilon\to 0} \frac{V^{a^{\varepsilon}}(t) - V^{a}(t)}{\varepsilon} = E\left[\nabla\Lambda_{t,T}^{a} R^{a}(t) + \Lambda_{t,T}^{a} e^{\rho t}\nabla\mathcal{U}_{t,T}^{a}\right] \triangleq \nabla V^{a}(t) . \tag{38}$$

**Step 2:** We are now in a position to derive the necessary condition. Consider total earnings as of date $0$

$$I^{a}(t) \triangleq E_t^{a}\left[\int_0^T U\left(s, \overline{Y}_s, a_s\right) ds + W\left(T, \overline{Y}_T\right)\right] = \int_0^t U\left(s, \overline{Y}_s, a_s\right) ds + e^{-\rho t} V^{a}(t) . \tag{39}$$

By definition, it is a $Q^{a}-$martingale. According to the extended Martingale Representation Theorem[28] of Fujisaki et al. (1972), all square integrable $Q^{a}-$martingales are stochastic integrals of $\{Z_t^{a}\}$ and there exists a unique process $\zeta$ in $L^2(Q^{a})$ such that

$$I^{a}(T) = I^{a}(t) + \int_t^T \zeta_s \sigma dZ_s^{a} . \tag{40}$$

This decomposition allows us to solve for $\nabla V^{a}(t)$. Reinserting $(35)$, $(36)$ and $(37)$ into $(38)$ yields[29]

$$\begin{aligned}\nabla V^{a}(t) &= E_t\left[\Lambda_{t,T}^{a} R^{a}(t)\,\sigma^{-1}\int_t^T \left(-\frac{\sigma^{-2}}{h_s}\nabla A_s + \Delta a_s\right) dZ_s^{a} + \Lambda_{t,T}^{a} e^{\rho t}\left(\int_t^T U_a \Delta a_s ds\right)\right] \\ &= e^{\rho t} E_t^{a}\left[I^{a}(T)\sigma^{-1}\int_t^T \left(-\frac{\sigma^{-2}}{h_s}\nabla A_s + \Delta a_s\right) dZ_s^{a} + \int_t^T U_a \Delta a_s ds\right] .\end{aligned}$$

[28]We cannot directly apply the standard Martingale Representation theorem because we are considering weak solutions, so that $\{Z_t^{a}\}$ does not necessarily generate $\{\mathcal{F}_t^{Y}\}$.

[29]The additional expectation term vanishes because both $\left(\frac{h_\varepsilon}{h_s}\right)\nabla A_s$ and $\Delta a_s$ are bounded and so

$$\left(\int_0^t U\left(\tau, \overline{Y}_\tau, a_\tau\right) d\tau\right) E_t^{a}\left[\int_t^T \left(-\left(\frac{h_\varepsilon}{h_s}\right)\nabla A_s + \Delta a_s\right) dZ_s^{a}\right] = 0 .$$

where subscripts denote derivatives and arguments are omitted for brevity. Given the law of motion (40), applying Ito's rule to the first term yields

$$d\left(I^a(\tau)\int_t^\tau\left(-\frac{\sigma^{-2}}{h_s}\nabla A_s+\Delta a_s\right)dZ_s^a\right)=\left[\zeta_\tau\sigma\left(-\left(\frac{\sigma^{-2}}{h_\tau}\right)\nabla A_\tau+\Delta a_\tau\right)\right]d\tau$$
$$+\left[\zeta_\tau\sigma\int_t^\tau\left(-\frac{\sigma^{-2}}{h_s}\nabla A_s+\Delta a_s\right)dZ_s^a+I_t^a(\tau)\left(-\left(\frac{\sigma^{-2}}{h_\tau}\right)\nabla A_\tau+\Delta a_\tau\right)\right]dZ_\tau^a\ .$$

Hence $\nabla V^a(t)$ can be represented as

$$e^{-\rho t}\nabla V^a(t)=E_t^a\left[\int_t^T\Gamma_s^1ds+\int_t^T\Gamma_s^2dZ_s^a\right]\ ,$$

where

$$\Gamma_s^1\ \triangleq\ \zeta_s\left[-\frac{\sigma^{-2}}{h_s}\int_0^s\Delta a_\tau d\tau+\Delta a_s\right]+U_a\left(s,\overline{Y}_s,a_s\right)\Delta a_s\ ,$$
$$\Gamma_s^2\ \triangleq\ \zeta_s\left[\int_t^s\left(-\frac{\sigma^{-2}}{h_\tau}\int_0^\tau\Delta a_rdr+\Delta a_\tau\right)dZ_\tau^a\right]+I_t^a(s)\left(-\frac{\sigma^{-2}}{h_s}\int_0^s\Delta a_\tau d\tau+\Delta a_s\right)\ .$$

Given that $\Gamma_s^2$ is square integrable,[30] we have

$$E_t^a\left[\int_t^T\Gamma_s^2dZ_s^a\right]=0\ .$$

As for the deterministic term, collecting the effect of each perturbation $\Delta a_s$ yields

$$e^{-\rho t}\nabla V^a(t)=E_t^a\left[\int_t^T\left(-\int_s^T\zeta_\tau\left(\frac{\sigma^{-2}}{h_\tau}\right)d\tau+\zeta_s+U_a\left(s,\overline{Y}_s,a_s\right)\right)\Delta a_sds\right]\ .$$

Finally, noticing that $\Delta a_s$ was arbitrary leads to

$$\left(E_t^a\left[-\int_t^T\zeta_s\frac{\sigma^{-2}}{h_s}ds\right]+\zeta_t+U_a\left(t,\overline{Y}_t,a_t^*\right)\right)(a_t-a_t^*)\leq 0\ . \tag{41}$$

**Step 3:** We now rewrite our solution as a function of the promised value $v_t$. Differentiating (39) with respect to time yields

$$e^{-\rho t}dv_t-\rho e^{-\rho t}v_t+U\left(t,\overline{Y}_t,a_t\right)=dI^a(t)=\zeta_t\sigma dZ_t^a\ ,$$

so that

$$dv_t=\left(\rho v_t-U\left(\overline{Y}_t,a_t\right)\right)dt+\gamma_t\sigma dZ_t^a\ ,$$

[30] Square integrability of $\Gamma_s^2$ can be established for any $\varepsilon\in[0,\varepsilon_0)$ following the same steps as in Lemma 7.3 of Cvitanić *et al.* (2009).

with $\gamma_t \triangleq \zeta_t e^{\rho t}$. Collecting the exponential terms in $(41)$ leads to $(11)$. ■

**Proof. Proposition 2:** The sufficient conditions are established comparing the equilibrium path $\{a_t^*\}_{t=0}^T$ with an arbitrary effort path $\{a_t\}_{t=0}^T$. We define $\delta_t \triangleq a_t - a_t^*$ and $\Delta_t \triangleq \int_0^t \delta_s ds = A_t - A_t^*$ as the differences in current and cumulative effort between the arbitrary and recommended paths. We also attach a star superscript to denote the value of the $\mathbb{F}^Y-$measurable stochastic processes along the equilibrium path. The Brownian motions generated by the two effort policies are related by

$$\begin{aligned}\sigma dZ_t^{a^*} &= \sigma dZ_t^a + \left[\hat{\eta}\left(Y_t - A_t, t\right) + a_t - \hat{\eta}\left(Y_t - A_t^*, t\right) - a_t^*\right] dt \\ &= \sigma dZ_t^a + \left[\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t\right] dt \,.\end{aligned}$$

By definition, the total reward from the optimal policy reads

$$\begin{aligned}I^{a^*}(T) &= \int_0^T U\left(t, \overline{Y}_t, a_t^*\right) dt + W\left(\overline{Y}_T\right) = V^{a^*}(0) + \int_0^T \zeta_t^* \sigma dZ_t^{a^*} \\ &= V^{a^*}(0) + \int_0^T \zeta_t^* \left[\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t\right] dt + \int_0^T \zeta_t^* \sigma dZ_t^a \,.\end{aligned}$$

Hence, the total reward from the arbitrary policy is given by

$$\begin{aligned}I^a(T) &= \int_0^T \left[U\left(t, \overline{Y}_t, a_t\right) - U\left(t, \overline{Y}_t, a_t^*\right)\right] dt + I^{a^*}(T) \\ &= \int_0^T \left[U\left(t, \overline{Y}_t, a_t\right) - U\left(t, \overline{Y}_t, a_t^*\right)\right] dt + V^{a^*}(0) \\ &\quad + \int_0^T \zeta_t^* \left[\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t\right] dt + \int_0^T \zeta_t^* \sigma dZ_t^a \,.\end{aligned}$$

Let us focus on the third term on the right hand side

$$\begin{aligned}-\int_0^T \zeta_t^* \frac{\sigma^{-2}}{h_t}\Delta_t dt &= -\int_0^T \zeta_t^* \frac{\sigma^{-2}}{h_t}\left(\int_0^t \delta_s ds\right) dt = \int_0^T \delta_t \left(-\int_t^T \zeta_s^* \frac{\sigma^{-2}}{h_s} ds\right) dt \\ &= \int_0^T \delta_t \left(e^{-\rho t}\frac{\sigma^{-2}}{h_t} p_t^* + \int_t^T \xi_s^* \sigma dZ_s^{a^*}\right) dt \,,\end{aligned}$$

where the last equality follows from the definition of $p$ and $\xi$.[31] Changing the

[31] Observe that this additional step is linked to the introduction of private information. Then the volatility $\zeta$ of the continuation value will differ on and off the equilibrium path. To the contrary, in problems without private information, the volatility remains constant because it only depends on observable output and not on past actions. This is why sufficiency holds without any restriction in, e.g., Schättler and Sung (1993) or Sannikov (2008).

Brownian motion and taking expectation yields

$$\begin{aligned} V^{a}(0)-V^{a^{*}}(0) &= E_0^a\left[I^a(T)\right]-V^{a^{*}}(0) \\ &= E_0^a\left[\int_0^T\left(U\left(t,\overline{Y}_t,a_t\right)-U\left(t,\overline{Y}_t,a_t^*\right)+\delta_t\left(\zeta_t^*+e^{-\rho t}\frac{\sigma^{-2}}{h_t}p_t^*\right)\right)dt\right] \\ &\quad +E_0^a\left[\int_0^T\left(\int_t^T\xi_s^*\left(\delta_s-\frac{\sigma^{-2}}{h_s}\Delta_s\right)ds\right)dt\right] \\ &= E_0^a\left[\int_0^T e^{-\rho t}\left(U\left(w_t,a_t\right)-U\left(w_t,a_t^*\right)+\delta_t\left(\gamma_t^*+\frac{\sigma^{-2}}{h_t}p_t^*\right)\right)dt\right] \\ &\quad +E_0^a\left[\int_0^T e^{\rho t}\xi_t^*\Delta_t\left(\delta_t-\frac{\sigma^{-2}}{h_t}\Delta_t\right)dt\right]. \end{aligned}$$

We know from the optimization property of $a_t^*$ that the first expectation term is at most equal to zero. On the other hand, the sign of the second expectation term is ambiguous. In order to bound it, we introduce the predictable process[32] $\chi_t^* \triangleq \zeta_t^* - e^{\rho t}\xi_t^* A_t^*$ and define the function[33]

$$H\left(t,a,A;\chi^*,\xi^*,p^*\right)\triangleq U\left(w,a\right)+\left(\chi^*+e^{\rho t}\xi^*A\right)a-e^{\rho t}\xi^*\frac{\sigma^{-2}}{h_t}A^2+\frac{\sigma^{-2}}{h_t}p^*a\,.$$

Taking a linear approximation of $H(\cdot)$ around $A^*$ yields

$$\begin{aligned} & H_t\left(a_t,A_t\right)-H_t\left(a_t^*,A_t^*\right)-\frac{\partial H_t\left(a_t^*,A_t^*\right)}{\partial A}\Delta_t \\ = \; & U\left(w_t,a_t\right)-U\left(w_t,a_t^*\right)+\delta_t\left(\underbrace{\chi_t^*+e^{\rho t}\xi_t^*A_t^*}_{=\zeta_t^*}+\frac{\sigma^{-2}}{h_t}p_t^*\right)+e^{\rho t}\xi_t^*\Delta_t\left(\delta_t-\frac{\sigma^{-2}}{h_t}\Delta_t\right), \end{aligned}$$

so that

$$V^{a}(0)-V^{a^{*}}(0)=E_0^a\left[\int_0^T e^{-\rho t}\left(H_t\left(a_t,A_t\right)-H_t\left(a_t^*,A_t^*\right)-\frac{\partial H_t\left(a_t^*,A_t^*\right)}{\partial A}\Delta_t\right)dt\right]$$

is negative when $H(\cdot)$ is jointly concave. Given that the agent seeks to maximize expected returns, imposing concavity ensures that $a^*$ dominates any alternative effort path. Concavity is established considering the Hessian matrix of $H(\cdot)$

$$\mathcal{H}\left(t,a,A\right)=\begin{pmatrix} U_{aa}\left(w_t,a_t\right) & e^{\rho t}\xi_t \\ e^{\rho t}\xi_t & -2e^{\rho t}\xi_t\frac{\sigma^{-2}}{h_t}\end{pmatrix},$$

[32] $\chi^*$ is predictable since both $\xi^*$ and $A^*$ are $\mathbb{F}^Y-$predictable.

[33] We use $H(\cdot)$ to denote the function because it is equivalent to the Hamiltonian of the optimal control problem which can be derived following Williams' (2008) method based on the stochastic maximum principle.

which is negative semi-definite when $-2\frac{\sigma^{-2}}{h_t}U_{aa}\left(w_t,a_t\right)\geq e^{\rho t}\xi_t$, as stated in $(16)$. ■

**Proof. Corollary 1:** Let $b_t$ be defined as

$$b_t\triangleq E\left[-\int_0^T e^{-\rho s}\gamma_s\left(\frac{h_\varepsilon}{h_s}\right)ds\middle|\mathcal{F}_t^a\right]=b_0+\int_0^t\xi_s\sigma dZ_s,\ \text{for all } t\in[0,T]\,,$$

where the second equality follows from $(17)$. Then the definition of $p_t$ in $(15)$ implies that

$$p_t=e^{\rho t}\sigma^2h_t\left[b_t+\int_0^t e^{-\rho s}\gamma_s\frac{\sigma^{-2}}{h_s}ds\right]\ ,$$

and so, as $T$ goes to infinity, $p_t$ solves the SDE[34]

$$dp_t=\left[\rho p_t+\frac{d\left(\sigma^2h_t\right)}{dt}\frac{\sigma^{-2}}{h_t}p_t+\gamma_t\right]dt+e^{\rho t}\sigma^2h_tdb_t=\left[p_t\left(\rho+\frac{\sigma^{-2}}{h_t}\right)+\gamma_t\right]dt+\vartheta_t\sigma dZ_t\ ,$$

with $\vartheta_t\triangleq e^{\rho t}\sigma^2h_t\xi_t$. ■

**Proof. Claim 1**: The Incentive Constraint $(14)$ allows us to express $\gamma$ as a function of the state and control variables $\{t,v,w,a\}$. Given that effort levels lie in a compact set, its recommended value satisfies

$$e^{-\rho t}-\frac{\partial j}{\partial v}U_a\left(w,a\right)+\sigma^2\frac{\partial^2 j}{\partial v^2}\gamma\left(t,v,w,a\right)\frac{\partial\gamma\left(t,v,w,a\right)}{\partial a}\geq 0\ ,$$

whereas wages take value over the real line and so fulfill the optimality condition

$$-e^{-\rho t}-\frac{\partial j}{\partial v}U_w\left(w,a\right)+\sigma^2\frac{\partial^2 j}{\partial v^2}\gamma\left(t,v,w,a\right)\frac{\partial\gamma\left(t,v,w,a\right)}{\partial w}=0\ .$$

Under our premise that the Incentive Constraint $(14)$ holds with equality, we obtain $\partial\gamma/\partial w=-\lambda\partial\gamma/\partial a>-\partial\gamma/\partial a$, which implies in turn that, when the optimality condition for wages binds, the one for effort is slack. It follows that optimal effort is constant and set equal to the upper-bound $a_t^*=1$. ■

**Proof. Proposition 3:** Assume that

$$\begin{aligned}\rho j\left(v,t\right)&=f\left(t\right)+j_1\ln\left(-\rho v\right),\\ w(t,v)&=-\frac{\ln(k_tv)}{\theta}+\lambda\Rightarrow U(w,1)=-k_tv\ .\end{aligned}$$

[34]The change with respect to time of $\sigma^{-2}/h_t$ is given by

$$\frac{d\left(\sigma^{-2}/h_t\right)}{dt}=\frac{d\left(\sigma^{-2}\left(h_0+t\sigma^{-2}\right)^{-1}\right)}{dt}=-\sigma^{-4}\left(h_0+t\sigma^{-2}\right)^{-2}=-\left(\frac{\sigma^{-2}}{h_t}\right)^2<0\ .$$

Observe that our guess implies that

$$\gamma_t(v,w,a) = -U_a(w(t,v),1) - \frac{\sigma^{-2}}{h_t}\lambda\theta v = -\lambda\theta U(w(t,v),1) - \frac{\sigma^{-2}}{h_t}\lambda\theta v = \lambda\theta v\left(k_t - \frac{\sigma^{-2}}{h_t}\right) .$$

Hence, differentiating the Incentive Constraint yields

$$\frac{\partial\gamma_t(v,w,a)}{\partial w} = -U_{aw}(w,a) = -\theta\gamma_t(v,w,a) - \frac{\sigma^{-2}}{h_t}\lambda\theta^2 v_t \ .$$

Therefore, the FOC for wages is equivalent to

$$\begin{aligned} & -1 - \frac{\partial j}{\partial v}\theta v k_t - \sigma^2\frac{\partial^2 j}{\partial v^2}\left[\left(\lambda\theta v\left(k_t - \frac{\sigma^{-2}}{h_t}\right)\right)^2 + (\lambda\theta v)^2\left(k_t - \frac{\sigma^{-2}}{h_t}\right)\frac{\sigma^{-2}}{h_t}\right]\theta \\ = \ & \frac{1}{\rho}\left(-\rho - j_1\theta k_t + \sigma^2 j_1\left[\left(k_t - \frac{\sigma^{-2}}{h_t}\right)^2 + \left(k_t - \frac{\sigma^{-2}}{h_t}\right)\frac{\sigma^{-2}}{h_t}\right]\lambda^2\theta^3\right) \\ = \ & \frac{1}{\rho}\left(-\rho - j_1\theta k_t + \sigma^2 j_1\left[k_t\left(k_t - \frac{\sigma^{-2}}{h_t}\right)\right]\lambda^2\theta^3\right) = 0 \, , \end{aligned}$$

implying the following quadratic equation for $k_t$

$$-\rho - k_t\left(j_1\theta + \sigma^2 j_1\frac{\sigma^{-2}}{h_t}\lambda^2\theta^3\right) + k_t^2\left(\sigma^2 j_1\lambda^2\theta^3\right) = 0 \ . \tag{42}$$

The remaining step consists in checking that the HJB equation is indeed satisfied

$$\begin{aligned} \rho j_t &= 1 - w + \frac{\partial j}{\partial t} + \frac{\partial j}{\partial v}(\rho v - U(w,1)) + \left(\frac{\sigma^2}{2}\right)\frac{\partial^2 j}{\partial v^2}\gamma^2 \\ &= 1 + \frac{\ln(-v)}{\theta} + \frac{\ln(-k_t)}{\theta} - \lambda - [f(t) + j_1\ln(-\rho v)] + \frac{f'(t)}{\rho} \\ &\quad + \frac{\rho + k_t}{\rho}j_1 - \left(\frac{\sigma^2}{2}\right)\frac{j_1}{\rho}\left(\lambda\theta\left(k_t - \frac{\sigma^{-2}}{h_t}\right)\right)^2 \end{aligned}$$

when $j_1 = \theta^{-1}$ and

$$f'(t) - \rho f(t) = -\rho\left(1 - \lambda + \frac{\ln(-k_t/\rho)}{\theta}\right) - \frac{\rho + k_t}{\theta} + \frac{\theta(\sigma\lambda)^2}{2}\left(k_t - \frac{\sigma^{-2}}{h_t}\right)^2 \ . \tag{43}$$

The quadratic equation (24) is obtained reinserting $j_1 = \theta^{-1}$ in (42)

$$-\rho - k_t\left(1 + \frac{(\lambda\theta)^2}{h_t}\right) + k_t^2(\sigma\lambda\theta)^2 = 0 \ .$$

The relevant solution is unique and given by the negative root because wages are not well defined when $k_t > 0$. The ODE described in the Proposition is obtained noticing that the quadratic equation above implies that

$$\frac{(\sigma\theta\lambda)^2}{2}\left(k_t - \frac{\sigma^{-2}}{h_t}\right)^2 = \rho + k_t + \frac{(\sigma\theta\lambda)^2}{2}\left(\left(\frac{\sigma^{-2}}{h_t}\right)^2 - k_t^2\right) ,$$

and reinserting this expression into (43).

As usual, the unique solution to the ODE is pinned down by its boundary condition. The value function as $t \to \infty$ must converge to the solution of the problem without quality uncertainty, i.e., when $h_t$ is infinite. It can be derived solving the following HJB

$$\rho l\left(v\right) = \max_{\{a,w\}}\left\{a - w + \frac{\partial l}{\partial t} + \frac{\partial l}{\partial v}\left(\rho v - U\left(w,a\right)\right) + \left(\frac{\sigma^2}{2}\right)\frac{\partial^2 l}{\partial v^2}\gamma\left(v,w,a\right)^2\right\} ,$$

with

$$\gamma\left(v,w,a\right) \geq -U_a\left(a,w\right) , \text{ for all } a > 0 .$$

The solution is of the form $\rho l\left(v\right) = F + \ln\left(-\rho v\right)/\theta$ with

$$\rho F = \rho\left(1 - \lambda + \frac{\ln(-K/\rho)}{\theta}\right) + \frac{\theta\left(\sigma\lambda K\right)^2}{2} ,$$

where $K \triangleq \lim_{t\to\infty} k\left(t\right) = \left(1 - \sqrt{1 + 4\rho\left(\sigma\lambda\theta\right)^2}\right) / \left[2\left(\sigma\lambda\theta\right)^2\right]$. One can easily verify that the desired convergence of $f\left(t\right)$ to $F$ as $t \to \infty$ holds true when the boundary condition $\lim_{t\to\infty} f'\left(t\right) = 0$ is satisfied. ■

**Proof. Corollary 2:** Let the function $\psi\left(t\right)$ be defined as

$$\psi\left(t\right) \triangleq \rho\left((1-\lambda) + \frac{\ln(-k_t/\rho)}{\theta}\right) - \frac{\left(\sigma\lambda\right)^2\theta}{2}\left[\left(\frac{\sigma^{-2}}{h_t}\right)^2 - k_t^2\right] . \tag{44}$$

Differentiating $\psi\left(t\right)$ with respect to time yields[35]

$$\psi'\left(t\right) = \left(\frac{\rho}{\theta}\right)\frac{\dot{k}_t}{k_t} - \left(\sigma\lambda\right)^2\theta\left(-\frac{\sigma^{-2}}{h_t} - \dot{k}_t k_t\right) > 0 .$$

Observe that $\psi\left(t\right)$ has been defined so as to satisfy the differential equation $f'\left(t\right) = \rho f\left(t\right) - \psi\left(t\right)$. In order to reach a contradiction, assume that $\rho f\left(t\right) < \psi\left(t\right)$. Then $f'\left(t\right) < 0$ and so $\rho f\left(s\right) < \psi\left(t\right) < \psi\left(s\right)$ for all $s \geq t$. But this contradicts the boundary condition $\lim_{s\to\infty} \rho f\left(s\right) = \lim_{s\to\infty}\psi\left(s\right) > 0$. We can therefore conclude that $\rho f\left(t\right) > \psi\left(t\right)$ which implies in turn that $f'\left(t\right) > 0$. ■

[35]Remember that both $\dot{k}_t$ and $k_t$ are negative.

**Proof. Corollary 3:** Given that $U_{aa}(w_t, a_t^*) = -k_t v (\lambda\theta)^2$ and $\vartheta_t^* = \lambda\theta\gamma_t^*(v) = (\lambda\theta)^2 v \left(k_t - \frac{\sigma^{-2}}{h_t}\right)$, the sufficient condition of Proposition 2 are satisfied when

$$2k_t v - v\left(k_t - \frac{\sigma^{-2}}{h_t}\right) = v\left(k_t + \frac{\sigma^{-2}}{h_t}\right) > 0 \Leftrightarrow -k_t > \frac{\sigma^{-2}}{h_t}\,. \tag{45}$$

Differentiating the explicit solution of the quadratic equation for $k_t$ yields

$$\frac{dk(t)}{dt} = \frac{1}{2}\left[1 - \frac{\frac{1}{(\sigma\lambda\theta)^2} + \frac{\sigma^{-2}}{h_t}}{\sqrt{\left(\frac{1}{(\sigma\lambda\theta)^2} + \frac{\sigma^{-2}}{h_t}\right)^2 + \frac{4\rho}{(\sigma\lambda\theta)^2}}}\right]\underbrace{\frac{d\left(\sigma^{-2}h_t^{-1}\right)}{dt}}_{<0} < 0\,. \tag{46}$$

Since $\sigma^{-2}/h_t$ is decreasing in $t$, condition $(45)$ is satisfied for all $t$ provided that $-k_0 > \sigma^{-2}/h_0$, i.e.

$$-\frac{1}{(\sigma\lambda\theta)^2} - 3\left(\frac{\sigma^{-2}}{h_0}\right) + \sqrt{\left(\frac{1}{(\sigma\lambda\theta)^2} + \left(\frac{\sigma^{-2}}{h_0}\right)\right)^2 + \frac{4\rho}{(\sigma\lambda\theta)^2}} > 0\,,$$

which, after some straightforward simplifications, leads to requirement $(26)$. ■

**Proof. Proposition 4:** Consider an arbitrary strategy such that $(14)$ does not hold over some time interval $[t, t+\varepsilon]$ with $\varepsilon > 0$.[36] The cheapest way for the principal to provide any given value to the agent in this time frame is by setting wages constant. Let $w^{\Delta}(v)$ be defined as

$$w^{\Delta}(v) \triangleq \frac{\ln(-\rho(v+\Delta))}{\theta} \Rightarrow U(w^{\Delta}(v), 0) = \rho(v+\Delta)\ ,$$

so that the promise keeping constraint holds when

$$\int_t^{t+\varepsilon} e^{-\rho(t-s)} U(w^{\Delta}(v_t), 0)dt + e^{-\rho\varepsilon} v_{t+\varepsilon} = v\ ,$$

that is if

$$v_{t+\varepsilon} = e^{\rho\varepsilon} v_t - (e^{\rho\varepsilon} - 1)(v_t + \Delta) = v_t - (e^{\rho\varepsilon} - 1)\Delta\,.$$

Now suppose that the principal expects the agent to provide full effort $a = 1$ at every point in time following $t + \varepsilon$. Let $i(\Delta, \varepsilon; t, v)$ denote the returns to the principal of this strategy. It is equal to

$$\begin{aligned} i(\Delta, \varepsilon; t, v) &= \int_t^{t+\varepsilon} e^{-\rho(s-t)} w^{\Delta}(v)\, ds + e^{-\rho\varepsilon} j(t+\varepsilon, v - (e^{\rho\varepsilon} - 1)\Delta) \\ &= \frac{1}{\rho}\left[\left(1 - e^{-\rho\varepsilon}\right)\frac{\ln(-\rho(v+\Delta))}{\theta} + e^{-\rho\varepsilon} f(t+\varepsilon) + e^{-\rho\varepsilon}\frac{\ln(-\rho(v - (e^{\rho\varepsilon} - 1)\Delta))}{\theta}\right]\,. \end{aligned}$$

[36] The proof easily extends to arbitrary strategies where the incentive constraint does not hold over a *finite* number of time intervals $[t_i, t_i + \varepsilon_i]$ with $\varepsilon_i > 0$, $t_{i+1} > t_i + \varepsilon_i$ and $0 < i \leq I < \infty$. One simply has to consider the last interval $[t_I, t_I + \varepsilon_I]$ and follow the logic of the proof to reach a contradiction.

Totally differentiating this expression with respect to $\Delta$ shows that it is concave in $\Delta$ and that it reaches its maximum when $\Delta = 0$. Then, differentiating $i(0, \varepsilon; t, v)$ with respect to the length $\varepsilon$ of the interval where the worker is perfectly insured yields

$$\begin{aligned} \frac{\partial i(0, \varepsilon; t, v)}{\partial \varepsilon} &= \frac{\partial}{\partial \varepsilon}\left(\frac{1}{\rho}\left[e^{-\rho\varepsilon} f(t+\varepsilon) + \frac{\ln(-\rho v)}{\theta}\right]\right) \\ &= \frac{1}{\rho}\left[e^{-\rho\varepsilon}\left(f'(t+\varepsilon) - \rho f(t+\varepsilon)\right)\right] = -\frac{e^{-\rho\varepsilon}}{\rho}\psi(t+\varepsilon) \ , \end{aligned}$$

where $\psi(t)$ is defined in $(44)$. Let us now consider the cases identified in the proposition.

(i) $F > 0$: The boundary condition $\lim_{t\to\infty} f'(t) = 0$ implies that $\lim_{h\to\infty} \psi(h) = \rho F$. Furthermore, it follows from $\lim_{h\to 0} k(h) = 0$ that $\lim_{h\to 0} \psi(h) = -\infty$. Hence $\psi(h)$ must switch sign at least once when $F$ is positive. However, there can only be one precision such that $\psi(h) = 0$ since we have shown in the proof of Corollary 2 that $\psi'(h) > 0$. We can therefore conclude from the equation above that $\partial i(0, \varepsilon; t, v)/\partial\varepsilon > 0$ when $h(t) < \tilde{h}$,, thus showing that it is optimal to insure workers. Conversely, when $h(t) > \tilde{h}$, $\partial i(0, \varepsilon; t, v)/\partial\varepsilon < 0$, showing that it cannot be optimal to insure workers within any time interval of positive finite measure. We still have to consider cases where $\varepsilon \to \infty$ so that workers are perfectly insured after a given date $t$. But this is clearly sub-optimal since $f(t) > \psi(t) > 0$ and so

$$\rho i(0, \infty; t, v) = \frac{\ln(-\rho v)}{\theta} < f(t) + \frac{\ln(-\rho v)}{\theta} = \rho j(t, v) \ .$$

(ii) $F \leq 0$ : Then $\psi(h) < \lim_{h\to 0} \psi(h) = F \leq 0$, implying that $\partial i(0, \varepsilon; t, v)/\partial\varepsilon > 0$ for all $t$. In other words, provision of full insurance always maximizes profits. ■

**Proof. Corollary 4:** The statement immediately follows from

$$\frac{1}{2} > \frac{dk(\sigma^{-2}/h_t)}{d(\sigma^{-2}/h_t)} = \left(\frac{1}{2}\right)\left[1 - \frac{\frac{1}{(\sigma\lambda\theta)^2} + \frac{\sigma^{-2}}{h_t}}{\sqrt{\left(\frac{1}{(\sigma\lambda\theta)^2} + \frac{\sigma^{-2}}{h_t}\right)^2 + \frac{4\rho}{(\sigma\lambda\theta)^2}}}\right] > 0 \ ,$$

and the solution for wages $w_t^*(v) = -\ln(k_t v)/\theta + \lambda$ . ■

**Proof. Corollary 5:** Reinserting the law of motion $(29)$ for $v$ into $(30)$ and applying Ito's lemma yields[37]

$$dw_t^* = -\left(\frac{1}{\theta}\right)\left[\left(\left(\frac{1}{k_t}\right)\frac{dk_t}{dt} - \frac{(\sigma\theta\lambda)^2}{2}\left(\left(\frac{\sigma^{-2}}{h_t}\right)^2 - k_t^2\right)\right)dt + \lambda\theta\left(k_t - \frac{\sigma^{-2}}{h_t}\right)\sigma dZ_t\right] .$$

[37] See the proof of Proposition 4 for the intermediate step linking the two equalities.

The statement for the volatility term is established reinserting $dk(\sigma^{-2}/h_t)/d\left(\sigma^{-2}/h_t\right)$ into

$$-\frac{d\left(k(t)-\frac{\sigma^{-2}}{h(t)}\right)}{dt}=-\left(\frac{dk(t)}{dt}-\frac{d\left(\sigma^{-2}/h\left(t\right)\right)}{dt}\right)=-\left(\underbrace{\frac{dk(\sigma^{-2}/h\left(t\right))}{d\left(\sigma^{-2}/h\left(t\right)\right)}}_{\in(0,1/2)}-1\right)\underbrace{\frac{d\left(\sigma^{-2}/h\left(t\right)\right)}{dt}}_{<0}<0\,.$$

The sign of the deterministic trend is established remembering that the sufficient condition $(16)$ holds if and only if $-k_t>\sigma^{-2}/h_t$. Hence, $\left(\sigma^{-2}/h_t\right)^2-k_t^2<0$, and so the trend is negative. ■

**Proof. Proposition 5:** We prove each point in turn:
**(a):** We shall establish that

$$\frac{\partial\mathcal{U}\left(t,\hat{\eta}_t\right)}{\partial a_t}\gtreqless 0\Longleftrightarrow R_t\gtreqless\lambda\exp\left(-\theta(1-\lambda)a_t^*\right)\ , \tag{47}$$

where

$$R_t\triangleq\int_t^{\infty}e^{-\rho(s-t)}\frac{\sigma^{-2}}{h_s}\exp\left(\frac{\theta^2}{2}\left(h_t^{-1}-h_s^{-1}\right)-\theta(1-\lambda)a_s^*\right)ds\,.$$

By definition, $\mathcal{U}\left(t,\hat{\eta}_t\right)$ when evaluated at time $t$ is given by

$$\mathcal{U}\left(t,\hat{\eta}_t\right)=\int_t^{+\infty}e^{-\rho(s-t)}E_t\left[U\left(\hat{\eta}_s,a_s^*\right)\right]ds\,. \tag{48}$$

In order to solve for it, we need to derive the expressions of the expectations terms. Equilibrium beliefs $\hat{\eta}$ satisfy the following law of motion $d\hat{\eta}_t=\left(\sigma^{-1}/h_t\right)dZ_t$. Hence $\hat{\eta}_s$ is normally distributed with mean $\hat{\eta}_t$ and variance $Var_t\left(\hat{\eta}_s\right)=E_t\left[\left(d\hat{\eta}_s\right)^2\right]=h_t^{-1}-h_s^{-1}$, which implies in turn that

$$\begin{aligned}E_t\left[U\left(\hat{\eta}_s,a_s^*\right)\right]&=-E_t\left[\exp(-\theta\hat{\eta}_s)\right]\exp\left(-\theta(1-\lambda)a_s^*\right)\\&=-\exp(-\theta\hat{\eta}_t)\exp\left(\frac{\theta^2}{2}\left(h_t^{-1}-h_s^{-1}\right)-\theta(1-\lambda)a_s^*\right)\,.\end{aligned}$$

Reinserting this expression into $(48)$ yields

$$\mathcal{U}\left(t,\hat{\eta}_t\right)=-\exp(-\theta\hat{\eta}_t))\left[\int_t^{\infty}e^{-\rho(s-t)}\exp\left(\frac{\theta^2}{2}\left(h_t^{-1}-h_s^{-1}\right)-\theta(1-\lambda)a_s^*\right)ds\right]\,. \tag{49}$$

Thus the FOC for current effort $a_t$ reads

$$\frac{\partial\mathcal{U}\left(t,\hat{\eta}_t\right)}{\partial a_t}=\frac{\partial U\left(\hat{\eta}_t,a_t\right)}{\partial a_t}+\int_t^{\infty}e^{-\rho(s-t)}\frac{\partial E_t\left[U\left(\hat{\eta}_s,a_s^*\right)\right]}{\partial a_t}ds\,.$$

Given that $\hat{\eta}_s = \left(h_0 m_0 + \sigma^{-2}(Y_s - A_s^*)\right)/h_s$, we have $\partial\hat{\eta}_s/\partial a_t = \left(\partial\hat{\eta}_s/\partial Y_s\right)\left(\partial Y_s/\partial a_t\right) = \sigma^{-2}/h_s$. Replacing this expression in the derivatives above yields

$$\begin{aligned}\frac{\partial\mathcal{U}(t,\hat{\eta}_t)}{\partial a_t} &= \frac{\partial U}{\partial a_t}(\hat{\eta}_t, a_t) - \int_t^\infty e^{-\rho(s-t)}\frac{\theta\sigma^{-2}}{h_t}E_t\left[U(\hat{\eta}_s, a_s)\right]ds \\ &= -\exp\left(-\theta\hat{\eta}_t\right)\left[\begin{array}{c}\theta\lambda\exp\left(-\theta(1-\lambda)a_t^*\right) \\ -\int_t^\infty e^{-\rho(s-t)}\frac{\theta\sigma^{-2}}{h_s}\exp\left(\frac{\theta^2}{2}\left(h_t^{-1}-h_s^{-1}\right)-\theta(1-\lambda)a_s^*\right)ds\end{array}\right].\end{aligned}$$

The deterministic nature of effort follows because the sign of $\partial\mathcal{U}(t,\hat{\eta}_t)/\partial a_t$ is independent of the equilibrium belief $\hat{\eta}_t$. More precisely, marginal returns to effort are positive whenever $(47)$ is positive.

**(b):** We now prove that there exists a precision $\bar{h}$ such that $a_t^* = 0$ if $h(t) \geq \bar{h}$. Let $R_t^0$ be defined as

$$R_t^0 \triangleq \int_t^\infty e^{-\rho(s-t)}\frac{\sigma^{-2}}{h_s}\exp\left(\frac{\theta^2}{2}\left(h_t^{-1} - h_s^{-1}\right)\right)ds\,,$$

so that

$$\frac{\partial\mathcal{U}(t,\hat{\eta}_t)}{\partial a_t} = -\theta\exp\left(-\theta\hat{\eta}_t\right)\left[\lambda\exp\left(-\theta(1-\lambda)a_t^*\right) - R_t^0\right] \text{ if } a_s^* = 0 \text{ for all } s > t. \quad (50)$$

We wish to establish that $R_t^0$ is a decreasing function of time. Differentiating its expression with respect to $t$ yields

$$\frac{dR_t^0}{dt} = R_t^0\left[\rho - \frac{1}{2}\left(\frac{\theta\sigma^{-1}}{h_t}\right)^2\right] - \frac{\sigma^{-2}}{h_t}\,. \quad (51)$$

When $\rho < \frac{1}{2}\left(\theta\sigma^{-1}/h_t\right)^2$, the derivative is obviously negative. To show that this is also true when $\rho > \frac{1}{2}\left(\theta\sigma^{-1}/h_t\right)^2$, we observe that

$$R_t^0 < \frac{\sigma^{-2}}{h_t}\int_t^\infty e^{-\rho(s-t)}\exp\left(\frac{\theta^2}{2}\left(h_t^{-1} - h_s^{-1}\right)\right)ds = \frac{\sigma^{-2}}{h_t}\left[\rho - \frac{1}{2}\left(\frac{\theta\sigma^{-1}}{h_t}\right)^2\right]^{-1},$$

whenever $\rho - \frac{1}{2}\left(\theta\sigma^{-1}/h_t\right)^2 > 0$. Reinserting this inequality into $(51)$ shows that $dR_t^0/dt < 0$ with $\lim_{t\to\infty} R_t^0 = 0$. Hence there exists a unique precision $\bar{h}$ where $R_t^0 \lesseqgtr \lambda\exp\left(-\theta(1-\lambda)\right)$ if $h(t) \gtreqless \bar{h}$. But then, the fact that $R_t \leq R_t^0$ for all possible equilibrium paths and $(47)$ imply in turn that $\partial\mathcal{U}(t,\hat{\eta}_t)/\partial a_t < 0$ for all $t$ such that $h(t) > \bar{h}$ and $a_t \in [0,1]$. In other words, recommended effort is set equal to its lower bound $a_t^* = 0$ whenever $h(t) \geq \bar{h}$. ■

# Appendix B: Additional results

**Derivation of (15):** We first change variable and define $\tilde{p}_t \triangleq (\sigma^{-2}/h_t)\, p_t$. Then $\tilde{p}_t = -E\int_t^T e^{-\rho(s-t)}\gamma_s \frac{\sigma^{-2}}{h_s} ds$, so that differentiating with respect to time leads to

$$\frac{d\tilde{p}_t}{dt} = \rho\tilde{p}_t + \frac{\sigma^{-2}}{h_t}\gamma_t = \rho\tilde{p}_t - \frac{\sigma^{-2}}{h_t}\left(U_a\left(w_t, a_t\right) + \tilde{p}_t\right),$$

where the second equality follows after substitution of $\gamma_t = -U_a\left(w_t, a\right) - \tilde{p}_t$. Integrating this expression, we obtain

$$\tilde{p}_t = E_a\left[\int_t^T e^{\left[-\rho(s-t)+\int_t^s \frac{\sigma^{-2}}{h_\tau} d\tau\right]}\frac{\sigma^{-2}}{h_s}U_a\left(w_s, a_s\right) ds\right].$$

To simplify the integral in the exponent, we observe that

$$\frac{\sigma^{-2}}{h_\tau} = \frac{\sigma^{-2}}{h_0 + \tau\sigma^{-2}} = \frac{d\ln h_t}{d\tau} \Longrightarrow \exp\left(\int_t^s \frac{\sigma^{-2}}{h_\tau} d\tau\right) = \exp\left(\ln h_s - \ln h_t\right) = \frac{h_s}{h_t}.$$

Therefore

$$\tilde{p}_t = E_a\left[\int_t^T e^{-\rho(s-t)}\left(\frac{h_s}{h_t}\right)\left(\frac{\sigma^{-2}}{h_s}\right)U_a\left(w_s, a_s\right) ds\right] = \frac{\sigma^{-2}}{h_t}E_a\left[\int_t^T e^{-\rho(s-t)}U_a\left(w_s, a_s\right) ds\right],$$

which, given the definition of $\tilde{p}_t$, is equivalent to (15). Observe, however, that when $a_t = 0$ for some $t$ then (13) is not representable as (15).

**Extending the HJB eq.**(19) **to include $\hat{\eta}$:** The HJB equations defined in (19) and can be extended to include $\hat{\eta}$ and would still be satisfied. To see this, define $X_t \triangleq Y_t - A_t$ and $g\left(X_t, t\right) \triangleq e^{-\rho t}\hat{\eta}(X_t, t)/\rho$. This function satisfies the HJB equations below because

$$\begin{aligned} e^{-\rho t}\hat{\eta}(X_t, t) + \frac{\partial g}{\partial t} + \hat{\eta}_X(X_t, t)\frac{\partial g}{\partial X_t} + \frac{\sigma_t^2}{2}\frac{\partial^2 g}{\partial X_t^2} &= e^{-\rho t}\left[\begin{array}{c} \hat{\eta}(X_t, t) - \hat{\eta}(X_t, t) \\ +\frac{1}{\rho}\hat{\eta}_t(X_t, t) + \frac{1}{\rho}\hat{\eta}_{X_t}(X_t, t)\hat{\eta}(X_t, t) \end{array}\right] \\ &= \left(\frac{e^{-\rho t}}{\rho}\right)\left[\hat{\eta}_t(X_t, t) + \hat{\eta}_{X_t}(X_t, t)\hat{\eta}(X_t, t)\right] = 0, \end{aligned}$$

where the last equality follows from (7).