Risk Sharing with (Dis)Aggregate Shocks

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Abstract

In this paper we examine conditions under which optimal risk sharing may not fully insure individuals against idiosyncratic shocks to their endowments or income, even when markets are complete. We analyze the benchmark risk-sharing model, but allow for the possibility that idiosyncratic shocks can induce fluctuations in aggregate resources, affecting, in this way, the Pareto optimal allocation of consumption. In particular, we show that idiosyncratic shocks affect consumption under optimal risk-sharing, when endowments are drawn from a class of power law distributions or when output is produced through input supply networks with star-shape or scale-free architecture. Under these conditions, we document two important features of optimal risk sharing. First, the effect of idiosyncratic risk on consumption growth is heterogeneous, depending on each agent's contribution to the aggregate resource, and, second, risk sharing involves an exposure to systemic risk that is composite of the undiversified components of idiosyncratic shocks. Additionally, we show that the frequently used empirical tests of risk-sharing which ignore the distributional aspects of income and the network structure of production, may suffer from a specification bias when idiosyncratic shocks do not dissipate in the aggregate. We offer empirical evidence to show that individual consumption growth is affected by a systemic component of risk, such as, for example, the shocks to the top percentiles of the income distribution. Our results have important implications for empirical studies of risk-sharing carried out at the individual, household and national level, irrespective of the size of the reference group, be it the household, the village, or the country.

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1 Introduction

How effective are markets and institutions in *fully* insuring agents against idiosyncratic risk? Under complete markets, individual consumption should respond only to aggregate risk and not to idiosyncratic risk. This is the principal implication of full risk-sharing. The complete markets case, therefore, provides a useful benchmark in assessing whether market imperfections play any role in explaining consumption allocations (Mace 1991). Empirical tests based on the key implications of risk-sharing (Mace 1991; Cochrane 1991; Townsend 1994) have relied heavily on this benchmark, so that an empirical rejection of risk-sharing is taken to indicate the presence of market imperfections and the absence of optimal risk sharing arrangements. In this paper we revisit this issue and show that, under certain conditions, optimal risk sharing and full risk sharing are not equivalent, so that a rejection of full risk sharing in the data may not indicate an absence of complete markets or optimal risk-sharing.

We are primarily interested in evaluating one central aspect of the benchmark risk-sharing model: the (asymptotic) restriction that *aggregate* idiosyncratic risk approaches zero as the size of the risk sharing group tends to infinity. However, such an approximation property, essentially implied by the law of large numbers, need not always hold even when idiosyncratic shocks are exogenous and independent over agents. A recent literature on the microeconomic origins of aggregate fluctuations has shown that in many cases aggregate shocks originate from idiosyncratic shocks and that, in fact, idiosyncratic shocks do not necessarily die out in the aggregate. It is this feature that appeals to us as an alternative to 'imperfections' as an explanation for the failure of full risk sharing. The absence of a prior restriction on the aggregate behaviour of idiosyncratic shocks may produce a fundamental *identification problem* such that the effect of an idiosyncratic shock on consumption cannot be separately recovered empirically. The objective of our research is, therefore, to characterize conditions under which the optimal risk sharing rule *may not fully insure* its agents against idiosyncratic shocks to their endowments or income and derive empirical implications for the same.

Our work is motivated by two observations. The first is that the share of income accruing to the top percentiles of the income distribution is disproportionately large. In fact, as pointed out by Alvaredo et al. (2013), "most of the action has been at the very top". In the United States alone, the authors estimate that the incomes of the top 1% earners account for approximately 22% of its total income. A consequence of this feature is that the pool of aggregate income is extremely fragile to the perturbations at the extremes so that the combined volatility of aggregate income is proportional to the volatility of the top share holders. Our second observation is that income production processes are typically interlinked, such that producers are affected by the integrated risk of input providers. Diamond (1967) and Newbery (1977) were among the first to incorporate the notion of multiplicative production disturbances in models of risk sharing. In such cases, it is easy to see that the risk borne by the most central supplier in a highly centralized production network is non-negligible.

Both these insights have been developed and brought to attention, most recently, in the macroeconomic context of firms and sectors by the works of Gabaix (2011) and Acemoglu et al. (2012).

Gabaix (2011) shows that the distribution of firm sizes is fat-tailed in the United States and demonstrates empirically that a significant part of aggregate fluctuations can be explained by shocks to large firms which account for a disproportional share of total output. By examining the input-output structure of the US economy, Acemoglu et al. (2012) offer a supplementary reason as to why idiosyncratic shocks might not average out in the aggregate according to the law of large numbers. More specifically, they show that inter-sectoral linkages may propagate sector, or firm, specific shocks when, for example, there exist general purpose inputs that are used by many other sectors or when general purpose inputs are provided by a few major suppliers. Early work by Jovanovic (1987) has also found that, in specific contexts, strong strategic complementarities between agents can lead to the failure of the diversification of idiosyncratic shocks in the aggregate. The contribution of sectoral shocks to output volatility has also been emphasized by Horvath (1998).

One immediate implication of the fact that the volatility of aggregate income is affected by the volatility of the top share holders is that the aggregate resource portfolio is subject to diversification constraints, even as the number of its members grow. Our model is, therefore, based on the simple idea that when insurance is provided, consumption volatility, which is driven by the volatility of the aggregate resource (traditionally an aggregate shock) is also as a result, exposed to the volatility of the top share holders. We show that consumption growth depends on a systemic, undiversifiable, component of risk, that is equal to the weighted sum of each agents' volatility. The weights represent the share of aggregate income contributed by each agent. Consider, for example, an agent holding 10% of total income. This corresponds to her weight in the income portfolio. An idiosyncratic shock to this agent affects every other agent's consumption growth including her own by a proportion 0.1, in-spite of optimal insurance, precisely because the planner is unable to diversify away this proportion.

Clearly, the empirical relevance of the risk-sharing model hinges crucially on the properties of the income portfolio. Extreme level of skewness in income shares are, for instance, characteristic of power law distributions with exponents less than two (infinite variance). In comparison, the maximum income share obtained from a Gaussian income distribution, with a finite mean and variance, are typically much smaller in magnitude, ranging between 0.02-0.001. This illustrates two aspects of our analysis. First, risk-sharing models that do not accommodate the role of idiosyncratic shocks in affecting the aggregate are compatible with the latter characterization of income shares rather than the former. In practice however, income distributions are often found to resemble the former, exhibiting fat tails with power-law parameters ranging between 1.4-1.8 (Atkinson et al. 2011; Feenberg and Poterba 1993; Levy and Solomon 1997; Mandelbrot 1960). Second we show that there is an important distinction between full insurance and optimal insurance. Key features of this result are that an idiosyncratic income shock can affect consumption growth, heterogeneously, even when risk is optimally shared, and that optimal risk sharing involves an exposure to systemic risk.

In examining this issue, we combine two strands of literature. The first strand, which we have discussed above, is primarily of macroeconomic relevance and analyzes the nature and composition of aggregate fluctuations. The second develops testable implications of risk sharing based on theoretical models that do not allow idiosyncratic shocks to play any role in the aggregate in the presence of complete markets or other mechanisms that ensure a full-information Pareto-optimal allocation[1]. These models account separately for economy-wide aggregate uncertainty, but maintain that individual idiosyncratic shocks cancel out in the aggregate. Based on this, a large empirical literature has developed and tested the key theoretical implication of the risk sharing model across individuals, households, or even countries. The empirical design usually involves regressing a measure of idiosyncratic shock against the change (or growth) in individual consumption and testing whether the coefficient of the shock is equal to zero.

The first contribution of our paper is to examine the implications of the failure of large numbers and the diversification argument on optimal risk-sharing. We consider the social planner's problem of optimally allocating consumption in a series of models with complete markets under various assumptions regarding the cross-sectional distribution of endowments and the production network structure of the economy. First, we consider the case of endowments that are distributed uniformly or with a finite variance across individuals in the economy. Our results are in agreement with the ones obtained in the classic studies of risk-sharing, where idiosyncratic shocks are not expected to induce changes in individual consumption under optimal consumption insurance.

Next, we replace our assumption of narrow cross-sectional differences in the distribution of endowments with the more factual assumption of endowments distributed as a power law. Under the power law distribution assumption, we find that the law of large numbers may fail and idiosyncratic shocks may affect aggregate consumption. Depending on the type of the power-law distribution under consideration, we show that the partial effect of an idiosyncratic shock to the largest endowments in the economy on the optimal consumption of any individual may decay at a rate that is much slower than the one implied by the law of large numbers.

Subsequently, we relax the context of an endowment-based economy to consider a production economy where the social planner allocates efficiently both consumption and production ensuring that both income and consumption risk are jointly insured. We employ various assumptions regarding the network asymmetries between different sectors in the input-output structure of the economy. We find that idiosyncratic shocks to central sectors may lead to fluctuations in aggregate consumption affecting in this way optimal consumption allocations. This is the case for certain types of scale-free networks and star-type networks, where some sectors are highly central, in that that they are major suppliers of the input used by other sectors.

Our second contribution is to recast the optimal risk sharing solution into a tractable empirical form that can be taken to the data. We show, first, that a basic measurement error issue emerges when idiosyncratic shocks cannot be fully diversified away. In such cases, the total idiosyncratic shock acts as a noisy measure of potentially insurable risk as it contains components of both insur-

[1] The solution to the risk-sharing problem is derived from a model, usually cast in the setting of a social planner who maximizes a weighted sum of individual utilities. As emphasized in the early work by Wilson (1968) and Diamond (1967), optimal resource allocation is achieved by pooling all individual endowments to obtain a distribution of the aggregate endowment that equalizes the weighted marginal utilities across individuals.

able and uninsurable risk. This results in a misspecified model that attenuates the estimates for insurance. We assess the extent of the specification bias in finite samples through a monte carlo study. The magnitude of bias depends on the choice of the utility function; we find that, while, the CRRA based specification performs well for any given value of the power-law parameter, the CARA specification performs quite poorly, yielding extremely biased results for skewed distributions (i.e., with power law parameters less than two) even over fairly large (risk-sharing) group sizes.

Finally, we provide empirical evidence to show that the consumption growth of any individual is exposed to a systemic component of risk under optimal risk sharing, i.e., it is partially affected by the weighted idiosyncratic shocks of other individuals in the risk sharing group. Using data on household income (PSID) and US states' output (BEA) we estimate the household and regional risk sharing models respectively. We decompose the aggregate component of the risk sharing model to isolate the idiosyncratic shocks attributable to the the top share-holders of the distribution. This variable, termed as the granular residual by Gabaix (2011), is based on both the aggregate income growth of top earners as well as the aggregate income growth of individuals employed in the most central sector of the economy (manufacturing). In both cases, we find that the residual explains a significant fraction of the individuals' consumption growth; a one standard deviation increase in the idiosyncratic shocks of the very top, lead to a 0.02-0.01 standard deviation increase in consumption growth.

This paper is organized as follows. In Section 2 we consider the variants of the endowment and production economies that we discussed in the introduction. Section 3 discusses the empirical implications of our theoretical findings. Section 4 describes and tests the principal implications of our analysis and Section 5 concludes.

2 Risk sharing model

Consider an economy with N infinitely lived agents and a single consumption good. Let $c_{it}\left(s_{\tau t}\right)$ be the consumption of agent $i \in \{1, ..., N\}$ in the state $s_{\tau t}$ of the economy, where $\tau \in \{1, ..., S\}$ indexes the event at date t. The agents share common information over the state variable $s_{\tau t}$ which occurs with probability $\pi\left(s_{\tau t}\right) \in [0,\, 1]$ with $\Sigma_{\tau=1}^{S}\pi\left(s_{\tau t}\right) = 1$, $\forall t \in \{0, ..., \infty\}$. The expected lifetime utility function of agent i is:

$$\sum_{t=0}^{\infty} \left(\rho_i\right)^t \sum_{\tau=1}^{S} \pi(s_{\tau t})\, u\left[c_{it}\left(s_{\tau t}\right),\, b_{it}\left(s_{\tau t}\right)\right], \tag{2.1}$$

where $u\left(\cdot\right)$ represents the time-separable utility function for the consumption good, $b_{it}\left(s_{\tau t}\right)$ denotes a preference shock and $\rho_i \in (0,\, 1)$ is the rate of time preference.

Each agent is endowed with $y_{it}(s_{\tau t})$ units of the consumption good. The endowment y_{it} is exogenously given[2] and follows a stochastic growth process[3]:

$$\frac{\Delta y_{it+1}}{y_{it}} = \frac{y_{it+1} - y_{it}}{y_{it}} = \sigma_i \varepsilon_{it+1}, \tag{2.2}$$

where σ_i is the variance of the shocks to agent's i endowment and ε_{it+1} is a random variable with mean value $\mathbb{E}\left(\varepsilon_{it+1}\right) = 0$ and variance $Var\left(\varepsilon_{it+1}\right) = 1$. The aggregate endowment of the consumption good at date t is given by:

$$Y_t(s_{\tau t}) = \sum_{i=1}^{N} y_{it}(s_{\tau t}), \tag{2.3}$$

Combining (2.2) and (2.3), we can express the growth rate of the aggregate endowment of the consumption good as:

$$\Delta \ln\left(Y_{t+1}\right) \simeq \frac{\Delta Y_{t+1}}{Y_t} = \frac{\sum_{i=1}^{N} \Delta y_{it+1}}{Y_t} = \frac{\sum_{i=1}^{N}\left(y_{it}\sigma_i\varepsilon_{it+1}\right)}{Y_t}. \tag{2.4}$$

The social planner has the objective of allocating the consumption of the endowments available in the economy in order to maximize the weighted sum of the expected utilities of the N agents provided by:

$$\sum_{i=1}^{N} \lambda_i \sum_{t=0}^{\infty} \left(\rho_i\right)^t \sum_{\tau=1}^{S} \pi(s_{\tau t})\, u\left[c_{it}\left(s_{\tau t}\right), b_{it}\left(s_{\tau t}\right)\right], \tag{2.5}$$

where $\lambda_i \in (0,\, 1)$ is agent i's Pareto weight in the planner's utility function that satisfies $\Sigma_{i=1}^{N}\lambda_i = 1$.

The resource constraint in the social planner's problem is represented in (2.6), where the aggregate consumption of the agents cannot exceed the level of the aggregate endowment:

$$\sum_{i=1}^{N} c_{it}\left(s_{\tau t}\right) \leq Y_t\left(s_{\tau t}\right). \tag{2.6}$$

The first order conditions for the consumption of agent i yield:

$$\lambda_i \left(\rho_i\right)^t \pi(s_{\tau t}) u_C\left[c_{it}\left(s_{\tau t}\right), b_{it}\left(s_{\tau t}\right)\right] = \mu_t\left(s_{\tau t}\right), \tag{2.7}$$

for all $s_{\tau t}$, where μ_t is the Lagrange multiplier associated with the resource constraint (2.6) and $u_C\left(\cdot\right)$ denotes the partial derivative of the agent's utility function with respect to consumption. Denote $\kappa_t\left(s_{\tau t}\right) \equiv \frac{\mu_t(s_{\tau t})}{\pi(s_{\tau t})}$ and suppress the state notation from now on, so that $c_{it} \equiv c_{it}\left(s_{\tau t}\right)$, $b_{it} \equiv b_{it}\left(s_{\tau t}\right)$ and $\kappa_t \equiv \kappa_t\left(s_{\tau t}\right)$. The first order conditions in (2.7) can be rewritten as:

$$\lambda_i \left(\rho_i\right)^t u_C\left(c_{it},\, b_{it}\right) = \kappa_t. \tag{2.8}$$

[2]This assumption is modified in Section 2.4 which introduces an input supply network where the social planner is allowed to optimally allocate both production and consumption across the agents in the economy.

[3]The stochastic growth specification for the income process in widely adopted in the literature on income dynamics and consumption smoothing. See for example Banks et al. (2001) & MaCurdy (1982)

Suppose that preferences are represented by the following Constant Relative Risk Aversion (CRRA) or power utility[4] function:

$$u\left(c_{it},\, b_{it}\right) = b_{it}\frac{\left(c_{it}\right)^{1-\gamma} - 1}{1-\gamma}, \tag{2.9}$$

where $\gamma > 0$ is the coefficient of relative risk aversion[5].

Substituting the partial derivative of the CRRA utility function with respect to consumption into the first order condition (2.8) and solving for c_{it} we get:

$$c_{it} = \left(\frac{b_{it}\lambda_i\left(\rho_i\right)^t}{\kappa_t}\right)^{\frac{1}{\gamma}}. \tag{2.10}$$

Aggregating over the N agents of the economy, we have:

$$\begin{aligned} \sum_{i=1}^{N} c_{it} &= \sum_{i=1}^{N}\left(\frac{b_{it}\lambda_i\left(\rho_i\right)^t}{\kappa_t}\right)^{\frac{1}{\gamma}} \\ \Rightarrow Y_t &= \left(\kappa_t\right)^{-\frac{1}{\gamma}}\sum_{i=1}^{N}\left(b_{it}\lambda_i\left(\rho_i\right)^t\right)^{\frac{1}{\gamma}}, \end{aligned} \tag{2.11}$$

where the last equation follows from substituting the resource constraint (2.6).

The last equality can be used to obtain an expression for $\ln\left(\kappa_t\right)$:

$$\ln\left(\kappa_t\right) = -\gamma\ln\left(Y_t\right) + \gamma\ln\left[\sum_{i=1}^{N}\left(b_{it}\lambda_i\left(\rho_i\right)^t\right)^{\frac{1}{\gamma}}\right]. \tag{2.12}$$

Denote $\Theta_{it} \equiv b_{it}\lambda_i\left(\rho_i\right)^t$ and use equations (7.25) and (2.12) to obtain the following expression for the logarithmic growth of optimal individual consumption:

$$\begin{aligned} \Delta\ln\left(c_{it+1}\right) &= -\frac{1}{\gamma}\Delta\left(\ln\kappa_{t+1}\right) + \frac{1}{\gamma}\Delta\ln\left(\Theta_{it+1}\right) \\ &= \Delta\ln\left(Y_{t+1}\right) - \Delta\ln\left[\sum_{i=1}^{N}\left(\Theta_{it+1}\right)^{\frac{1}{\gamma}}\right] + \frac{1}{\gamma}\Delta\ln\left(b_{it+1}\right) + \frac{1}{\gamma}\ln\left(\rho_i\right). \end{aligned} \tag{2.13}$$

The logarithmic growth of total endowments in the expression above can be calculated based

[4]The power utility assumption is maintained throughout the main body of the paper and is relaxed in the appendix where we examine other utility functions that belong to the Hyperbolic Absolute Risk Aversion (HARA) class, as well as to the Constant Absolute Risk Aversion (CARA) class. Our main results remain robust to considering alternative types of utility functions.

[5]In Appendix (7.3) we relax this assumption and consider the case where agent have heterogenous risk preferences. We find that the asymptotic predictions are similar. The key difference is that the social planner will allocate consumption according to individual preferences for risk. Proportional allocation is able to provide some insurance against aggregate shocks. However, when aggregate shocks are themselves affected by idiosyncratic shocks, the implication is that that consumption growth is affected less by the idiosyncratic shocks of risk averse agents and more by the idiosyncratic shocks of risk loving agents.

on equation (2.4) as follows:

$$\Delta \ln (Y_{t+1}) \simeq \frac{\Delta Y_{t+1}}{Y_t} = \frac{\sum_{i=1}^{N} (\Delta y_{it+1})}{Y_t} = \frac{\sum_{i=1}^{N} (y_{it}\sigma_i\varepsilon_{it+1})}{Y_t}. \tag{2.14}$$

Substituting (2.14) into (2.13), we get:

$$\Delta \ln (c_{it+1}) = \frac{\sum_{i=1}^{N} (y_{it}\sigma_i\varepsilon_{it+1})}{Y_t} - \Delta \ln \left[\sum_{i=1}^{N} (\Theta_{it+1})^{\frac{1}{\gamma}}\right] + \frac{1}{\gamma}\Delta \ln (b_{it+1}) + \frac{1}{\gamma} \ln (\rho_i). \tag{2.15}$$

We demonstrate in what follows that idiosyncratic shocks dissipate in the aggregate, for large enough group sizes, when the size of endowments is distributed uniformly across agents or with a finite variance. However, when endowments are distributed with fat tails, idiosyncratic shocks affect the aggregate pool of resources and, as a result, the consumption growth of each individual. In the last subsection, we consider the case of a production economy, where the input supply network is allowed to be balanced, star-like and scale free, and we provide conditions under which the network structure matters for the lack of diversification of idiosyncratic shocks and, therefore, for optimal risk sharing.

2.1 Identical-sized endowments

Assuming that $\overline{y}_t = y_{it} = (1/N)Y_t$ for every agent i, then the logarithmic growth of total endowments can be written as:

$$\Delta \ln (c_{it+1}) = \frac{\overline{y}_t}{N\overline{y}_t} \sum_{i=1}^{N} (\sigma_i\varepsilon_{it+1}) - \Delta \ln \left[\sum_{i=1}^{N} (\Theta_{it+1})^{\frac{1}{\gamma}}\right] + \frac{1}{\gamma}\Delta \ln (b_{it+1}) + \frac{1}{\gamma} \ln (\rho_i). \tag{2.16}$$

The partial derivative of the changes in the optimal consumption of individual i with respect to the j-th agent's idiosyncratic endowment shock, ε_{jt+1}, is equal to:

$$\frac{\partial (\Delta \ln (c_{it+1}))}{\partial \varepsilon_{jt+1}} = \frac{\sigma_j}{N}.$$

The partial effect of ε_{jt+1} becomes 0 as $N \to \infty$.

2.2 Endowments distributed with finite variance

If endowments are distributed with a finite variance, then $N^{-1} \sum_{i=1}^{N} (y_{it}) \xrightarrow{a.s.} \mathbb{E}(y_t)$, where $\mathbb{E}(y_t)$ is a finite number. Thus, we can express total endowments Y_t as follows:

$$Y_t = \sum_{i=1}^{N} (y_{it}) = N\mathbb{E}(y_t).$$

The partial derivative of equation (2.15) with respect to the idiosyncratic endowment shock

ε_{jt+1} can, then, be written as:

$$\frac{\partial\left(\Delta \ln\left(c_{it+1}\right)\right)}{\partial \varepsilon_{jt+1}} = \frac{\sigma_j y_{jt}}{Y_t} = \frac{\sigma_j y_{jt}}{N\mathbb{E}\left(y_t\right)}.$$

Under the finite variance assumption, the term $\frac{\sigma_j y_{jt}}{\mathbb{E}(y_t)}$ remains a positive real number and the partial effect of ε_{jt+1} becomes 0 as $N \to \infty$.

Combining the results of Sections 2.1 and 2.2, we derive the following proposition:

Proposition 1. *Assuming CRRA preferences, the partial effect of an idiosyncratic shock to the j-th agent's endowment on the optimal consumption of the i-th individual becomes 0 as $N \to \infty$ when: (i) endowments are distributed uniformly across agents or (ii) endowments are distributed with a finite variance across agents.*

2.3 Endowments following a power law distribution

Denote $\mathbb{P}\left(\cdot\right)$ the frequency distribution of endowment sizes in the economy. The size for endowments is assumed in this section to follow a power law distribution according to which:

$$\mathbb{P}\left(y\right) = cy^{-\zeta},$$

where $c > 0$ is a normalizing constant, $y > c^{1/\zeta}$, and $\zeta \geq 1$ denotes the exponent of the power law distribution of endowments. Following Gabaix (2012: p.742), we normalize the constant so that $c = 1$ and we note that the random variable $y^{-\zeta}$ is uniformly distributed.[6] We show in Appendix (6.3) that, the j-th largest endowment in the population of N agents is approximately equal to:

$$y_{jt} = \left(\frac{j}{N}\right)^{-\frac{1}{\zeta}}. \tag{2.17}$$

The size of aggregate endowments as the size of the economy grows large (i.e., $N \to \infty$) will depend on the value of parameter ζ.

- if $\zeta > 1$, then the mean value of the endowment size takes a finite value, as the size of the economy grows large (i.e., $N^{-1}\sum_{i=1}^{N}\left(y_{it}\right) \xrightarrow{a.s.} \mathbb{E}\left(y_t\right)$). Thus, for large N, total endowments Y_t can be expressed as $Y_t = \sum_{i=1}^{N}\left(y_{it}\right) = N\mathbb{E}\left(y_t\right)$ and the logarithmic growth of total endowments can be written as:

$$\Delta \ln\left(c_{it+1}\right) = \frac{1}{N\mathbb{E}\left(y_t\right)}\sum_{i=1}^{N}\left(\left(\frac{i}{N}\right)^{-\frac{1}{\zeta}}\sigma_i\varepsilon_{it+1}\right) - \Delta \ln\left[\sum_{i=1}^{N}\left(\Theta_{it+1}\right)^{\frac{1}{\gamma}}\right] + \frac{1}{\gamma}\Delta \ln\left(b_{it+1}\right) + \frac{1}{\gamma}\ln\left(\rho_i\right). \tag{2.18}$$

[6] Heuristically, note that if y follows a power law distribution and $c = 1$, then $\mathbb{P}\left(y > x\right) = x^{-\zeta}$. Thus, $\mathbb{P}\left(y^{-\zeta} > x\right) = \mathbb{P}\left(y > x^{-\frac{1}{\zeta}}\right) = \left(x^{-\frac{1}{\zeta}}\right) = x$.

The partial derivative of equation (2.18) with respect to the idiosyncratic endowment shock ε_{jt+1} is, then, equal to:

$$\frac{\partial\left(\Delta \ln\left(c_{it+1}\right)\right)}{\partial \varepsilon_{jt+1}} = \frac{1}{N^{\frac{\zeta-1}{\zeta}}} \frac{\sigma_j}{j^{\frac{1}{\zeta}} \mathbb{E}\left(y_t\right)}.$$

Given that $\zeta > 1$ and that $\frac{\sigma_j}{j^{\frac{1}{\zeta}} \mathbb{E}(y_t)}$ remains a positive finite value, then the partial effect of ε_{jt+1} becomes 0 as $N \rightarrow \infty$.

- if $\zeta = 1$, then $y_{jt} = \left(\frac{N}{j}\right)$ and the expected value of endowments (for a large N) is equal to $\mathbb{E}\left(y_t\right) = \int_1^N y f\left(y\right) dy = \int_1^N y y^{-2} dy = \ln N$, where $f\left(y\right) = y^{-2}$ is the probability density function of the distribution of endowment sizes. Therefore, the aggregate endowment is equal to $Y_t = N\mathbb{E}\left(y_t\right) = N \ln N$ and the logarithmic growth of total endowments can be expressed as:

$$\Delta \ln\left(c_{it+1}\right) = \frac{\sum_{i=1}^N \left(\frac{N}{i} \sigma_i \varepsilon_{it+1}\right)}{N \ln N} - \Delta \ln\left[\sum_{i=1}^N \left(\Theta_{it+1}\right)^{\frac{1}{\gamma}}\right] + \frac{1}{\gamma} \Delta \ln\left(b_{it+1}\right) + \frac{1}{\gamma} \ln\left(\rho_i\right). \quad (2.19)$$

The partial derivative of equation (2.19) with respect to the idiosyncratic endowment shock ε_{jt+1} is equal to:

$$\frac{\partial\left(\Delta \ln\left(c_{it+1}\right)\right)}{\partial \varepsilon_{jt+1}} = \frac{\sigma_j}{j \ln N}. \quad (2.20)$$

Equation (2.20) leads to the following proposition:

Proposition 2. *Assuming CRRA preferences, the partial effect of an idiosyncratic shock to the j-th largest endowment on the optimal consumption of the i-th individual decays according to:*

(i) $\frac{1}{j \ln N}$ *when the size of endowments follows a power-law distribution with* $\zeta = 1$ *(i.e., a Zipf distribution) and*

(ii) $\frac{1}{j^{\frac{1}{\zeta}} N^{\frac{\zeta-1}{\zeta}}}$ *when the size of endowments follows a power-law distribution with* $\zeta > 1$.

In contrast to Proposition (1), Proposition (2) suggests that idiosyncratic shocks may matter for changes in optimal consumption. This result is also to be contrasted with the theoretical and empirical literature on risk sharing which does not consider the possibility that idiosyncratic shocks may not cancel out in the aggregate.

2.4 Networked production economy

In this section, we adopt the assumption that each agent is an entrepreneur that produces an intermediate product x_{it} which can be either used as an input in the production of the aggregate consumption good Y_t, or used in the production of the intermediate products of the other agents in the economy. More specifically, it is assumed that entrepreneurs use a Cobb-Douglas production technology to produce their intermediate output:

$$x_{it} = z_{it}^{\alpha} l_{it}^{\alpha} \prod_{j=1}^{N} x_{ijt}^{(1-\alpha) w_{ijt}}, \quad \forall\, i \in \{1, \ldots, N\} \tag{2.21}$$

where z_{it} denotes an idiosyncratic productivity shock to i's production at time t, l_{it} is the amount of labor used, $\alpha \in (0,\, 1)$ is the relative share of labor in the production of x_{it}, and w_{ijt} is the relative share of input j in the total mix of intermediate inputs for the production of the $i-$th good.

The intermediate sectors in the economy are related through an exogenous input-output relationship represented in the matrix $W_t = [w_{ijt}]$, according to which the output of agent j is used as an input in the production of the intermediate good x_i of agent i when $w_{ijt} > 0$.

Productivity shocks z_i are assumed to be distributed independently across entrepreneurs and to follow a stochastic growth process:

$$\Delta \left(\ln \left(z_{it+1}\right)\right) \backsimeq \frac{\Delta z_{it+1}}{z_{it}} = \sigma_i \varepsilon_{it+1}, \tag{2.22}$$

where $\sigma_i \in (\underline{\sigma},\, \overline{\sigma})$ denotes the variance of the shocks to entrepreneur's i productivity (with $0 < \underline{\sigma} < \overline{\sigma}$) and ε_{it+1} is a random variable with mean value $\mathbb{E}\left(\varepsilon_{it+1}\right) = 0$ and variance $Var\left(\varepsilon_{it+1}\right) = 1$.

Additionally, we assume constant returns to scale in the production function of each agent i so that the input shares of all agents sum up to 1, i.e., $\sum_{j=1}^{N} w_{ijt} = 1$, in every time period t. Following Acemoglu et al. (2012), we define the *influence vector* of the economy as the vector υ such that for every i:

$$\upsilon_{it} \equiv \frac{\alpha}{N} + (1-\alpha) \sum_{j=1}^{N} \upsilon_{jt} w_{jit}. \tag{2.23}$$

The following feasibility constraint holds for every entrepreneur i:

$$y_{it} + \sum_{j=1}^{N} x_{jit} \leq x_{it}, \quad \forall\, i \in \{1, \ldots, N\}$$

where y_{it} is the amount of the intermediate good i that is used as an input in the production of the aggregate consumption good Y_t. The aggregate consumption good in the economy is also produced using a Cobb-Douglas production technology:

$$Y_t = A_t \prod_{i=1}^{N} \left(y_{it}\right)^{\frac{1}{N}}, \tag{2.24}$$

where A_t is an aggregate productivity shock and y_{it} is the intermediate input for the production of Y_t that is provided by agent i. For simplicity, we assume that all agents participate equally in the production of the aggregate consumption good.

Regarding the total amount of labor L available in the economy, we assume that every agent is endowed with an equal amount of labor units l_i^e, which are supplied inelastically[7] and which satisfy the following equality:

$$\sum_{i=1}^{N} l_{it}^e = L_t.$$

The social planner's problem in this section's production economy is (i) to optimally allocate the aggregate consumption good among the agents in the economy and (ii) to optimally choose the intermediate production level x_{it}, the quantity of labor l_{it}, the quantity of intermediate inputs x_{ijt} supplied to every agent j and the quantity of intermediate input y_{it} supplied to the production of the aggregate consumption good Y_t. The social planner can achieve these objectives in a two-stage process, by first choosing c_{it} for every agent i at each point in time t to maximize the weighted sum of the expected utilities of the N agents provided by (2.5). In the second stage, the social planner maximizes the aggregate consumption good Y_t via the optimal choice of inputs and production for every entrepreneur i. The solution to the first stage optimization problem of the planner was obtained in the introduction of section (2) where we derived the optimal allocation of consumption for a given level of the aggregate consumption good. In this section, we obtain the planner's optimal conditions for maximizing the aggregate consumption good. In mathematical terms, the second stage objective[8] of the planner at each point in time t is:

$$\max_{\{Y_t, y_{it}, x_{ijt}, x_{it}, l_{it}\}} \ln(Y_t)$$

subject to the following technological and resource allocation constraints:

$$Y_t = A_t \prod_{i=1}^{N} (y_{it})^{\frac{1}{N}} \tag{2.25}$$

$$y_{it} + \sum_{j=1}^{N} x_{jit} \leq z_{it}^{\alpha} l_{it}^{\alpha} \prod_{j=1}^{N} x_{ijt}^{(1-\alpha) w_{ijt}}, \quad \forall\, i \in \{1, \ldots, N\} \tag{2.26}$$

$$\sum_{i=1}^{N} l_{it} \leq L_t. \tag{2.27}$$

Let μ_{it}^y and μ_t^L denote the Lagrange multipliers of the constraints provided in (2.26) and (2.27) respectively. Substituting the technological constraint (2.25) into the objective function and taking the first order conditions of the planner's problem with respect to y_{it}, x_{ijt} and l_{it}, we obtain the following equalities:

$$y_{it} = \frac{1}{N \mu_{it}^y} \tag{2.28}$$

[7]The disutility of labor is ignored in the analysis of the social planner's problem under the assumption that the preferences of agents are additively separable between leisure and consumption.

[8]For analytical convenience, the objective of the planner is expressed in terms of the logarithm of the aggregate consumption good, $\ln(Y_t)$, where $Y_t = A_t \prod_{i=1}^{N} (y_{it})^{\frac{1}{N}}$.

$$x_{ijt} = (1-\alpha)\, w_{ijt} x_{it} \frac{\mu_{it}^y}{\mu_{jt}^y} \tag{2.29}$$

$$l_{it} = \alpha x_{it} \frac{\mu_{it}^y}{\mu_t^L}. \tag{2.30}$$

Substituting (2.29) and (2.30) into the logarithmic transformation of agent i's production function (2.21), we have:

$$\ln(x_{it}) = \alpha \ln(z_{it}) + \alpha \ln(l_{it}) + (1-\alpha) \sum_{j=1}^{N} w_{ijt} \ln(x_{ijt})$$

$$\Rightarrow \ln(x_{it}) = \alpha \ln(z_{it}) + \alpha \ln\left(\alpha x_{it} \frac{\mu_{it}^y}{\mu_t^L}\right) + (1-\alpha) \sum_{j=1}^{N} w_{ijt} \ln\left[(1-\alpha)\, w_{ijt} x_{it} \frac{\mu_{it}^y}{\mu_{jt}^y}\right] \tag{2.31}$$

Rearrange (2.31) to obtain:

$$\ln(\mu_{it}^y) = -\alpha \ln(z_{it}) + \alpha \ln\left(\mu_t^L\right) - B - (1-\alpha) \sum_{j=1}^{N} w_{ijt} \ln(w_{ijt}) + (1-\alpha) \sum_{j=1}^{N} w_{ijt} \ln\left(\mu_{jt}^y\right), \tag{2.32}$$

where $B \equiv \alpha \ln(\alpha) + (1-\alpha) \ln(1-\alpha)$.

Multiply (2.32) with the i-th element of the influence vector υ and sum over all agents to get:

$$-\frac{1}{N} \sum_{j=1}^{N} \ln\left(\mu_{jt}^y\right) = \sum_{i=1}^{N} \left(\upsilon_i \ln(z_{it})\right) + \Psi_t, \tag{2.33}$$

where $\Psi_t \equiv \frac{1}{\alpha} B + \frac{(1-\alpha)}{\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[\upsilon_i w_{ijt} \ln(w_{ijt})\right] - \ln\left(\mu_t^L\right)$. The complete proof of (2.33) is provided in the appendix and is based on Acemoglu et al. (2012)

Taking the logarithm of both sides of equation (2.24) and combining it with equations (2.28) and (2.33) we obtain the following equivalent expressions:

$$\begin{aligned}
\ln(Y_t) &= \ln\left[A_t \prod_{i=1}^{N} (y_{it})^{\frac{1}{N}}\right] \\
&= \ln\left[A_t \prod_{i=1}^{N} \left(\frac{1}{N \mu_{it}^y}\right)^{\frac{1}{N}}\right] \\
&= \ln(A_t) - \ln(N) - \frac{1}{n} \sum_{i=1}^{N} \ln(\mu_{it}^y) \\
&= \ln(A_t) - \ln(N) + \sum_{i=1}^{N} \left(\upsilon_{it} \ln(z_{it})\right) + \Psi_t, \qquad (2.34)
\end{aligned}$$

Equation (2.34) provides an expression for the maximal aggregate consumption good that the social planner can reach by optimally organizing the production of all agents in the economy.

Substituting (2.34) in equation (2.13), we obtain an expression that relates the changes in optimal individual consumption to changes in the aggregate consumption good:

$$\begin{aligned}\Delta \ln \left(c_{it+1}\right) &= \Delta \ln \left(Y_{t+1}\right) - \Delta \ln \left[\sum_{i=1}^{N} \left(\Theta_{it+1}\right)^{\frac{1}{\gamma}}\right] + \frac{1}{\gamma} \Delta \ln \left(b_{it+1}\right) + \frac{1}{\gamma} \ln \left(\rho_i\right) \\ &= \sum_{i=1}^{N} \left[\Delta \left(\upsilon_{it} \ln \left(z_{it+1}\right)\right)\right] + \Delta \ln \left(A_{t+1}\right) + \Delta \Psi_{t+1} - \Delta \ln \left[\sum_{i=1}^{N} \left(\Theta_{it+1}\right)^{\frac{1}{\gamma}}\right] + \frac{1}{\gamma} \Delta \ln \left(b_{it+1}\right) + \frac{1}{\gamma} \ln \left(\rho_i\right).\end{aligned} \tag{2.35}$$

The expression in (2.35) demonstrates that changes in optimal individual consumption may be affected by idiosyncratic productivity shocks $\ln\left(z_{it+1}\right)$ depending on the values that the elements of the influence vector υ_i may take.

The *degree* d_{it} of agent i in period t is defined as the share of agent j's output in the intermediate input supply of the economy normalized by $(1-\alpha)$:

$$d_{it} \equiv \sum_{j=1}^{N} w_{jit}. \tag{2.36}$$

The influence vector provided in (2.23) can be alternatively written in a matrix form, i.e.,

$$\upsilon'_{nt} \equiv \frac{\alpha}{N} \left[I - \left(1-\alpha\right) W'_{nt}\right]^{-1} \mathbf{1}. \tag{2.37}$$

Given that the input shares of any agent in the economy are assumed to sum up to one, i.e., $\sum_{j=1}^{N} w_{ijt} = 1$, then the inverse $\left[I - \left(1-\alpha\right) W'_{nt}\right]^{-1}$ exists and υ_{nt} can be expressed into the following convergent power series form:

$$\upsilon'_{nt} = \frac{\alpha}{N} \mathbf{1}' \sum_{k=0}^{\infty} \left(1-\alpha\right)^k W_{nt}^k. \tag{2.38}$$

From (2.38), we conclude that:

$$\upsilon'_{nt} \quad \geq \quad \frac{\alpha\left(1-\alpha\right)}{N} \mathbf{1}' W_{nt}. \tag{2.39}$$

The product $\mathbf{1}'W_{nt}$ yields the vector of degrees $d'_{nt} \equiv \left[d_{1t}\, d_{2t}\, \ldots d_{Nt}\right]$, given that the sum of each column i of W_{nt} is the degree of agent j defined in equation (2.36) as $d_{it} = \sum_{j=1}^{N} w_{jit}$. Thus, (2.39) can be rewritten as:

$$\upsilon'_{nt} \quad \geq \quad \frac{\alpha\left(1-\alpha\right)}{N} d'_{nt}. \tag{2.40}$$

The effect of an idiosyncratic shock ε_{jt+1} on aggregate output as the number of sectors becomes infinitely large will depend on the asymptotic behaviour of υ_j which denotes the Bonacich centrality of the j-th entrepreneur defined in (2.23). The following examples demonstrate cases in which the partial effect of ε_{jt+1} on $\Delta \ln\left(c_{it+1}\right)$ does not vanish as $N \to \infty$ or decays at a rate that is lower than $1/\sqrt{N}$.

The following definitions will be used throughout this section: Consider two sequences of positive numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$. In what follows, (i) $a_n = \mathcal{O}(b_n)$ holds when the two sequences satisfy $\limsup_{n\to\infty}(a_n/b_n) < \infty$, (ii) $a_n = \Omega(b_n)$ holds when the two sequences satisfy $\liminf_{n\to\infty}(a_n/b_n) > 0$, and (iii) $a_n = \Theta(b_n)$ holds if $a_n = \mathcal{O}(b_n)$ and $a_n = \Omega(b_n)$ are satisfied simultaneously.

In what follows, we document how idiosyncratic shocks may have a different impact on the aggregate consumption good and its optimal allocation depending on the network structure of the production economy. To simplify our exposition, we assume that the network structure and the influence vector do not change over time.

2.4.1 Balanced networks

A balanced network is defined as follows:

Definition 1. A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is balanced if $\max_{i\in\{1,\ldots,N\}} d_i^n = \Theta(1)$.

Applying the sup norm on both sides of (2.39), we have:

$$\|\upsilon_n\|_\infty \geq \frac{\alpha(1-\alpha)}{N}\|d_n\|_\infty\,, \tag{2.41}$$

Note that $\max_{i\in\{1,\ldots,N\}} d_i = \|d_n\|_\infty = \Theta(1)$ holds by definition under our assumption of a balanced network. Thus, from inequality (2.41) we can conclude that the maximal element of the influence vector of the economy tends to zero[9] as $N \to \infty$.

More generally, according to (2.40), the following element-wise inequality holds:

$$\begin{aligned} \upsilon_n' &\geq \frac{\alpha(1-\alpha)}{N}d_n' \\ \Rightarrow [\upsilon_1\,\upsilon_2\,\upsilon_3\ldots\,\upsilon_n] &\geq \frac{\alpha(1-\alpha)}{N}[d_1\,d_2\,d_3\ldots\,d_n]\,. \end{aligned}$$

Thus, we conclude that asymptotically every element of the influence vector υ_n is not bounded away from zero as $N \to \infty$.

Proposition 3. *Assuming CRRA preferences and a balanced inter-sectoral production structure, the partial effect of an idiosyncratic productivity shock to the j-th agent's sector on the optimal consumption of the i-th individual becomes 0 as $N \to \infty$.*

2.4.2 Star type networks

Assume that entrepreneur $j = 1$ is the sole supplier of the intermediate input for production of the rest of the agents in the economy so that $w_{i1} = 1\,\forall\, i \in \{1,\ldots,N\}$. Then, the input-output network structure takes a star shape.

[9]Note that $\alpha(1-\alpha)$ is constant and that $\|d_n\|_\infty$ is asymptotically bounded away from zero and different than ∞.

The influence vector in this case is $\upsilon_n' = \frac{a}{N}\mathbf{1}' + [(1-\alpha)\ 0 \ldots 0]$ and the Bonacich centrality of agent 1 is $\upsilon_1 = \frac{\alpha}{N} + 1 - \alpha$, whereas the centrality of every other entrepreneur is $\upsilon_i = \frac{\alpha}{N}\,\forall\, i \in \{2, \ldots, N\}$. Consider how agent i's optimal consumption is affected by the idiosyncratic shocks to agent 1:

$$\begin{aligned}\lim_{N\to\infty}\frac{\partial\left(\Delta\ln\left(c_{it+1}\right)\right)}{\partial\varepsilon_{1t+1}} &= \lim_{N\to\infty}\left(\upsilon_1\sigma_1\right)\\ &= \left(1-\alpha\right)\sigma_1,\ \forall\, i\in\{1,\ldots,N\}\,. \end{aligned}\tag{2.42}$$

It is apparent from (2.42) that every agent's optimal consumption is affected by the idiosyncratic shocks to agent 1. Idiosyncratic shocks to agent 1 are propagated to the entire economy and matter for the optimal allocation of consumption because this entrepreneurial activity occupies a central position in the economy's input-output network structure.

Denote $\|\cdot\|_\infty$ to be the sup-norm of a vector. Then, $\|\upsilon_n\|_\infty$ yields the largest element of the influence vector in absolute value. We use the following definitions suggested by Acemoglu et al. (2012):

Definition 2. A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ has dominant entrepreneurs if $\|\upsilon_n\|_\infty = \Theta(1)$.

Definition 2 implies that the largest share of output to be found in the economy is asymptotically different from zero as the number of agents in the economy increases.

Definition 3. A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ has a star-type network structure if $\max_{i\in\{1,\ldots,N\}} d_i = \Theta(N)$, where d_i denotes the degree of agent i.

Using definitions 2 and 3, Acemoglu et al. (2010) have shown that dominant agents always exist in economies characterized by a star-like structure.

Applying the sup norm on both sides of (2.39), we have:

$$\|\upsilon_n\|_\infty \geq \frac{\alpha\left(1-\alpha\right)}{N}\|d_n\|_\infty\,,\tag{2.43}$$

Note that $\max_{i\in\{1,\ldots,N\}} d_i = \|d_n\|_\infty = \Theta(N)$ holds by definition under our assumption of a star-like network. Thus, from inequality (2.43) we can conclude that the maximal element of the influence vector of the economy remains different from zero for all $n\in\mathbb{N}$, as well as for $N\to\infty$. Given that $\|\upsilon_n\|_\infty \nrightarrow 0$ in a star-like network, we conclude that the idiosyncratic shocks to the productivity of agents who occupy a central position in the input supply network will not cancel out in the aggregate. Assuming that agent $j=1$ is a central agent in the economy, then the optimal allocation of consumption to any agent i in the economy is responsive to $1'$s idiosyncratic shocks, i.e.,

$$\lim_{N\to\infty}\frac{\partial\left(\Delta\ln\left(c_{it+1}\right)\right)}{\partial\varepsilon_{1t+1}} = \lim_{N\to\infty}\left(\upsilon_1\sigma_1\right) > 0.$$

This result leads to the following proposition.

Proposition 4. *Assuming CRRA preferences and a star-type inter-sectoral production structure, the partial effect of an idiosyncratic productivity shock to the dominant sector on the optimal consumption of the i-th individual remains different from 0 as $N \to \infty$. The corresponding partial effect of idiosyncratic productivity shocks to the non-dominant sectors of the economy converges to zero as $N \to \infty$.*

2.4.3 Scale free networks

Let $\mathbb{P}(\cdot)$ be the frequency distribution of the degrees in the input supply network. Then, the following definition characterizes power law networks:

Definition 4. A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ has a scale free network structure if the degree distribution in the degree sequence $\left\{d^{(n)} = (d_1,\, d_2,\, ,\, d_n)\right\}_{n\in\mathbb{N}}$ follows a power-law distribution of the following form:

$$\mathbb{P}(d) = cd^{-\zeta},$$

where $c > 0$ is a normalizing constant, $d > c^{1/\zeta}$, and $\zeta \geq 1$. In what follows, we normalize the constant to be equal to one, i.e., $c = 1$.

Under the assumption that degrees follow a power law distribution in scale free networks, the j-th largest degree in the input supply network of N agents is approximately equal to:[10]

$$d^j = \left(\frac{j}{N}\right)^{-\frac{1}{\zeta}}. \tag{2.44}$$

The asymptotic behavior of the elements of the influence vector υ_n as the size of the economy grows large (i.e., $N \to \infty$) will depend on the value of ζ, the exponent of the power law distribution of degrees:

- if $\zeta > 1$, then the degree of the most central agent, i.e., agent 1, is $d_1 \equiv \max_{i\in\{1,\dots,N\}} d_i = N^{\frac{1}{\zeta}}$. Thus, from inequality (2.40) we can conclude that:

$$\begin{aligned} \|\upsilon_n\|_\infty &\geq \frac{\alpha(1-\alpha)}{N}\|d_n\|_\infty \\ &\geq \frac{\alpha(1-\alpha)}{N^{\frac{\zeta-1}{\zeta}}}. \end{aligned} \tag{2.45}$$

  In words, the maximal element of the influence vector of the economy decays to zero at a rate that is at most $N^{\frac{\zeta-1}{\zeta}}$. More generally, shocks to any agent $j's$ productivity will asymptotically have no influence on changes in the optimal consumption allocation of agent i:

$$\lim_{N\to\infty}\frac{\partial(\Delta\ln(c_{it+1}))}{\partial\varepsilon_{jt+1}} = \lim_{N\to\infty}(\upsilon_j\sigma_j) = 0,\, \forall\, j \in \{1,\dots,N\}.$$

[10]See footnote 6 for a derivation of equation (2.44).

- if $\zeta = 1$, then the degree of any agent j in the economy is $d_j = \frac{N}{j}$. The degree of the most central agent 1 is $d_1 \equiv \max_{i \in \{1,\dots,N\}} d_i = N$ and from inequality (2.43) we can conclude that:

$$\begin{aligned} \|\upsilon_n\|_\infty &\geq \frac{\alpha(1-\alpha)}{N}\|d_n\|_\infty \\ &\geq \alpha(1-\alpha) \\ &> 0. \end{aligned} \tag{2.46}$$

According to (2.46) the largest element in the influence vector υ_n is bounded away from zero as $N \to \infty$. Therefore, the number of agents increases. More generally, the following element-wise inequality holds, when $d_j = \frac{N}{j}$:

$$\begin{aligned} &\upsilon_n' \geq \frac{\alpha(1-\alpha)}{N} d_n' \\ \Rightarrow &[\upsilon_1\, \upsilon_2\, \upsilon_3 \dots\, \upsilon_n] \geq \frac{\alpha(1-\alpha)}{N}[d_1\, d_2\, d_3 \dots\, d_n] \\ \Rightarrow &[\upsilon_1\, \upsilon_2\, \upsilon_3 \dots\, \upsilon_n] \geq \frac{\alpha(1-\alpha)}{N}\left[N\, \frac{N}{2}\, \frac{N}{3} \dots\right]. \end{aligned}$$

Thus, we conclude that asymptotically every element of the influence vector υ_n is bounded away from zero as $N \to \infty$, the lower bound being a function of the ranking of each agent's degree in the input supply network of the economy. Hence, idiosyncratic shocks do not cancel out in the aggregate and the optimal allocation of consumption to any agent i in the economy is affected by the idiosyncratic productivity shock ε_{jt+1} of agent j:

$$\lim_{N\to\infty} \frac{\partial(\Delta \ln(c_{it+1}))}{\partial \varepsilon_{jt+1}} = \lim_{N\to\infty}(\upsilon_j \sigma_j) > 0.$$

The discussion above can be summarized in the following proposition:

Proposition 5. *Assuming CRRA preferences and a scale-free inter-sectoral production structure, the partial effect of an idiosyncratic productivity shock to the j-th largest sector on the optimal consumption of the i-th individual:*

(i) decays to 0 at a rate that is slower than $N^{\frac{\zeta-1}{\zeta}}$ as $N \to \infty$, when the degree of inter-sectoral input supply connections follows a power-law distribution with $\zeta > 1$ and (ii) remains different from 0 as $N \to \infty$, when the degree of inter-sectoral input supply connections follows a power-law distribution with $\zeta = 1$ (i.e., a Zipf distribution).

3 Empirical implications

3.1 Empirical Tests of Risk Sharing

In this section we assess the extent of mis-specification in frequently used tests of risk-sharing, under different endowment or income generating processes. In developing our argument, we now allow for the possibility that the agent is affected by an economy-wide aggregate shock η that is common to all agents, in addition to the idiosyncratic shock ε_i. In effect, we assume the following decomposition of the composite shock Φ_i to agent i's income growth:

$$\Phi_i = \underbrace{\eta}_{Aggregate} + \underbrace{\varepsilon_i}_{Idiosyncratic} . \tag{3.1}$$

The benchmark risk sharing model shows that under optimal risk sharing, agents are insured up to the economy-wide aggregate shock, η, because the idiosyncratic shock ε_i can be diversified away by pooling income resources together. Consider now the situation where an element, s_i, of the idiosyncratic shock ε_i does not dissipate in the aggregate, contributing in this way to an aggregate shock $\sum_{j=1}^{N} s_j\varepsilon_j$, as described in Sections (2.3) and (2.4). In this case, a further decomposition of the shock to agent $i's$ income growth, as expressed in equation (3.1) can be made in terms of its insurable and uninsurable components:

$$\begin{aligned} \Phi_i &= \eta + (s_i\varepsilon_i + (1-s_i)\varepsilon_i) \\ &= \underbrace{\eta + s_i\varepsilon_i}_{Uninsurable} + \underbrace{(1-s_i)\varepsilon_i}_{Insurable} . \end{aligned} \tag{3.2}$$

In words, the overall shock Φ_i to agent $i's$ income growth process comprises of an (i) economy-wide shock, η, (ii) the systemic undiversifiable component of the idiosyncratic shock to agent i's income, $s_i\varepsilon_i$, and (iii) the diversifiable component of the idiosyncratic shock $(1-s_i)\varepsilon_i$. Under optimal consumption insurance, agents are only insured against, the diversifiable component of the idiosyncratic shock, $(1-s_i)\varepsilon_i$.

It is important to emphasize that while the common aggregate shock η can be absorbed by an aggregate variable (such as a constant or a time fixed effect), thereby eliminating the 'noise' in the measurement of the idiosyncratic shock, the undiversifiable component $s_i\varepsilon_i$ of the composite shock Φ_i persists within the idiosyncratic shock and cannot be tackled in a similar way[11]. One reason for this is that the measurement error induced by this form of mis-specification is multiplicative rather than additive. It is essential, therefore, to difference out this component from the measure of idiosyncratic shocks, so that there is complete separability between the insured and uninsured components of risk in any empirical specification that aims to test for risk-sharing.

[11] Ravallion and Chaudhuri (1997) have previously brought to attention the fact that the presence of an economy-wide aggregate shock can add noise to the measurement of the idiosyncratic component, therefore biasing parameter estimates. As a result, they propose the use of time fixed effects, rather than using demeaned consumption growth as the dependent variable (the specification employed, for example, by Townsend (1994)).

The empirical test for optimal risk sharing is based on the following equation[12], which has been developed in Section 2:

$$\Delta \ln (c_{it+1}) = \frac{\sum_{i=1}^{N} (y_{it}\sigma_i\varepsilon_{it+1})}{Y_t} - \Delta \ln \left[\sum_{i=1}^{N} (\Theta_{it+1})^{\frac{1}{\gamma}} \right] + \frac{1}{\gamma}\Delta \ln (b_{it+1}) + \frac{1}{\gamma} \ln (\rho_i) . \tag{3.3}$$

To test for optimal risk sharing, the above equation suggests an empirical specification of the form[13]:

$$\Delta \ln (c_{it+1}) = \beta_1 \frac{\sum_{i=1}^{N} (y_{it}\sigma_i\varepsilon_{it+1})}{Y_t} + \beta_2 \left(1 - \frac{y_{it}\sigma_i}{Y_t}\right) \varepsilon_{it+1} + u_{it+1}, \tag{3.4}$$

where β_1 and β_2 are the (population) coefficient parameters of the variables $\frac{\sum_{i=1}^{N}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t}$ and $\left(1 - \frac{y_{it}\sigma_i}{Y_t}\right)\varepsilon_{it+1}$, respectively. Note that the term $\left(1 - \frac{y_{it}\sigma_i}{Y_t}\right)\varepsilon_{it+1}$ measures the component of idiosyncratic shock that is optimally insurable by the social planner. As a result, $\widetilde{\beta}_2$ measures the effect of an idiosyncratic shock on consumption growth after differencing out the contribution of each individual shock on the aggregate pool of resources.

Commonly used empirical tests, on the other hand, employ an over-specification of equation (3.3) and estimate instead[14]:

$$\Delta \ln (c_{it+1}) = \widetilde{\beta}_1 \frac{\sum_{i=1}^{N} (y_{it}\sigma_i\varepsilon_{it+1})}{Y_t} + \widetilde{\beta}_2 \varepsilon_{it+1} + u_{it+1}. \tag{3.5}$$

It is evident that when, idiosyncratic shocks do not dissipate in the aggregate, the above specification does not allow for the full separability of the idiosyncratic shock such that its total effect on consumption growth remains unidentified. This is because the shock does only appear as a separate variable, but also features as part of the aggregate endowment function. The econometrician will treat ε_{it+1} as an idiosyncratic shock that can be insured by the social planner whereas only a part of it can be optimally insured. The estimated $\widetilde{\beta}_2 = \beta_2 \left(1 - \frac{y_{it}\sigma_i}{Y_t}\right)$ will underestimate the 'true' insurance parameter as it captures a combination of (optimal) insurance and the granular fluctuation to aggregate resources. Ignoring the distributional aspects of the income generating process will

[12] Throughout this section, we consider preferences represented by the CRRA utility function but our results extend to utility functions that belong to the HARA class as well.

[13] We abstract from the issue of measurement error in endowments or consumption. In principle, our main results hold regardless of the presence of measurement error in these variables. Additionally, we only consider a single risk sharing group. However, our suggested test can be implemented for multiple risk sharing groups using panel time series estimators which allow for heterogeneous slope coefficients across groups (Pesaran 2006; Coakley et al. 2006). The estimated parameters would, then, be group specific and can be viewed as estimates from panel regressions for each group. An average parameter estimate, with the use of a group fixed effect, is typically not recommended as it would be difficult to interpret.

[14] Note that the empirical specifications used in this section are based on aggregate income, rather than aggregate consumption which is used more often in the empirical risk-sharing literature (see, though, Ravallion and Chaudhuri (1997) for a notable exception). Optimal risk-sharing implies that the aggregate resource constraint (2.6) is binding. Therefore, aggregate income and aggregate consumption can be used interchangeably. It is also possible to employ time fixed effects (panel data) or a constant (cross-sectional data) that replaces and absorbs the aggregate endowment variable. Further the use of aggregate endowments rather than aggregate consumption ensures that all parameters of our specification are structurally identified and does not suffer from the the identification issue due to 'reflection' as noted by Manski (1993).

mean that these two components cannot be separately identified.

We now derive analytical expressions for the extent of bias that results from estimating equation 3.5, rather than equation 3.4. Note that the specification takes the form akin to that of a non-classical multiplicative measurement error in the explanatory variable.To derive the bias, we take the example of a one-variable risk sharing model and suppress the constant and time subscripts for convenience. In fact, it can be shown that the bias is further exacerbated by the correlation of the mis-measured variable with the aggregate resources variable, $\frac{\sum_{i=1}^{N}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t}$, included in equation (3.5) as is or proxied by a constant or time fixed effects [15].

Denote $\varepsilon_i^* = \left(1 - \frac{y_i\sigma}{Y_t}\right)\varepsilon_i$. Then, equation (3.4), the true data generating model under CRRA preferences, can be written as:

$$\Delta \ln (c_i) = \beta_2 \varepsilon_i^* + u_i. \tag{3.6}$$

Consider the following mis-specified model:

$$\Delta \ln (c_i) = \widetilde{\beta}_2 \varepsilon_i + u_i. \tag{3.7}$$

The estimator $\widetilde{\beta}_2$ from the mis-specified model suffers from a measurement error that is multiplicative. Then, the least squares estimator based on the mis-specified model represents only a proportion of the true parameter β_2. The bias is approximately:

$$\widetilde{\beta}_2 = \beta_2 \left(\frac{\sigma_{\varepsilon^*,\varepsilon}}{\sigma_\varepsilon^2}\right) \tag{3.8}$$

The proportion by which the coefficient is biased is akin to the reliability ratio or signal-to-noise ratio, where $\sigma_{\varepsilon^*,\varepsilon}$, the covariance between the composite idiosyncratic shock ε and the diversifiable component of the idiosyncratic shock ε^* represents the quality of the 'signal'. The bias from mis-specification will be lower as the quality of the signal increases, i.e., as the covariance between the composite idiosyncratic shock and the diversifiable component of the idiosyncratic shock increases. In a relatively large economy where all agents have an equal share of the total output, it is easy to see this covariance will tend towards 1 and $\widetilde{\beta}_2$ will converge to the true parameter value β_2.

In the appendix, we present results on the specification bias for different endowment and income generating processes. As an example, we develop these results for the case in which the investigator directly estimates equation (3.5) instead of equation (3.4) using cross-sectional data. However, the main results continue to hold in principle, even with the use of panel data or when the aggregate endowment is replaced and absorbed by the use of a constant or time fixed effect[16]. The results can be summarized as follows:

Proposition 6. *(i) When all endowments are identically sized or drawn from a distribution with finite variance, the bias decays to zero at $1/N$ as $N \to \infty$.*

[15] This will also result in all coefficients representing the aggregate resource constraint to be inconsistent.

[16] For the specification where a constant or time fixed effect is used, the bias is amplified by a factor proportional to the estimated constant or time fixed effect.

(i) For the CRRA specification, when endowments are follow a power-law distribution with $\zeta \geq 1$, the bias is approximately:

$$\widetilde{\beta}_2 - \beta_2 \sim \beta_2 \left(\frac{-\sigma}{N}\right)$$

In other words, the bias decays to zero at $1/N$ as $N \to \infty$.

(ii) For the CRRA specification, when income is generated through a network-production structure with a scale free degree distribution the lower bound of the bias is approximately:

$$\widetilde{\beta}_2 - \beta_2 \geq \beta_2 \left(\frac{-\sigma}{N}\left[\frac{a}{N}\sum_{i=1}^{N}\varepsilon_i^2 + \frac{a}{N}(1-a)\sum_{i=1}^{N}\left(d_i\varepsilon_i^2\right)\right]\right)$$

When the degree distribution follows a power law with $\zeta > 1$, the bias decays to zero at $1/N$ as $N \to \infty$. For $\zeta = 1$ bias decays to zero at $\ln N/N$ (slower than $1/N$) as $N \to \infty$. Finally, when the degree distribution follows a power law with $\zeta < 1$, the bias decays to zero at $1/N^{\left(2-\frac{1}{\zeta}\right)}$ (slower than $1/N$), as $N \to \infty$.

(iii) For the CARA specification, the bias is approximately:

$$\widetilde{\beta}_2 - \beta_2 \sim \beta_2 \left(\frac{-\sigma\mathbb{E}[y_i]}{N}\right)$$

For, $\zeta > 1$, the bias decays to zero at $1/N$ as $N \to \infty$. However when $\zeta = 1$, a $(\ln N)$ correction applies in the numerator (but not the denominator, as in the CRRA case) and bias decays to zero at $\ln N/N$ (slower than $1/N$) as $N \to \infty$. Further, when endowments are distributed with finite variance or follow a power-law distribution with $\zeta < 1$, the bias is approximately:

$$N^{2-\frac{1}{\zeta}}(\widetilde{\beta}_2 - \beta_2) \xrightarrow{d} \beta_2(-\sigma g_\zeta)$$

where g_ζ is a random variable that follows a Lévy distribution with exponent ζ. The bias decays to zero at $1/N^{\left(2-\frac{1}{\zeta}\right)}$ (slower than $1/N$), as $N \to \infty$.

3.2 Finite Sample Simulations

In this sub-section we conduct a small simulation study to assess the extent of the specification bias in finite samples. In the first case, we consider the true data generating model under CRRA preferences[17]:

$$\Delta \ln (c_{it+1}) = \beta_1 \frac{\sum_{i=1}^{N}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t} + \beta_2\left(1 - \frac{y_{it}\sigma_i}{Y_t}\right)\varepsilon_{it+1} + u_{it+1}, \tag{3.9}$$

and evaluate the performance of the estimator $\widetilde{\beta}_2$ from the misspecified model:

$$\Delta \ln (c_{it+1}) = \widetilde{\beta}_1 \frac{\sum_{i=1}^{N}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t} + \widetilde{\beta}_2\varepsilon_{it+1} + u_{it+1}. \tag{3.10}$$

[17]We simulate the model with two time periods, suppressing the constant.

We obtain draws for y_{it} by simulating a power law distribution[18] with the power-law exponent ranging between 1.1 and 2.0[19]. Both the iid shock ε_{it+1} and measurement error u_{it+1} are drawn from normal distribution with mean 0 and standard deviation 1. For simplicity, we homogenize the variance of the shocks to be common across all agents and set it as equal to 0.6.

We also, in the second case, consider the true data generating model, given CARA preferences:

$$\Delta \ln (c_{it+1}) = \beta_1 \frac{\sum_{i=1}^{N} (y_{it}\sigma_i \varepsilon_{it+1})}{N} + \beta_2 \left(1 - \frac{y_{it}\sigma_i}{N}\right) \varepsilon_{it+1} + u_{it+1}, \tag{3.11}$$

and evaluate the performance of the estimator $\widetilde{\beta}_2$ from the misspecified model:

$$\Delta \ln (c_{it+1}) = \widetilde{\beta}_1 \frac{\sum_{i=1}^{N} (y_{it}\sigma_i \varepsilon_{it+1})}{N} + \widetilde{\beta}_2 \varepsilon_{it+1} + u_{it+1}. \tag{3.12}$$

A total of 250 draws are obtained for the independent variables and error terms. The misspecified model is estimated via OLS. In order to asses the quality of the estimators in the simulations, we calculate the root mean squared error (RMSE) which is the square-root of the mean square error[20]

Tables 1 & 2 report results from the simulations for three coefficient values; $\beta_2 = 0$, $\beta_2 = 0.5$ and $\beta_2 = 1$ over a range of parameter values governing the income distribution, $\zeta \in [1.1, 2]$. It can be seen from Tables 1 that the bias from mis-specification tends to be quite low for the CRRA specification for any given value of the power-law parameter. Further, the extent of this bias declines rapidly as the size of the risk sharing group becomes larger. On the other hand, Table 2, shows that the CARA specification performs quite poorly when the income distribution exhibits fat tails, i.e. for the lowest values of ζ. The bias from mis-specification are extremely large and persists even over fairly large (risk-sharing) group sizes.

4 Features of Optimal Risk Sharing and Implications for Full Insurance

In Section 2 we have shown conditions under which optimal risk sharing may not result in perfect or full insurance. Thus, under optimal risk sharing, agents are insured up to the systemic undiversifiable factor $s_i\varepsilon_i$ resulting from the uninsurable portion of the idiosyncratic shock to income, as well as any other economy-wide aggregate shock, η. Under partial risk sharing, there will be an additional factor that induces changes in own consumption, which, for the case of agent i, arises

[18] While our Monte Carlo study simulates an endowment structure by varying the tail parameters of the distribution, it is possible to also simulate an income distribution generated through a network production structure. This can be done by first simulating a scale-free degree distribution for the network structure to obtain the centrality scores v_{it}. The risk sharing specification can then be based on equation (2.35).

[19] We set the minimum threshold of 20 for simulating the power law distribution

[20] The MSE for the parameter β is calculated is given by: MSE(β)=$\Sigma_r \frac{(\hat{\beta_r} - \bar{\hat{\beta}})^2}{R} + \left[\frac{\Sigma_r(\beta_s - \hat{\beta_r})}{R}\right]^2$, where $\hat{\beta_r}$ is the estimate in replication r, $\bar{\hat{\beta}}$ is the mean of the estimate for all replications, β_s is the true value of the parameter, and R is the number of replications in the simulation experiment.

from the uninsured portion of the diversifiable component of i's idiosyncratic risk, i.e., $\beta_2 s_i \varepsilon_i$, where $\beta_2 \in [0, 1]$ denotes the extent of partial insurance. Denoting $\Phi_i^{Uninsured}$ the uninsured component of the composite risk factor defined in equation (3.2), we have:

$$\Phi_i^{Uninsured} = \eta + s_i \varepsilon_i + \beta_2 (1 - s_i) \varepsilon_i.$$

Therefore, under partial risk sharing the following general specification holds:

$$\Delta \ln (c_{it+1}) = \eta + \sum_{j=1}^{N} s_j \varepsilon_j + \beta_2 (1 - s_i) \varepsilon_{ij} + u_{it+1}$$

There are two main implications of this result. First, we can see that the effect of an idiosyncratic income shock on consumption growth is ***heterogeneous***. The total effect[21] depends on the agent specific undiversifiable factor s_i. The factor s_i can be related to the (inverse) rank of agent i in the endowment distribution or her centrality in the income production network structure. To illustrate the extent of heterogeneity, we measure the share of the maximal element (denote this agent by m), $\|s\|_\infty = \max_i |s_i|$, in a simulated income distribution with varying power law parameters, using the procedure described in Section (3.2). Using this, we calculate the partial effect of an idiosyncratic shock to agent m, on her own consumption, given by[22] $\frac{\delta \Delta c_{\|s\|_\infty}}{\delta \varepsilon_{\|s\|_\infty}} = \|s\|_\infty + (1 - \|s\|_\infty) \beta_2$. Table (3) reports results from this exercise for three coefficient values; $\beta_2 = 0$, $\beta_2 = 0.5$ and $\beta_2 = 1$. It can be seen that the total effect of an idiosyncratic shock to m's consumption growth is higher than what is expected from optimal insurance, as implied by β_2. For example, with a power-law distribution with $\zeta = 1.1$, the $\frac{\delta \Delta c_{\|s\|_\infty}}{\delta \varepsilon_{\|s\|_\infty}} \sim 0.3$ when $\beta_2 = 0$, over all sample sizes. This means that the idiosyncratic shock of an agent holding the maximum income share affects her consumption growth by upto 0.3%, even when there is complete optimal risk-sharing. In contrast, when income is distributed with $\zeta = 2$ (finite variance), $\frac{\delta \Delta c_{\|s\|_\infty}}{\delta \varepsilon_{\|s\|_\infty}} \sim 0.08$ and this effect decreases as sample size increases. This discrepancy between optimal and full risk sharing varies proportionately with the amount of optimal insurance provided. For instance, when $\beta_2 = 0.9$, the difference between the total effect of an idiosyncratic shock and what is expected due to optimal insurance is only 0.03.

Secondly optimal risk sharing involves an exposure to ***systemic risk*** that is composite of the undiversifiable components of idiosyncratic shocks. This means that every agent i's consumption

[21]The total effect of agent i's idiosyncratic shock on her consumption growth is given by $s_i + \beta_2(1 - s_i)$. Inference on this test statistic is complicated by the fact that its distribution, which depends on the shape of the income distribution, can be non-standard. For example in the power-law case, the test-statistic will depend on the power law exponent ζ, which for values less than two, is non-gaussian. One possibility is to use simulation methods (Krinsky and Robb 1991) to derive the distribution of the test statistic for inference. In the first step, we can estimate the power law parameter, ζ given the observed distribution of endowments y_t. In the next step, one can use the given empirical distribution of y_t to estimate equation 3.4 to obtain a consistent estimate for β_2. Given the mean and variance of both parameters we can draw observations from a large bvariate sample, $(\beta_2, \zeta)_r$ where $r = 1, ..., R$, from a normal distribution for β_2 and a normal gaussian distribution for ζ. For each simulated draw we can construct a measure of the test statistic. Finally, the quantiles of the sample of draws, for example, the .025th and .975th quantiles, can be used to estimate the boundaries of a confidence interval of the functions.

[22]The variable s_i is typically normalized by each agents' standard deviation of volatility. In our simulations, we have assumed this to be a constant 0.6 for all agents.

is affected by a proportion of agents j's idiosyncratic shock. A consequence of this feature is that shocks to the top 1% of the income distribution or the most central agents in the production process, have economy-wide externalities, affecting the consumption growth profile of every agent. To gauge this effect, we calculate the partial effect of an idiosyncratic shock to agent m, on the consumption of any other agent j, given by, $\frac{\delta \Delta c_j}{\delta \varepsilon_{\|s\|_\infty}} = (1-\beta_2)\|s\|_\infty$. It can be seen, in Table (3), that the effect of agent m's idiosyncratic shock to agent j's consumption growth is non-negligible. For example, with a power-law distribution with $\zeta = 1.1$, the $\frac{\delta \Delta c_j}{\delta \varepsilon_{\|s\|_\infty}} \sim 0.3$ when $\beta_2 = 0$, over all sample sizes. This means that the idiosyncratic shock of an agent holding the maximum income share affects every other agent's consumption growth by a proportion, 0.3, as a result of risk-pooling. In contrast, when income is distributed with $\zeta = 2$ (finite variance), $\frac{\delta \Delta c_{\|s\|_\infty}}{\delta \varepsilon_{\|s\|_\infty}} \sim 0.08$ and this effect decreases as sample size increases. The extent of systemic exposure to risk varies proportionately with the amount of optimal insurance provided. For example, when $\beta_2 = 0.9$, the effect of agent m's idiosyncratic shock to agent j's consumption growth is relatively small, at approximately, 0.033 for heavily skewed income distributions.

In the next section, we provide empirical evidence on exposure to systemic risk, using data from two, conceptually different, risk sharing groups in the United States.

4.1 Testing Granularity in Data

4.1.1 Empirical Specification

In this section we test whether individuals are exposed to to a degree of ***systemic risk*** as a result of risk-sharing. Borrowing terminology from Gabaix (2011), we term this exercise as a test for 'granularity'. We have shown in the previous sections that under optimal risk-sharing every agent i's consumption is affected by a proportion of agents j's idiosyncratic shock and that this proportion depends inversely on the rank of agent j in the endowment distribution or her centrality in the income production network structure. We use two different risk sharing groups. of differing sizes, to test our hypothesis. Our first risk-sharing group is the set of US households ($N \sim 3 \times 10^6$) for which there has been evidence of a significant amount of risk-sharing. Our second, much smaller, risk-sharing group is that of US regions/states ($N = 50$). In contrast to the households, Kalemli-Ozcan et al. (2003) show that the extent of partial insurance is quite low, at approximately 0.5. To proceed with the analysis, we first present descriptive statistics to characterize the distributional aspects of the income data and then develop a specification that enables us to explore the extent of systemic risk exposure.

As we have shown, when endowments are drawn from a power-law distribution, i.i.d shocks can accumulate rather than dissipate in the aggregate. We now explore the distributional aspects of the endowments or income distribution in data. Figure (1) plots the distribution of household income (household risk sharing), obtained from the PSID sample in 1996, and gross state product (regional risk-sharing), obtained from the US Bureau of Economic Analysis in 2006. Figures (2(a)) & (2(c)) plot the histograms of the respective distributions while Figures (2(b)) & (2(d)) plots thier empirical counter-cumulative distribution functions. We can see in both cases, that the tail of the distribution is well-approximated by a power law distribution, as shown by the approximate linear relationship in the solid grey line. As an alternative, we also consider a log-normal distribution but find that it fits the data poorly, as shown by the dashed grey line. Rough estimates for the shape parameter, based on the modified log rank-log size regression suggested by Gabaix and Ibragimov (2011) are as follows: for the PSID sample of household income, we estimate $\zeta = 0.982$, with a standard error of 0.0002; for gross state product of US states, we estimate $\zeta = 0.855$, with a standard error of 0.0142.

Our estimates for household income should be treated with caution as they are based on a sample of households. There exists, however, a large literature that provide support for the power-law distribution of individual incomes in different countries. The empirical investigation of whether individual incomes satisfy a power-law dates back to Vilfred Pareto (1897) who showed that the distribution of income in the upper tail has a 'pareto' or a power-law distribution. Feenberg and Poterba (1993) calculate ζ for the United States, between 1951 and 1990 and find that the exponent ranges between 2.5 (1970) and 1.6 (1990). Atkinson, Piketty, and Saez (2011) update this study and estimate the power law parameter for top percentiles of the population income distribution in the United States and find that it ranges between 1.4-1.67 for the most recent year of 2007[23]. In fact, the authors find that there is a steady decline in the estimates of ζ through the years and that inequality is increasing. Note, that although we have evaluated the risk-sharing model keeping the distribution of endowments as fixed, a more flexible model incorporating the dynamics of the growth process and thereby the evolution of the income distribution, can yield potentially similar results. For instance, Gabaix (1999) shows that even with some, arbitrary, initial distribution, the distribution of endowments will converge to a power-law distribution if there exists a mechanism that prevents each agent from becoming infinitesimally small (in terms of endowments). From a practical viewpoint, there are many mechanisms that exist in society, for example social safety nets, unemployment insurance, which prevent individual incomes from falling below a minimum threshold level (such as, for example, a poverty line)[24].

Although we do not explicitly calculate the centrality measures or plot the degree distributions for the production network[25], there is substantial support in the literature for the fact that many real-world networks are scale-free (see for example, Barabási and Albert (1999)). Acemoglu et al.

[23]The estimate for the year 2007, is approximately 1.67 when capital gains are excluded and 1.4 when capital gains are included

[24]Reed (2001) provides an alternative perspective. He argues that the distribution of time itself is stochastic and follows an exponential distribution. As a result, the current distribution of incomes should be that of a geometric Brownian motion observed after an exponentially distributed time T, leading to a double Pareto distribution.

[25]Our theoretical section assumes for analytical convenience that each agent represents a sector of production. In practise, individual workers are employed in firms which belong to a certain sector and an overall production network. In this case, the extent of insurance would depend additionally on how much each agents' sector or firm is able to insure them via the wage contract (Guiso et al. (2005) provide some estimates of the degree of insurance at the firm level). If, for example, sector level shocks are fully passed on to the worker, then, her consumption volatility would be exactly proportional to her sector's volatility which ultimately depends on aggregate volatility, as we have shown in Section 2.4.

2012 provide extensive evidence to show that the degree distribution of the United States sectoral production network follows a power law with the exponent ranging between 1.2 and 1.4. Finally, note that we have shown that the centrality of each sector is equal to the sales ratio for each sector (see Appendix 6.2). Consequently, for the empirical analysis below, we use the observed sales ratio of the corresponding sector rather than calculate centrality scores from the underlying network.

We now move on to describing our test for granularity. Our test involves decomposing the aggregate component of Equation (3.3) into two parts: the aggregate income growth of the top percentiles of the income distribution and the aggregate income growth of the corresponding bottom percentile, both normalised by total income. For example, the equation below decomposes the aggregate component into $\frac{\sum_{i=1}^{100}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t}$, the normalised aggregate income growth of the top 100 income earners in the PSID sample and the $\frac{\sum_{i=1}^{N-100}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t}$ the normalised aggregate income growth of the remaining income earners;

$$\Delta \ln(c_{it+1}) = \beta_{1a}\frac{\sum_{i=1}^{N-100}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t} + \beta_{1b}\frac{\sum_{i=1}^{100}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t} + \beta_2\left(1-\frac{y_{it}\sigma_i}{Y_t}\right)\varepsilon_{it+1} + u_{it+1} \quad (4.1)$$

Our hypothesis is that the consumption growth of any individual are exposed to a systemic component of risk under optimal risk sharing, i.e. it is partially affected by the proportional idiosyncratic shocks of other individuals in the risk sharing group. The proportion by which idiosyncratic shocks of other agents enter the consumption growth profile depend on the specific utility function. In the above equation, we have assumed a CRRA utility function which implies that the aggregate idiosyncratic shocks of, for example, the top 100 income earners expressed as a ratio of total income affects consumption growth of individuals not belonging to the top 100.

In testing this hypothesis, we allow for the possibility that the aggregate components contain a covariate shock η, previously assumed away in the analysis[26]. Our test for granularity requires that the covariate shock be differenced out from aggregate income growth of the top income earners so that we are left with a measure of their aggregate idiosyncratic shocks (independent of the covariate shock). To achieve this, we undertake different demeaning schemes to difference out the covariate shock from the income growth of each individual belonging to the top income group and then aggregate the resulting demeaned measure. For instance, the demeaned measure of $\frac{\sum_{i=1}^{100}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t}$ is obtained as:

$$r_t = \frac{\sum_{i=1}^{100}\left(y_{it}(g_{it}-\bar{g}_t^{100})\right)}{Y_t} \quad (4.2)$$

Here r_t represents the 'granular residual' as termed by Gabaix (2011). The term ε_{it+1}, which may contain the covariate shock, has been replaced by $(g_{it}-\bar{g}_t^{100})$; where g_{it} refers to income growth of individual i (a measure of both idiosyncratic and covariate shock) and $\bar{g}_t^{100}$ is the mean income

[26] All our theoretical results hold regardless of the inclusion of a covariate risk term.

growth of the top 100 income earners. Other demeaning schemes include differencing using the mean income growth of the top 1000 income earners.

We construct the granular residual based on both the aggregate income growth of top earners in the PSID sample (the top 100 sample income earners) as well as the aggregate income growth of the top 1, 0.5 and 0.1 percentiles of the US income distribution which is the relevant risk sharing group.

The general specification is the following:

$$\Delta \ln \left(c_{it+1}\right) = \beta_{1a} \frac{\sum_{i=1}^{N-\mathbf{T}} \left(y_{it}\sigma_i\varepsilon_{it+1}\right)}{Y_t} + \beta_{1b} r_t^{\mathbf{T}} + \beta_2 \left(1 - \frac{y_{it}\sigma_i}{Y_t}\right)\varepsilon_{it+1} + u_{it+1} \tag{4.3}$$

where $\mathbf{T}$ is the relevant top income earners group (sample top 100, population 1%, 0.5%, 0.1%). Equation (4.3) is estimated for the sample of individuals excluding those that comprise the defined top income earners. Note that $\beta_{1a} \neq \beta_{1b}$ because of the second part of the decomposition in Equation (4.1) is now demeaned to represent the granular residual.

We can conduct a similar analysis to test whether the shocks to the incomes of individuals employed in the most central sector of the economy affect consumption growths of other individuals. This is the case when income is generated from a scale-free or star like network production structure as outlined in Section (2.4) The specification for this test is:

$$\Delta \ln \left(c_{it+1}\right) = \beta_{1a} \frac{\sum_{i=1}^{\text{non-manf}} \left(y_{it}\sigma_i\varepsilon_{it+1}\right)}{Y_t} + \beta_{1b} r_t^{\text{manf}} + \beta_2 \left(1 - \frac{y_{it}\sigma_i}{Y_t}\right)\varepsilon_{it+1} + u_{it+1} \tag{4.4}$$

where r_t^{manf} is the granular residual representing the shocks of the manufacturing sector employees, demeaned using the average income growth of non-manufacturing sector employees. The normalised aggregate income growth of the non-manufacturing sector employees is denoted by $\frac{\sum_{i=1}^{\text{non-manf}}(y_{it}\sigma_i\varepsilon_{it+1})}{Y_t}$.

4.1.2 Results: Household Risk Sharing, $N \sim 3 \times 10^6$

We use three data sources for our empirical analysis. The first dataset is the Panel Study of Income Dynamics (PSID). We use data from the 1974-1996 waves of the PSID as the definitions of both the income and consumption variables remained roughly consistent over this period. Additionally, since the PSID primarily measured food consumption, we estimate the consumption risk sharing equations with respect to food consumption growth[27]. We follow Schulhofer-Wohl (2011) in constructing the PSID variables: income is defined as the household's total money income excluding aid (food stamps,

[27] Schulhofer-Wohl (2011) points out that while food may not be an ideal proxy for total consumption, it is more likely to be time separable as assumed by the expected utility formulation. Similar to Schulhofer-Wohl (2011) we also drop the years 1988 and 1989 when no food consumption data was collected. Further, we assume away any measurement error associated with the consumption variable for convenience.

help from relatives etc.) and unemployment insurance. The income and food data is deflated using the CPI and the food & beverages component of CPI respectively.

We use two additional sources of data to construct the aggregate components four our granular hypothesis test. To measure the income growth of the *population* top percentiles of the US income distribution, we use the Top Incomes Database (Alvaredo et al. 2014). The top income database provides distributional statistics for top incomes in twenty-two countries. The distributional statistics used to estimate the series for the USA are produced by the Statistics of Income Division of the Internal Revenue Service. Income is defined as income as the sum of all income components reported on tax returns (wages and salaries, pensions received, profits from businesses, capital income such as dividends, interest, or rents, and realized capital gains) before individual income taxes. It excludes government transfers such as Social Security retirement benefits or unemployment compensation. Further details about the construction of the series can be found in Alvaredo et al. (2014).

Finally, we obtain the productivity growth rates for each sector if the US economy from the World KLEMS database. It is a databse on U.S. productivity growth for the period 1947-2010. The database uses the North American Industry Classification System (NAICS) to define sectors.

Table 5 presents regressions of consumption growth on the granular residual as defined in Equation (4.2). We report standardized coefficients in each column. These regressions are supportive of the granular hypothesis. The first two columns show that adding the granular residual to a basic regression that regresses own idiosyncratic shocks on consumption growth increases the adjusted R-square by 70%. Columns (3) & (4) reports results from the standard risk sharing specification together with the granular residual defined as the demeaned aggregate income growth of the top 1000 sample income earners. As noted earlier, this specification is estimated for the sample of income earners excluding the top 100 to avoid the reflection problem. The results show that consumption growth is affected by the granular residual. A one standard deviation increase in the idiosyncratic shock of the top 100 income earners increases consumption growth by approximately 0.013 standard deviations. Additionally, we report two information criterion (IC) based statistics, the adjusted R square and the Akaike Information Criteria (AIC) to assess the incremental explanatory power of the granular residual. Comparing Column (4) to Column (3) we can see that the granular residual adds significant explantory power to the risk sharing equation, over and above what is explained by the aggregate shock (proxied by the aggregate income growth of the sample excluding the top 100 income earners). Column (5) reports similar results but demeans the aggregate income growth of the top 100 using the mean income growth of the top 100 income earners.

Table 5 reports results from the same specification as described above, but uses the population income growth of the top percentile of the income distribution as obtained from the top incomes database. Column (2) reports results for the effect of the granular residual on consumption growth, constructed using the aggregate income growth of the top 1% of population income earners. Columns

(3) & (4) show results from a granular residual constructed using the top 0.5% and 0.1% of population income earners respectively. All the results show that the addition of the granular residual in different ways, increases the explanatory power of the specification compared to the baseline specification reported in Column (1). A one standard deviation increase in the idiosyncratic shock of the top (population) income percentiles increases consumption growth by approximately 0.008 standard deviations.

Finally, Table 5 shows how the shocks to the employees of the most central sector (manufacturing) affect the consumption growth of the non-manufacturing sector employees. The table reports results from constructing the granular residual using both the population and sample measures of aggregate idiosyncratic shocks to the manufacturing sector. It also includes a specification controlling for the aggregate income growth of the non-manufacturing sector. All the results indicate that shocks to the manufacturing sector explain a significant proportion of fluctuations to the consumption growth of the non-manufacturing households. A one standard deviation increase in the idiosyncratic shock of the manufacturing sector income earners increases consumption growth of those employed in the non-manufacturing sector by approximately 0.010 standard deviations.

4.1.3 Results: Regional Risk Sharing, $N = 50$

To test for granular effects in regional risk-sharing, we use state-level[28] data on Gross State Product (GSP) and personal income at current prices for the period 1963-2012 from the Bureau of Economic Analysis (BEA). Data are transformed to fixed price using the United States national CPI.

Table 5 reports results from regressions of consumption (state personal income) growth on the granular residual defined as GSP growth of the top percentiles of the GSP distribution. Column (2) & (3) report results for the effect of the granular residual on consumption growth, constructed using the aggregate GSP growth of the top 5% and 1% respectively. To begin with, we find that, in contrast to the households, there is limited risk sharing across regions in USA. The extent of partial insurance is approximately 0.5. Nevertheless, Our decomposition exercise shows that consumption growth is substantially affected by the shocks to the top performing states[29]. The addition of the granular residual in both columns, increases the explanatory power of the specification compared to the baseline specification reported in Column (1) by approximately 6%[30]. A one standard deviation increase in the idiosyncratic shock of the top-producing regions increases state consumption growth by approximately 0.006 standard deviations.

[28] We exclude Washington, DC, similar to Kalemli-Ozcan et al. (2003).

[29] The sates that comprise the top 5% are California, Illinois, New York and Texas. California and New York belong to the top 1% of the state GSP distribution. Their status is fairly stable over time.

[30] Note that the increase in explanatory power for the regional risk-sharing specifications is smaller compared to the household risk sharing specifications because the regions are much less insured compared to households. The explanatory power of the aggregate variables are therefore proportional to the extent of insurable risk.

5 Conclusion

Perfect insurance is traditionally postulated as the outcome of a fully-enforced optimal risk-sharing arrangement, such as under the complete markets paradigm, where consumption growth is perfectly robust to the effects of idiosyncratic shocks. This view is based on the premise that the aggregate resource constraint can be fully diversified against idiosyncratic risk when the size of the risk sharing group is sufficiently large. In this paper, we have evaluated the validity of this premise. We have examined conditions under optimal risk-sharing, whereby the asymptotic rate of decay of the shock's effect on consumption growth is significantly dampened. The maximum effect of an idiosyncratic shock on consumption growth scales at lnN when income is drawn from a distribution with no finite moments compared to N when income is drawn from a distribution with a finite mean and variance. As a consequence, the effects of idiosyncratic shocks are not mitigated even at fairly large group sizes. We have shown that frequently used empirical tests of risk-sharing, can suffer from a specification bias when idiosyncratic shocks do not dissipate in the aggregate. Further, we find that the data supports our hypothesis that the consumption growth of any individual is exposed to a systemic component of risk arising from undiversified idiosyncratic risks.

All together, our results indicate that the nature and composition of idiosyncratic shocks do matter for income and consumption smoothing, even when risk-sharing arrangements are perfect. To illustrate this result, we have relied on the existence of complete markets while it is true that in most practical applications there appears to be a mix of formal and informal risk-sharing arrangements ridden with enforcement and commitment issues (Dubois et al. 2008). Although our analysis has highlighted the limited diversification constraints that arise under optimal risk-sharing, it would be reasonable to expect that similar concerns may apply under other forms of formal and informal insurance where information asymmetry, limited commitment, and other market imperfections hold. In these cases, we envisage that our findings can also prove useful for comparing the efficiency of different risk-sharing arrangements.

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Table 1: Monte Carlo Results for CRRA specification

CRRA specification, $\beta_2 = 0$

ζ	N=10^1		N=10^2		N=10^3		N=10^4		N=10^5		N=10^6	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
1.1	-0.0226	0.3911	-0.0067	0.0989	0.0016	0.0316	-6.30E-05	0.0097	-8.70E-06	0.003	-1.80E-05	0.0011
1.2	-0.0115	0.375	0.0144	0.1117	1.00E-04	0.0316	3.30E-04	0.0094	-9.20E-05	0.0028	-3.40E-05	0.001
1.3	0.0557	0.4045	0.0065	0.1049	-7.40E-04	0.0332	3.10E-04	0.0096	-8.10E-05	0.003	-1.70E-04	0.0011
1.4	-0.0075	0.4014	-0.0015	0.1	0.0021	0.0303	-7.50E-05	0.0108	6.70E-05	0.0031	1.10E-05	0.001
1.5	-0.0094	0.3913	0.0068	0.1006	-5.10E-04	0.0353	3.70E-05	0.0101	1.90E-04	0.0033	-2.40E-05	1.00E-03
1.6	-0.0089	0.3746	0.0035	0.0898	0.001	0.0319	-0.0011	0.0098	3.20E-04	0.0032	-3.20E-05	9.70E-04
1.7	0.036	0.4085	6.50E-04	0.1007	3.10E-04	0.033	-1.40E-04	0.0096	2.10E-04	0.0034	9.00E-05	0.001
1.8	-0.0298	0.4043	-0.003	0.0952	-1.40E-04	0.0317	-7.40E-05	0.0097	7.30E-05	0.0031	-2.40E-05	9.90E-04
1.9	0.0477	0.3784	0.0034	0.0904	-0.0017	0.0307	1.20E-04	0.0107	1.20E-04	0.0031	-1.40E-06	0.001
2	-0.0108	0.3433	-0.0122	0.1036	-0.0017	0.0311	-1.60E-04	0.0101	1.60E-04	0.0034	-1.00E-05	9.90E-04

CRRA specification, $\beta_2 = 0.5$

ζ	N=10^1		N=10^2		N=10^3		N=10^4		N=10^5		N=10^6	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
1.1	0.4828	0.3971	0.5024	0.0987	0.5006	0.032	0.5	0.0108	0.5	0.0032	0.5	0.001
1.2	0.4735	0.3831	0.5007	0.1017	0.5014	0.0312	0.5008	0.0112	0.4999	0.003	0.4999	9.70E-04
1.3	0.4637	0.4013	0.4976	0.0949	0.4971	0.0321	0.5008	0.0105	0.5003	0.0029	0.5	0.001
1.4	0.4935	0.4015	0.4947	0.0962	0.4991	0.0309	0.5001	0.0108	0.4996	0.0031	0.5	9.80E-04
1.5	0.4864	0.3779	0.5085	0.1039	0.5016	0.0302	0.4994	0.0092	0.5	0.0032	0.4999	9.40E-04
1.6	0.4897	0.4023	0.4941	0.0951	0.5004	0.0318	0.5005	0.01	0.5001	0.0035	0.5001	0.001
1.7	0.439	0.394	0.5051	0.1033	0.4974	0.0323	0.5	0.0097	0.5001	0.0032	0.5	0.001
1.8	0.5129	0.3881	0.4974	0.099	0.4985	0.0272	0.4992	0.0102	0.5	0.0029	0.5	0.001
1.9	0.471	0.352	0.5098	0.1084	0.4984	0.0293	0.5001	0.01	0.5003	0.003	0.5001	0.0011
2	0.4614	0.3528	0.5069	0.1042	0.5012	0.0304	0.5001	0.01	0.5002	0.0031	0.5	0.0011

CRRA specification, $\beta_2 = 1$

ζ	N=10^1		N=10^2		N=10^3		N=10^4		N=10^5		N=10^6	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
1.1	0.9841	0.3439	0.9896	0.099	0.9986	0.0313	0.999	0.0111	0.9998	0.0031	1	0.001
1.2	0.8985	0.4078	0.9963	0.1045	0.9986	0.0318	1	0.0096	0.9998	0.0034	1	1.00E-03
1.3	0.9528	0.4274	0.9953	0.1063	1.002	0.0313	1	0.01	0.9999	0.0031	0.9999	0.001
1.4	0.9429	0.3663	0.9846	0.1007	0.9988	0.0311	1	0.0094	1	0.0032	1	9.60E-04
1.5	0.956	0.3236	0.9902	0.106	1.001	0.03	0.9996	0.0098	1	0.003	1	9.60E-04
1.6	0.9432	0.3464	0.9892	0.1053	1.003	0.0326	1.001	0.0107	1	0.0031	1	9.50E-04
1.7	0.9055	0.3853	1.011	0.1033	0.9994	0.0295	0.9987	0.0107	0.9998	0.0031	1	9.80E-04
1.8	0.9162	0.43	0.993	0.0959	0.998	0.0331	1	0.0092	1	0.0033	1	1.00E-03
1.9	0.9335	0.3742	0.9995	0.1027	0.9987	0.0304	0.9993	0.01	1	0.0029	1	9.90E-04
2	0.9567	0.3554	0.9896	0.0989	0.9985	0.028	0.9994	0.0106	0.9996	0.0031	1	8.90E-04

Table 2: Monte Carlo Results for CARA specification

CARA specification, $\beta_2 = 0$

ζ	N=10^1		N=10^2		N=10^3		N=10^4		N=10^5		N=10^6	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
1.1	-2.76E-02	4.10E-01	-2.00E-03	9.99E-02	-1.90E-04	3.43E-02	1.20E-04	1.03E-02	-2.70E-04	3.20E-03	-1.70E-05	1.10E-03
1.2	4.37E-02	3.91E-01	5.30E-03	1.01E-01	3.70E-05	3.12E-02	6.40E-04	1.09E-02	4.30E-05	3.30E-03	-2.60E-05	1.00E-03
1.3	-1.16E-02	3.88E-01	1.70E-03	1.01E-01	-5.60E-04	3.19E-02	-2.20E-04	1.02E-02	-5.20E-05	3.30E-03	2.90E-05	1.10E-03
1.4	1.93E-02	3.44E-01	5.10E-03	1.02E-01	3.50E-03	3.18E-02	2.40E-04	9.40E-03	-1.20E-04	3.00E-03	-9.90E-05	9.70E-04
1.5	-9.00E-03	3.76E-01	-0.0116	0.1062	0.0016	3.24E-02	2.20E-06	0.0096	-2.50E-05	3.20E-03	1.20E-04	1.00E-03
1.6	0.0068	3.77E-01	-0.0024	0.1028	0.0013	0.0301	-9.60E-04	0.0104	7.10E-05	0.0028	1.80E-05	0.0011
1.7	-0.0119	0.3732	-0.0022	0.1044	-0.0013	0.0313	-5.70E-04	0.0095	-1.40E-04	0.0031	-1.30E-05	9.50E-04
1.8	-0.0679	0.4069	0.0071	0.1054	-2.10E-04	0.0308	3.70E-04	0.0105	-6.00E-05	0.0033	-3.60E-06	9.60E-04
1.9	-0.0249	0.3532	-0.0018	0.1029	-8.60E-04	0.0296	-2.00E-04	0.0099	1.80E-04	0.0031	1.20E-04	9.90E-04
2	-0.0464	0.3764	-1.70E-04	0.0927	0.0013	0.0343	2.00E-04	0.0093	-3.30E-05	0.0031	4.90E-05	9.90E-04

CARA specification, $\beta_2 = 0.5$

ζ	N=10^1		N=10^2		N=10^3		N=10^4		N=10^5		N=10^6	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
1.1	7.80E+27	1.20E+29	-1.70E+41	2.70E+42	-9.80E+43	1.50E+45	-2.10E+53	3.30E+54	-1.20E+63	1.50E+64	-1.00E+70	1.60E+7
1.2	-3.90E+17	6.10E+18	-1.80E+17	2.50E+18	-5.40E+21	8.20E+22	-5.60E+21	6.90E+22	-2.20E+30	3.10E+31	-2.60E+29	4.10E+3
1.3	-1.30E+07	1.40E+08	-6.20E+11	9.80E+12	-2.50E+11	3.50E+12	-3.00E+14	4.70E+15	-6.20E+15	7.80E+16	-2.50E+14	3.80E+1
1.4	-8.00E+04	9.50E+05	-4.90E+08	7.80E+09	-1.30E+07	2.00E+08	-1.20E+07	1.30E+08	-4.40E+07	4.00E+08	-5.30E+07	8.00E+0
1.5	-1.60E+04	1.90E+05	-4.10E+04	5.70E+05	-5131	5.10E+04	-1.50E+04	1.50E+05	-5.60E+04	8.10E+05	-2.10E+05	3.30E+0
1.6	-104.1	590.9	-261.4	2137	-61.02	728.9	-45.26	542.3	-3.799	28.19	-1.954	18.45
1.7	-27.36	144.5	-7.216	56.99	-2.724	15.35	-3.694	36.99	-0.7888	13.67	0.4479	0.4616
1.8	-27.99	212.9	-4.226	36.41	-0.1744	2.68	-0.011	6.446	0.4872	0.0345	0.4947	0.0525
1.9	-7.256	20.91	-0.4462	3.388	0.1094	3.787	0.4816	0.0414	0.497	0.009	0.4996	0.0014
2	-8.7	56.04	-0.1875	2.017	0.4281	0.1147	0.4918	0.0149	0.4989	0.0036	0.4999	1.00E-03

CARA specification, $\beta_2 = 1$

ζ	N=10^1		N=10^2		N=10^3		N=10^4		N=10^5		N=10^6	
	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE	Mean	RMSE
1.1	-2.60E+31	4.10E+32	-1.60E+42	2.50E+43	-5.00E+59	7.90E+60	-1.20E+51	1.90E+52	-3.90E+62	6.20E+63	-1.60E+65	1.80E+6
1.2	-9.50E+12	2.80E+14	-1.70E+17	2.70E+18	-1.70E+23	2.70E+24	-5.40E+23	8.50E+24	-5.30E+26	5.20E+27	-1.40E+30	1.50E+3
1.3	-1.40E+09	1.50E+10	-1.70E+09	1.60E+10	-7.80E+09	7.00E+10	-6.50E+14	7.40E+15	-3.00E+11	2.40E+12	-9.40E+12	6.50E+1
1.4	-1.30E+05	1.80E+06	-2.90E+06	4.40E+07	-1.90E+08	2.90E+09	-1.70E+08	2.60E+09	-9.10E+06	8.40E+07	-1.30E+09	1.60E+1
1.5	-8.90E+05	1.40E+07	-1083	8174	-4247	3.60E+04	-661.4	3469	-5094	5.10E+04	-1.50E+04	2.10E+0
1.6	-3084	4.90E+04	-165.6	1557	-109.3	897	-48.48	413.5	-11.95	69.4	-7.81	77.13
1.7	-349.2	4626	-31.05	233.5	-5.005	26.82	-1.084	14.65	-0.2382	9.304	0.9294	0.2115
1.8	-56.23	441.8	-13.52	156.1	-4.607	65.71	0.7726	1.027	0.9792	0.0649	0.9941	0.0271
1.9	-10.6	31.41	-2.88	20.34	0.6427	1.209	0.9567	0.1028	0.9864	0.1323	0.9995	0.0014
2	-31.71	391.7	-0.3129	3.292	0.8529	0.3989	0.9821	0.0311	0.9976	0.0093	0.9998	0.0011

Table 3: Risk Sharing: Maximal element, CRRA specification

CRRA specification, $\beta_2 = 0$

ζ	N=10^1		N=10^2		N=10^3		N=10^4		N=10^5		N=10^6	
	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$
1.1	0.337	0.337	0.334	0.334	0.336	0.336	0.33	0.33	0.337	0.337	0.34	0.34
1.2	0.306	0.306	0.297	0.297	0.311	0.311	0.302	0.302	0.311	0.311	0.301	0.301
1.3	0.273	0.273	0.29	0.29	0.285	0.285	0.283	0.283	0.28	0.28	0.29	0.29
1.4	0.254	0.254	0.259	0.259	0.247	0.247	0.254	0.254	0.26	0.26	0.252	0.252
1.5	0.238	0.238	0.233	0.233	0.233	0.233	0.214	0.214	0.22	0.22	0.217	0.217
1.6	0.213	0.213	0.203	0.203	0.188	0.188	0.189	0.189	0.192	0.192	0.204	0.204
1.7	0.191	0.191	0.175	0.175	0.153	0.153	0.158	0.158	0.166	0.166	0.156	0.156
1.8	0.17	0.17	0.149	0.149	0.135	0.135	0.121	0.121	0.134	0.134	0.13	0.13
1.9	0.176	0.176	0.123	0.123	0.105	0.105	0.101	0.101	0.082	0.082	0.103	0.103
2	0.153	0.153	0.112	0.112	0.086	0.086	0.067	0.067	0.066	0.066	0.057	0.057

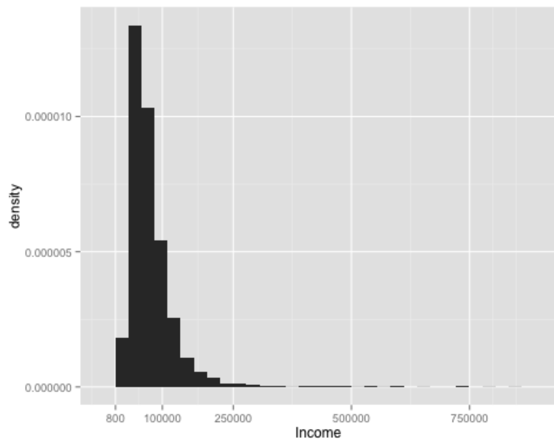
CRRA specification, $\beta_2 = 0.5$

ζ	N=10^1		N=10^2		N=10^3		N=10^4		N=10^5		N=10^6	
	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$
1.1	0.67	0.167	0.67	0.168	0.67	0.166	0.67	0.166	0.67	0.169	0.67	0.167
1.2	0.66	0.155	0.65	0.153	0.65	0.153	0.65	0.150	0.66	0.158	0.65	0.155
1.3	0.64	0.141	0.64	0.141	0.64	0.141	0.64	0.143	0.63	0.133	0.64	0.143
1.4	0.63	0.125	0.62	0.124	0.62	0.125	0.63	0.130	0.62	0.124	0.63	0.126
1.5	0.62	0.118	0.62	0.119	0.61	0.113	0.61	0.113	0.61	0.111	0.62	0.118
1.6	0.61	0.110	0.60	0.099	0.60	0.097	0.60	0.097	0.60	0.100	0.60	0.096
1.7	0.60	0.102	0.58	0.085	0.58	0.082	0.58	0.082	0.58	0.080	0.58	0.081
1.8	0.59	0.091	0.58	0.076	0.57	0.067	0.57	0.073	0.56	0.061	0.57	0.065
1.9	0.58	0.083	0.56	0.061	0.55	0.050	0.55	0.049	0.55	0.051	0.54	0.044
2	0.57	0.073	0.56	0.056	0.54	0.043	0.54	0.035	0.53	0.030	0.53	0.029

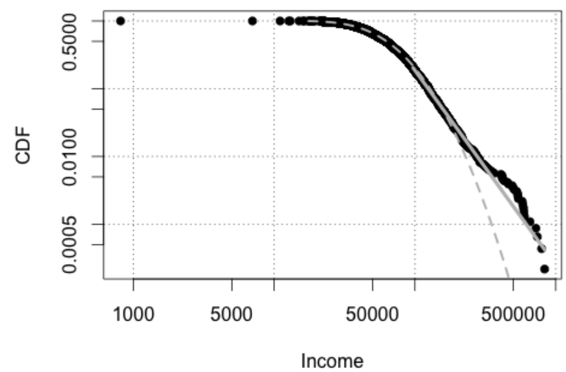
CRRA specification, $\beta_2 = 0.9$

ζ	N=10^1		N=10^2		N=10^3		N=10^4		N=10^5		N=10^6	
	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_{\|s\|_\infty}}{\delta\varepsilon_{\|s\|_\infty}}$	$\frac{\delta\Delta c_j}{\delta\varepsilon_{\|s\|_\infty}}$
1.1	0.93	0.034	0.93	0.034	0.93	0.034	0.93	0.033	0.93	0.033	0.93	0.033
1.2	0.93	0.031	0.93	0.031	0.93	0.031	0.93	0.031	0.93	0.031	0.93	0.031
1.3	0.93	0.029	0.93	0.028	0.93	0.028	0.93	0.028	0.93	0.028	0.93	0.027
1.4	0.93	0.026	0.93	0.025	0.93	0.026	0.93	0.025	0.93	0.026	0.93	0.026
1.5	0.92	0.024	0.92	0.023	0.92	0.022	0.92	0.024	0.92	0.022	0.92	0.022
1.6	0.92	0.021	0.92	0.020	0.92	0.020	0.92	0.019	0.92	0.018	0.92	0.020
1.7	0.92	0.021	0.92	0.017	0.92	0.016	0.92	0.015	0.92	0.017	0.92	0.016
1.8	0.92	0.017	0.92	0.015	0.91	0.015	0.91	0.013	0.91	0.013	0.91	0.012
1.9	0.92	0.017	0.91	0.012	0.91	0.011	0.91	0.010	0.91	0.009	0.91	0.010
2	0.92	0.016	0.91	0.011	0.91	0.009	0.91	0.007	0.91	0.006	0.91	0.007

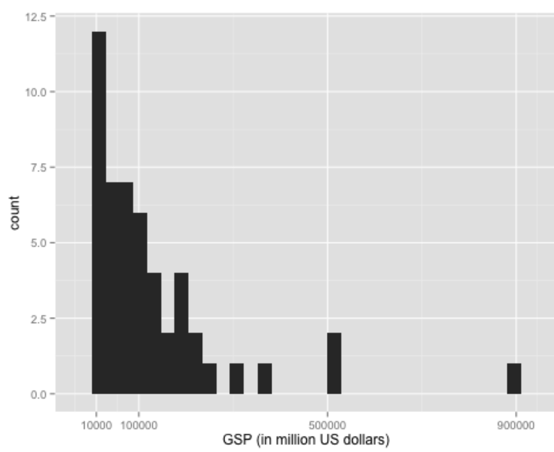
Figure 1: Power Law fits of HH Income and State Output Distributions



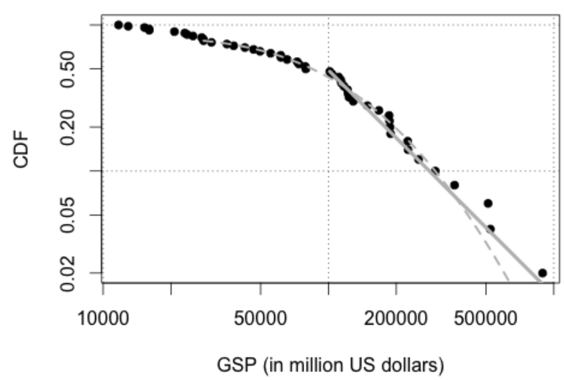
(a) Histogram, PSID Sample Income



(b) CDF, PSID Sample Income



(c) Histogram, US States GSP



(d) CDF, US States GSP

Table 4: Granularity Test, Household Risk Sharing - Top (100) Sample Income Growth, $s_{top100} = 0.12$

Dep. variable: consumption growth

	(1)	(2)	(3)	(4)	(5)
Agg. Income Growth of Sample			0.032***	0.026***	0.021***
(Excl. top 100)			(0.004)	(0.004)	(0.004)
Agg. Income Growth (top 100)		0.026***		0.013**	
(demeaned using mean of top 1000)		(0.004)		(0.004)	
Agg. Income Growth (top 100)					0.028***
(demeaned using mean of top 100)					(0.004)
Income Growth	0.032***	0.032***	0.032***	0.032***	0.032***
	(0.004)	(0.004)	(0.004)	(0.004)	(0.004)
N	58776	58776	58776	58776	58776
IC: Adjusted R-sq	0.00105	0.0017	0.0021	0.0022	**0.0027**
IC: AIC	74664.8	74626.6	74603.46	74598.14	**74564.15**

Table 5: Granularity Test, Household Risk Sharing - Population top percentiles Income Growth, $s_{top1\%} = 0.19$, $s_{top0.01\%} = 0.4$

Dep. variable: consumption growth

	(1)	(2)	(3)	(4)
Agg. Income Growth (bottom 99%)	0.004	0.002	0.002	0.003
	(0.004)	(0.004)	(0.004)	(0.004)
Agg. Income Growth (top 1%)		0.008**		
(demeaned using top 5%)		(0.004)		
Agg. Income Growth (top 0.5%)			0.009**	
(demeaned using top 1%)			(0.004)	
Agg. Income Growth (top 0.1%)				0.009**
(demeaned using top 0.5%)				(0.004)
Income Growth	0.031***	0.031***	0.031***	0.031***
	(0.004)	(0.004)	(0.004)	(0.004)
N	58776	58776	58776	58776
IC: Adjusted R-sq	0.00100	0.00106	**0.00108**	0.00106
IC: AIC	74097.82	74095.45	**74094.31**	74095.11

Table 6: Granularity Test, Household Risk Sharing - Manufacturing (top) Sector Growth, $s_{Manf.} \sim 0.21$

Dep. variable: consumption growth

	(1)	(2)	(3)
Agg. Income Growth (Non Manf.)	0.028***	0.028***	0.021***
	(0.004)	(0.004)	(0.05)
Agg. Sample Manf. Income Growth		0.009**	
(demeaned using sample Non-Manf.)		(0.004)	
Agg. Manf. Sector Growth			0.012**
			(0.004)
Income growth	0.030***	0.030***	0.030***
	(0.004)	(0.004)	(0.005)
N	49160	49160	49160
IC: Adjusted R-sq	0.0017	0.0018	**0.0020**
IC: AIC	63624.96	63622.33	**59692.6**

Table 7: Granularity Test, Regional Risk Sharing - US States, $s_{top5\%} = 0.25$, $s_{top1\%} = 0.125$

Dep. variable: Consumption Growth

	(1)	(2)	(3)
Agg. GSP Growth (bottom 95%)	0.173***	0.207**	0.199***
	(0.015)	(0.018)	(0.015)
Agg. GSP Growth (top 5%)		0.056**	
(demeaned using top 10%)		(0.016)	
Agg. GSP Growth (top 1%)			0.074***
(demeaned using top 5%)			(0.013)
Income Growth	0.688***	0.686***	0.693***
	(0.015)	(0.015)	(0.015)
N	2160	2160	2304
IC: Adjusted R-sq	0.619	0.641	**0.642**
IC: AIC	-10732.68	10838.82	**-10865.97**

6 Proofs

6.1 Proof of equation (2.33)

The proof in this section is based on Acemoglu et al. (2012). According to equation (2.32), we have:

$$\ln\left(\mu_{it}^{y}\right) = -\alpha \ln\left(z_{it}\right) + \alpha \ln\left(\mu_{t}^{L}\right) - B - (1-\alpha)\sum_{j=1}^{N} w_{ij}\ln\left(w_{ij}\right) + (1-\alpha)\sum_{j=1}^{N} w_{ij}\ln\left(\mu_{jt}^{y}\right), \tag{6.1}$$

where $B \equiv \alpha \ln(\alpha) + (1-\alpha)\ln(1-\alpha)$.

Multiplying (6.1) with the i-th element of the influence vector υ, we get:

$$\begin{aligned} \upsilon_i \ln\left(\mu_{it}^{y}\right) &= \alpha\upsilon_i \ln\left(z_{it}\right) - \alpha\upsilon_i \ln\left(\mu_{t}^{L}\right) \\ &\quad + B\upsilon_i + (1-\alpha)\,\upsilon_i\sum_{j=1}^{N} w_{ij}\ln\left(w_{ij}\right) + (1-\alpha)\,\upsilon_i\sum_{j=1}^{N} w_{ij}\ln\left(\mu_{jt}^{y}\right). \end{aligned} \tag{6.2}$$

Summing over all agents, we have:

$$\begin{aligned} \sum_{i=1}^{N}\left[\upsilon_i \ln\left(\mu_{it}^{y}\right)\right] &= \sum_{i=1}^{N}\left[\alpha\ln\left(z_{it}\right)\upsilon_i\right] - \sum_{i=1}^{N}\left[\alpha\ln\left(\mu_{t}^{L}\right)\upsilon_i\right] + \sum_{i=1}^{N}\left(B\upsilon_i\right) \\ &\quad + \sum_{i=1}^{N}\left[(1-\alpha)\,\upsilon_i\sum_{j=1}^{N} w_{ij}\ln\left(w_{ij}\right)\right] + \sum_{i=1}^{N}\left[(1-\alpha)\,\upsilon_i\sum_{j=1}^{N} w_{ij}\ln\left(\mu_{jt}^{y}\right)\right]. \end{aligned} \tag{6.3}$$

Note that the following equations hold:

$$\begin{aligned} \sum_{i=1}^{N}\left[(1-\alpha)\,\upsilon_i\sum_{j=1}^{N} w_{ij}\ln\left(w_{ij}\right)\right] &= (1-\alpha)\sum_{i=1}^{N}\left[\upsilon_i\sum_{j=1}^{N} w_{ij}\ln\left(w_{ij}\right)\right] \\ &= (1-\alpha)\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\upsilon_i w_{ij}\ln\left(w_{ij}\right)\right], \end{aligned} \tag{6.4}$$

$$\begin{aligned} \sum_{i=1}^{N}\left[(1-\alpha)\,\upsilon_i\sum_{j=1}^{N} w_{ij}\ln\left(\mu_{jt}^{y}\right)\right] &= (1-\alpha)\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\upsilon_i w_{ij}\ln\left(\mu_{jt}^{y}\right)\right] \\ &= (1-\alpha)\sum_{j=1}^{N}\left[\ln\left(\mu_{jt}^{y}\right)\sum_{i=1}^{N}\upsilon_i w_{ij}\right] \end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha)\sum_{j=1}^{N}\left[\ln\left(\mu_{jt}^{y}\right)\frac{1}{1-\alpha}\left(\upsilon_j-\frac{\alpha}{n}\right)\right] \\
&= \sum_{j=1}^{N}\left[\ln\left(\mu_{jt}^{y}\right)\left(\upsilon_j-\frac{\alpha}{n}\right)\right] \\
&= \sum_{j=1}^{N}\upsilon_j\ln\left(\mu_{jt}^{y}\right)-\frac{\alpha}{n}\left[\sum_{j=1}^{N}\ln\left(\mu_{jt}^{y}\right)\right].
\end{aligned}
\tag{6.5}
$$

The transition from the second line to the third line in the last series of equation made use of the definition of the influence vector $\upsilon_i \equiv \frac{\alpha}{N} + (1-\alpha)\sum_{j=1}^{N}\upsilon_j w_{ji}$.

Combining (6.3), (6.4) and (6.5), we get:

$$
\begin{aligned}
\sum_{i=1}^{N}\left[\upsilon_i\ln\left(\mu_{it}^{y}\right)\right] &= \alpha\sum_{i=1}^{N}\left[\ln\left(z_{it}\right)\upsilon_i\right]-\alpha\sum_{i=1}^{N}\left[\ln\left(\mu_{t}^{L}\right)\upsilon_i\right]+B\sum_{i=1}^{N}\left(\upsilon_i\right) \\
&\quad +(1-\alpha)\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\upsilon_i w_{ij}\ln\left(w_{ij}\right)\right]+\sum_{j=1}^{N}\upsilon_j\ln\left(\mu_{jt}^{y}\right)-\frac{\alpha}{n}\left[\sum_{j=1}^{N}\ln\left(\mu_{jt}^{y}\right)\right].
\end{aligned}
\tag{6.6}
$$

Note[31] that $B\sum_{i=1}^{N}\left(\upsilon_i\right)=B$. Rearranging (6.6), we obtain the expression (2.33) that is used in the main text:

$$
-\frac{1}{n}\sum_{j=1}^{N}\ln\left(\mu_{jt}^{y}\right) = \sum_{i=1}^{N}\left(\upsilon_i\ln\left(z_{it}\right)\right)+\frac{1}{\alpha}B+\frac{(1-\alpha)}{\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\upsilon_i w_{ij}\ln\left(w_{ij}\right)\right]-\ln\left(\mu_{t}^{L}\right).
$$

6.2 Competitive equilibrium in the networked production economy

In this section, we characterize the competitive equilibrium of the economy presented in Section 2.4.

Definition 5. A competitive equilibrium with n sectors of intermediate goods and one sector of final good in period t consists of the intermediate-goods prices $p_{1t}, \ldots, p_{Nt}$, the labour wage h_t, the

[31] Using the definition of the influence vector:

$$
\begin{aligned}
\sum_{i=1}^{N}\left(\upsilon_i\right) = \quad & \textstyle\sum_{i=1}^{N}\left[\frac{\alpha}{N}+(1-\alpha)\sum_{j=1}^{N}\upsilon_j w_{ji}\right] &&= \alpha+(1-\alpha)\sum_{i=1}^{N}\sum_{j=1}^{N}\upsilon_j w_{ji} \\
& &&= \alpha+(1-\alpha)\sum_{j=1}^{N}\left(\upsilon_j\sum_{i=1}^{N}w_{ji}\right) \\
& &&= \alpha+(1-\alpha)\sum_{j=1}^{N}\upsilon_j \\
\Rightarrow \quad & \textstyle\sum_{i=1}^{N}\left(\upsilon_i\right)=1. &&
\end{aligned}
$$

quantity of the final good Y_t, the quantities of inputs for the final good $y_{1t}, \ldots, y_{Nt}$, the quantities of intermediate goods x_{ijt}, and the quantity of labour $l_{1t}, \ldots, l_{Nt}$, such that:

(i) consumers maximize their utility,

(ii) firms in the intermediate and final good sectors maximize their profit,

(iii) labour and commodity markets clear.

Each consumer i solves the following utility maximization problem:

$$\mathbb{E}\left[\sum_{t=0}^{\infty} (\beta_i)^t \, u\left(c_{it}, \, b_{it}\right)\right], \tag{6.7}$$

subject to:

$$c_{it} \leq h_t l_{it}$$

$$0 \leq l_{it} \leq l_{it}^e.$$

We have assumed that consumers do not value consumption in their objective function (6.7) and, therefore, they supply labour inelastically. In addition, we have assumed away the possibility of savings. Thus, consumers optimally choose to consume all the income that they earn in each period t, i.e., $c_{it} = h_t l_{it}$, $\forall t \in \{0, ..., \infty\}$.

The profit maximization[32] for the producer of the final consumption good is:

$$\max_{\{Y_t, \, y_{it}\}} Y_t - \sum_{i=1}^{N} \left(p_{it} y_{it}\right),$$

subject to $Y_t = A_t \prod_{i=1}^{N} \left(y_{it}\right)^{\frac{1}{N}}$.

The first order conditions of this profit maximization problem yield:

$$y_{it} = \frac{Y_t}{N p_{it}}.$$

Let h_t denote the labour wage in period t. Then, the profit maximization problem for the producer of the intermediate good in sector i is:

$$\max_{\{l_{it}, \, x_{it}, \, x_{i1t}, \ldots, \, x_{iNt}\}} p_{it} x_{it} - \sum_{j=1}^{N} p_{jt} x_{ijt} - h_t l_{it}$$

subject to $x_{it} = z_{it}^{\alpha} l_{it}^{\alpha} \prod_{j=1}^{N} x_{ijt}^{(1-\alpha) w_{ijt}}$.

The first order conditions yield:

$$l_{it} = \frac{\alpha p_{it} x_{it}}{h_t} \tag{6.8}$$

[32]For simplicity, we treat the final consumption good as the numeraire good and normalize its price to unity.

$$x_{ijt} = \frac{(1-\alpha)\, p_{it} w_{ijt} x_{it}}{p_{jt}}. \tag{6.9}$$

Under market clearing for the labour and commodity markets, the following equalities hold:

$$\begin{aligned} \sum_{i=1}^{N} c_{it} &= Y_t \\ y_{it} + \sum_{j=1}^{N} x_{jit} &= x_{it} \\ \sum_{i=1}^{N} l_{it} &= L_t \end{aligned} \tag{6.10}$$

Substituting (6.8) and (6.9) into (6.10), we obtain:

$$\frac{Y_t}{N p_{it}} + \sum_{j=1}^{N} \left[\frac{(1-\alpha)\, p_{jt} w_{jit} x_{jt}}{p_{it}} \right] = x_{it}.$$

Multiplying both sides by p_{it} and denoting as ς_{it} the value of sales of firm i at time t, we obtain the following expression:

$$\varsigma_{it} = \frac{Y_t}{N} + (1-\alpha) \sum_{j=1}^{N} \left[w_{jit} \varsigma_{jt} \right]. \tag{6.11}$$

In vector form, equation (6.11) can be re-written as:

$$\begin{aligned} \varsigma_t' &= \frac{Y_t}{N} \mathbf{1}' \left[I - (1-\alpha) W_{nt} \right]^{-1} \\ &= \frac{Y_t}{\alpha} \upsilon_{nt}'. \end{aligned} \tag{6.12}$$

The last line in the equation above follows from applying the definition of the influence vector. The element-wise equalities implied by equation (6.12) can be used to obtain an expression for the sales ratio of each sector:

$$\varsigma_{it} = \frac{Y_t}{\alpha} \upsilon_{it} \tag{6.13}$$

$$\Rightarrow \sum_{i=1}^{N} \varsigma_{it} = \frac{Y_t}{\alpha} \sum_{i=1}^{N} \upsilon_{it} = \frac{Y_t}{\alpha}. \tag{6.14}$$

Dividing (6.13) by (6.14), we obtain:

$$\frac{\varsigma_{it}}{\sum_{j=1}^{N} \varsigma_{jt}} = \upsilon_{it}$$

Thus, the sales ratio for each sector is equal to the Bonacich centrality of the sector in the networked production economy.

6.3 Order Statistics for Power Law Distributions

We consider a pareto-type power law distribution with c.d.f,

$$P(y) = \int_y^\infty p(y')\mathrm{d}y' = 1 - cy^{-\zeta} \tag{6.15}$$

The *m-th* moment $\mathbb{E}(y_{k,n})^m$ of the *k-th* order statistic from this power law distribution is finite for $m < \zeta(n-k+1)$ and is given by (Nevzorov 2001, Assignment 6.2.),

$$\mathbb{E}(y_{k,n})^m = \frac{n! \cdot \Gamma\left(n-k+1-\frac{m}{\zeta}\right)}{(n-k)! \cdot \Gamma\left(n+1-\frac{m}{\zeta}\right)} \tag{6.16}$$

with $\Gamma(a)$, the standard Gamma function:

$$\Gamma(a) = \int_0^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$$

The first moment of the *k-th* order statistic is,

$$\mathbb{E}(y_{k,n}) = \frac{n!\left(n-k-\frac{1}{\zeta}\right)!}{(n-k)!\left(n-\frac{1}{\zeta}\right)!} \tag{6.17}$$

It is easy to see that for Zipf's law with $\zeta = 1$, Equation (6.17) reduces to, $\mathbb{E}(y_{k,n}) = \frac{n}{n-k}$. For, any $\zeta > 1$, and for a large n and large k[33] we can approximate the Γ-functions (and factorials) of Equation (6.17) using Sterling's formula,

$$\begin{aligned}
\mathbb{E}(y_{k,n}) &\sim \frac{n^{n+\frac{1}{2}}\left(n-k-\frac{1}{\zeta}\right)^{n-k-\frac{1}{\zeta}+\frac{1}{2}}}{(n-k)^{n-k+\frac{1}{2}}\left(n-\frac{1}{\zeta}\right)^{n-\frac{1}{\zeta}+\frac{1}{2}}} \\
&= \frac{n^{n+\frac{1}{2}}\left(n-k-\frac{1}{\zeta}\right)^{n-k+\frac{1}{2}}}{(n-k)^{n-k+\frac{1}{2}}\left(n-\frac{1}{\zeta}\right)^{n+\frac{1}{2}}} \cdot \left(\frac{n-k-\frac{1}{\zeta}}{n-\frac{1}{\zeta}}\right)^{-\frac{1}{\zeta}} \\
&\sim \left(\frac{n-k}{n}\right)^{-\frac{1}{\zeta}} = \left(\frac{j}{n}\right)^{-\frac{1}{\zeta}}
\end{aligned} \tag{6.18}$$

, where j is the rank (in descending order) of the observation in the sequence.

[33] The approximations can also be made with large n and a fixed value of k (for lower rank order statistics) with similar results

6.4 Limiting Distribution of the Bias

We first state Lévy's Theorem (adapted from Durrett (1996)) which is a generalized version of the central limit theorem that relaxes the assumption of the finiteness of the variance and identifies a new family of limiting distributions.

Theorem 1. *Lévy's Theorem: Suppose that $X_1, X_2, ...$ are i.i.d. with a distribution that satisfies (i) $l \lim_{x \overset{\infty}{\longrightarrow}} \mathbb{P}(X_1 > x)/\mathbb{P}(|X_1| > x) = \theta \in [0.1]$ and (ii) $\mathbb{P}(|X_1| > x) = x^{-\zeta} L(x)$ with zeta $\in (0, 2)$ and $L(x)$ slowly varying. Let $s_n = \sum_{i=1}^{\infty} X_i$, $a_n = \inf\{x : \mathbb{P}(|X_1| > x) \leq 1/n\}$, and $b_n = nE[X_1 1_{|X_1| \leq a_n}]$. As $n \overset{\infty}{\longrightarrow}$, $(s_n - b_n)/a_n$ converges in distribution to a nondegenerate random variable Y, which follows a Lévy distribution with exponent ζ.*

The true data generating model, given CRRA preferences and an endowment-based economy is shown in Section (3) to be:

$$\Delta \ln (c_i) = \beta_1 \frac{\sum_{i=1}^{N} (y_i \sigma \varepsilon_i)}{Y} + \beta_2 \left(1 - \frac{y_i \sigma}{Y}\right) \varepsilon_i + u_i. \tag{6.19}$$

Denote $\varepsilon_i^* = \left(1 - \frac{y_i \sigma}{Y_t}\right) \varepsilon_i$. Then, equation (6.20) can be written as:

$$\Delta \ln (c_i) = \beta_2 \varepsilon_i^* + u_i. \tag{6.20}$$

Consider the following misspecified model:

$$\Delta \ln (c_i) = \widetilde{\beta}_2 \varepsilon_i + u_i. \tag{6.21}$$

The estimator $\widetilde{\beta}_2$ from the misspecified model can be obtained from:

$$\begin{aligned}
\widetilde{\beta}_2 &= \frac{\sum\limits_{i=1}^{N} \varrho_i \varepsilon_i}{\sum\limits_{i=1}^{N} \varepsilon_i^2} \\
&= \frac{\sum\limits_{i=1}^{N} (\beta_2 \varepsilon_i^* + u_i) \varepsilon_i}{\sum\limits_{i=1}^{N} \varepsilon_i^2} \\
&= \beta_2 \left(\frac{\sum\limits_{i=1}^{N} \varepsilon_i^* \varepsilon_i}{\sum\limits_{i=1}^{N} \varepsilon_i^2} \right) + \frac{\sum\limits_{i=1}^{N} \varepsilon_i u_i}{\sum\limits_{i=1}^{N} \varepsilon_i^2}
\end{aligned}$$

The numerator of the first term (representing the bias) can be written as,

$$\sum_{i=1}^{N}(\varepsilon_i^*\varepsilon_i) = \sum_{i=1}^{N}\left[\left(1-\frac{y_{it}\sigma_i}{Y}\right)\varepsilon_i^2\right] \tag{6.22}$$
$$= \sum_{i=1}^{N}\varepsilon_i^2 - \frac{\sigma}{Y}\sum_{i=1}^{N} y_i\varepsilon_i^2$$

Since ε is normally distributed, the law of large number applies and the first term of the expression is equal to $N\mathbb{E}[\varepsilon_i^2] = N$. For the second term, when $\zeta > 1$, by the law of large numbers, $Y = \sum_{i=1}^{N} y_i = N\mathbb{E}\left[y_i\right]$. For, $\sum_{i=1}^{N} y_i\varepsilon_i^2$, observe that, for the variable $y_i^{\alpha_1}\varepsilon_i^{\alpha_2}$,

$$\mathbb{P}(|y_i^{\alpha_1}\varepsilon_i^{\alpha_2}| > x) = \mathbb{P}\left(y_i > \left(\frac{x}{|\varepsilon_i^{\alpha_2}|}\right)^{\frac{1}{\alpha_1}}\right) \tag{6.23}$$
$$\sim \mathbb{E}\left[|\varepsilon_i|^{\frac{\zeta\alpha_2}{\alpha_1}}\right] x^{\frac{-\zeta}{\alpha_1}} \tag{6.24}$$

Therefore, for $\zeta > 1$, Lévy's Theorem implies that,

$$\frac{1}{N^{\frac{1}{\zeta}}\mathbb{E}\left[|\varepsilon|_i^{2\zeta}\right]}\left(\sum_{i=1}^{N} y_i\varepsilon_i^2 - N\mathbb{E}(y_i\varepsilon_i^2)\right) \xrightarrow{d} g_\zeta \tag{6.25}$$

where g_ζ is a random variable that follows a Lévy distribution with exponent ζ with $a_N = N^{\frac{1}{\zeta}}\mathbb{E}\left[|\varepsilon|_i^{2\zeta}\right]$, $b_N = \mathbb{E}(y_i\varepsilon_i^2)$ is the scaling factor as per Equation (6.23).

Assuming independence between y_i and ε_i with $\mathbb{E}(\varepsilon_i^2) = 1$, Equation (6.22) converges to to,

$$\sum_{i=1}^{N}(\varepsilon_i^*\varepsilon_i) \to N - \frac{\sigma}{N\mathbb{E}\left[y_i\right]}N\mathbb{E}(y_i\varepsilon_i^2) \tag{6.26}$$
$$\to N - \sigma$$

The limiting distribution of the bias in the estimate of β_2 under the misspecified model can be obtained as:

$$\widetilde{\beta}_2 - \beta_2 = \beta_2\left(\frac{\sum_{i=1}^{N}\varepsilon_i^*\varepsilon_i}{\sum_{i=1}^{N}\varepsilon_i^2} - 1\right) + \frac{\sum_{i=1}^{N}\varepsilon_i u_i}{\sum_{i=1}^{N}\varepsilon_i^2} \tag{6.27}$$

We assume that ε_i and u_i are independent which implies that the second term of the expression converges in probability to zero. When $\zeta > 1$:

$$
\begin{aligned}
\widetilde{\beta}_2 - \beta_2 =& \beta_2 \left(\frac{\sum\limits_{i=1}^{N} \varepsilon_i^* \varepsilon_i}{\sum\limits_{i=1}^{N} \varepsilon_i^2} - 1 \right) + \frac{\sum\limits_{i=1}^{N} \varepsilon_i u_i}{\sum\limits_{i=1}^{N} \varepsilon_i^2} \\
\to& \beta_2 \left(\frac{N - \sigma}{N} - 1 \right) + 0 \\
\to& \beta_2 \left(-\frac{\sigma}{N} \right)
\end{aligned}
\tag{6.28}
$$

For zipf's law case, Lévy's Theorem applied to $\zeta = 1$, gives,

$$
\frac{1}{N^{\frac{1}{\zeta}} \mathbb{E}\left[|\varepsilon|_i^{2\zeta} \right]} \left(\sum_{i=1}^{N} y_i \varepsilon_i^2 - N \ln N \right) \xrightarrow{d} g_\zeta \tag{6.29}
$$

Applying the $(\ln N)$ correction to both numerator and denominator of Equation (6.22),

$$
\widetilde{\beta}_2 - \beta_2 \to \beta_2 \left(-\frac{\sigma}{N} \right) \tag{6.30}
$$

In contrast to the CRRA specification, one could also consider a CARA specification. In this case, it is easy to see, based on the above, that with $\zeta > 1$, the bias decays asymptotically, as the size of the risk-sharing group increases, and is of order $(1/N)$. However, with $\zeta = 1$, a $(\ln N)$ correction applies, but only to the numerator and the bias is or order $(\ln N/N)$. With $\zeta < 1$, the bias converges to a Lévy-distributed random variable, g_ζ with exponent ζ and scales at $\left(\frac{1}{N^{2-\frac{1}{\zeta}}} \right)$.

Instead of a distribution of endowments, we could also consider a production network structure. It can be seen from the above analysis that the rate of convergence of the bias depends on the term $N^{-1} \sum\limits_{i=1}^{N} \frac{y_i \sigma \varepsilon_i^2}{Y}$. Note that in the network production case, the term $\sum\limits_{i=1}^{N} \frac{y_i \sigma \varepsilon_i^2}{Y}$ is replaced by $\sum\limits_{i=1}^{N} \upsilon_i \sigma \varepsilon_i^2$, where υ_i denotes the centrality of each agent in the production network. This term can be expanded as,

$$
\sum_{i=1}^{N} \upsilon_i \sigma \varepsilon_i^2 \geq \sigma \sum_{i=1}^{N} \left[\frac{a}{N} + \frac{a}{N} (1-a) d_i \right] \varepsilon_i^2 \tag{6.31}
$$

$$
= \sigma \left[\frac{a}{N} \sum_{i=1}^{N} \varepsilon_i^2 + \frac{a}{N} (1-a) \sum_{i=1}^{N} \left(d_i \varepsilon_i^2 \right) \right] \tag{6.32}
$$

If the distribution of degrees is scale-free with a power-law distribution parameter $\zeta > 1$, then, as before, Lévy's Theorem implies that $\sum_{i=1}^{N} d_i \varepsilon_i^2$ converges in distribution to g_ζ with exponent ζ as per Equation (6.25). The second term scales at $\mathcal{O}_p(N)$. As a result, we can see that the bias, which depends on the term $N^{-1} \sum\limits_{i=1}^{N} \upsilon_i \sigma \varepsilon_i^2$, scales at $\mathcal{O}_p\left(\frac{1}{N}\right)$ when income is generated from a scale free network with parameter $\zeta > 1$.

If the distribution of degrees is scale-free with a power-law distribution parameter $\zeta = 1$ (zipf), note that by Lévy's Theorem, $\sum_{i=1}^{N} d_i \varepsilon_i^2$ converges in distribution to g_ζ with exponent ζ as per Equation (6.29), i.e. with $b_n = \ln N$. As a result, the second term in Equation (6.31) scales $\mathcal{O}_p(N \ln N)$. In this case, the bias, which depends on the term $N^{-1} \sum\limits_{i=1}^{N} \upsilon_i \sigma \varepsilon_i^2$, scales at $\mathcal{O}_p\left(\frac{\ln N}{N}\right)$ when income is generated from a scale free network with parameter $\zeta = 1$.

If the distribution of degrees is scale-free with a power-law distribution parameter $\zeta < 1$, note that by Lévy's Theorem, $\sum_{i=1}^{N} d_i \varepsilon_i^2$ converges in distribution to g_ζ with exponent ζ as per Equation (6.25) but with $b_n = 0$. As a result, the second term in Equation (6.31) scales $\mathcal{O}_p(N^{\frac{1}{\zeta}})$. In this case, the bias, which depends on the term $N^{-1} \sum\limits_{i=1}^{N} \upsilon_i \sigma \varepsilon_i^2$, scales at $\mathcal{O}_p\left(\frac{1}{N^{2-\frac{1}{\zeta}}}\right)$ when income is generated from a scale free network with parameter $\zeta < 1$.

7 Robustness checks

The risk-sharing literature has typically focused on three members of the family of hyperbolic absolute risk aversion (HARA) utility functions. Similarly to these studies, we include in our analysis all of these types of utility function to examine the effects of idiosyncratic shocks on the optimal risk-sharing allocations

First, we consider the Constant Absolute Risk Aversion (CARA) utility which is obtained when $\gamma \to -\infty$ and $\delta = -1$. Then, we examine the Constant Relative Risk Aversion (CRRA) utility function which is obtained when $\gamma > 0$ and $\delta = 0$. Finally, by assuming that $\delta = 1$ and $a = \gamma > 0$ we obtain the Decreasing Absolute Risk Aversion (DARA) preferences. The CARA and CRRA utility functions were adopted in the classic risk-sharing models of Mace (1991), Cochrane (1991), Townsend (1994) and Obstfeld (1995). DARA preferences was suggested by Ogaki and Zhang (2001) and more recently, Mazzocco (2012) incorporate full heterogeneity in risk preferences.

7.1 Constant Absolute Risk Aversion (CARA)

The CARA or exponential utility function adopted in this section takes the form:

$$u\left(c_{it},\, b_{it}\right) = -\frac{1}{\gamma} e^{-\gamma\left(c_{it} - b_{it}\right)}. \tag{7.1}$$

Taking the partial derivative with respect to consumption, substituting into the first order condition (2.8) and solving for c_{it} we obtain:

$$c_{it} = -\frac{1}{\gamma}\ln(\kappa_t) + \frac{1}{\gamma}\ln\left(\lambda_i (\beta_i)^t\right) + b_{it}. \tag{7.2}$$

Aggregating over the N agents of the economy, we get:

$$\begin{aligned}\sum_{i=1}^{N} c_{it} &= -\frac{1}{\gamma}\sum_{i=1}^{N}\ln(\kappa_t) + \frac{1}{\gamma}\sum_{i=1}^{N}\ln\left(\lambda_i (\beta_i)^t\right) + \sum_{i=1}^{N} b_{it} \\ \Rightarrow Y_t &= -\frac{N}{\gamma}\ln(\kappa_t) + \frac{1}{\gamma}\sum_{i=1}^{N}\ln\left(\lambda_i (\beta_i)^t\right) + \sum_{i=1}^{N} b_{it}, \end{aligned} \tag{7.3}$$

where the last equation follows from substituting the resource constraint (2.6) and from exploiting the fact that the (modified) Lagrange multiplier κ_t and the preference parameters are constant across individuals. Solving for $\ln(\kappa_t)$, we get:

$$\ln(\kappa_t) = -\frac{\gamma}{N}Y_t + \frac{1}{N}\sum_{i=1}^{N}\ln\left(\lambda_i (\beta_i)^t\right) + \frac{\gamma}{N}\sum_{i=1}^{N} b_{it}. \tag{7.4}$$

Take the first difference between two points in time of the optimal consumption derived in (7.2):

$$\Delta c_{it+1} = -\frac{1}{\gamma}\Delta\ln(\kappa_{t+1}) + \frac{1}{\gamma}\Delta\ln\left(\lambda_i (\beta_i)^{t+1}\right) + \Delta b_{it+1}. \tag{7.5}$$

Substituting for $\ln(\kappa_t)$ in the equation above, we have:

$$\Delta c_{it+1} = \frac{1}{N}\Delta Y_{t+1} - \frac{1}{N\gamma}\Delta\left[\sum_{i=1}^{N}\ln\left(\lambda_i (\beta_i)^{t+1}\right)\right] + \frac{1}{\gamma}\Delta\ln\left(\lambda_i (\beta_i)^{t+1}\right) - \frac{1}{N}\Delta\left(\sum_{i=1}^{N} b_{it+1}\right) + \Delta b_{it+1}. \tag{7.6}$$

Equation (7.6) relates the pareto optimal changes in consumption to changes in total endowments and, therefore, changes in total consumption. According to equation (2.4), the changes in aggregate endowments can be written as:

$$\Delta Y_{t+1} = \sum_{i=1}^{N}\Delta y_{it+1} = \sum_{i=1}^{N}\left(y_{it}\sigma_i\varepsilon_{it+1}\right). \tag{7.7}$$

Substituting the changes in total endowments into (7.6) and denoting $\Lambda_{it+1} \equiv \lambda_i (\beta_i)^{t+1}$ we have:

$$\Delta c_{it+1} = \frac{1}{N}\sum_{i=1}^{N}\left(y_{it}\sigma_i\varepsilon_{it+1}\right) - \frac{1}{N\gamma}\Delta\left[\sum_{i=1}^{N}\ln(\Lambda_{it+1})\right] + \frac{1}{\gamma}\Delta\ln(\Lambda_{it+1}) - \frac{1}{N}\Delta\left(\sum_{i=1}^{N} b_{it+1}\right) + \Delta b_{it+1}. \tag{7.8}$$

In the following three subsections, we first derive the asymptotic result that idiosyncratic shocks do not matter when the size of endowments is equal across the agents of the economy or when the endowments are distributed with a finite variance. Nevertheless, we also show in the last subsection that idiosyncratic shocks can cause aggregate fluctuations in total consumption when the initial distribution of endowments follows a power law distribution.

7.1.1 Identical-sized endowments

Assume that endowments are equal across individuals so that $y_{it} = (1/N)Y_t$ for every agent i. Then, we can rewrite (7.8) as follows:

$$\Delta c_{it+1} = \frac{Y_t}{N^2}\sum_{i=1}^{N}(\sigma_i \varepsilon_{it+1}) - \frac{1}{N\gamma}\Delta\left[\sum_{i=1}^{N}\ln(\Lambda_{it+1})\right] + \frac{1}{\gamma}\Delta\ln(\Lambda_{it+1}) - \frac{1}{N}\Delta\left(\sum_{i=1}^{N}b_{it+1}\right) + \Delta b_{it+1}. \quad (7.9)$$

Taking the partial derivative of the changes in the optimal consumption of individual i with respect to any idiosyncratic endowment shock ε_{jt+1} (including $j = i$), we get:

$$\frac{\partial(\Delta c_{it+1})}{\partial \varepsilon_{jt+1}} = \frac{\sigma_j y_{jt}}{N} = \sigma_j\left(\frac{Y_t}{N^2}\right).$$

The partial effect of ε_{jt+1} becomes 0 as $N \to \infty$.

7.1.2 Endowments distributed with finite variance

If endowments are distributed with a finite variance, then the partial derivative of equation (7.8) with respect to the idiosyncratic endowment shock ε_{jt+1} becomes equal to:

$$\frac{\partial(\Delta c_{it+1})}{\partial \varepsilon_{jt+1}} = \frac{\sigma_j y_{jt}}{N}.$$

Given the finite variance assumption, then y_{jt} remains a positive real number and the partial effect of ε_{jt+1} becomes 0 as $N \to \infty$.

Combining the results of Sections 7.1.1 and 7.1.2 we derive the following proposition:

Proposition 7. *Assuming CARA preferences, the partial effect of an idiosyncratic shock to the j-th agent's endowment on the optimal consumption of the i-th individual becomes 0 as $N \to \infty$ when:(i) endowments are distributed uniformly across agents or (ii) endowments are distributed with a finite variance across agents.*

7.1.3 Endowments following a power law distribution

Under the assumption that endowments follow a power law distribution, the j-th largest endowment in the population of N agents is approximately equal to:

$$y_{jt} = \left(\frac{j}{N}\right)^{-\frac{1}{\zeta}},$$

where $\zeta \geq 1$ denotes the exponent of the power law distribution of endowments (i.e., ζ is such that $\mathbb{P}(y > x) = x^{-\zeta}$). Substituting the above expression of the size of the endowment into equation (7.8), we get:

$$\Delta c_{it+1} = \frac{1}{N}\sum_{i=1}^{N}\left(\left(\frac{i}{N}\right)^{-\frac{1}{\zeta}}\sigma_i\varepsilon_{it+1}\right) - \frac{1}{N\gamma}\Delta\left[\sum_{i=1}^{N}\ln\left(\Lambda_{it+1}\right)\right] + \frac{1}{\gamma}\Delta\ln\left(\Lambda_{it+1}\right) - \frac{1}{N}\Delta\left(\sum_{i=1}^{N}b_{it+1}\right) + \Delta b_{it+1}. \tag{7.10}$$

Thus, the partial derivative of the changes in the optimal consumption of individual i with respect to any idiosyncratic endowment shock ε_{jt+1} becomes equal to:

$$\frac{\partial\left(\Delta c_{it+1}\right)}{\partial\varepsilon_{jt+1}} = \sigma_j\left(\frac{j^{-\frac{1}{\zeta}}}{N^{1-\frac{1}{\zeta}}}\right).$$

The effect of an idiosyncratic shock ε_{jt+1} as the number of agents becomes infinitely large will depend on the value of the parameter ζ.

- if $\zeta > 1$ then:

$$\frac{\partial\left(\Delta c_{it+1}\right)}{\partial\varepsilon_{jt+1}} = \sigma_j\left(\frac{j^{-\frac{1}{\zeta}}}{N^{\frac{\zeta-1}{\zeta}}}\right)$$

  Therefore, the partial effect on changes in individual consumption is positive for small N, but becomes 0 as $N \to \infty$. For example, the partial effect of ε_{jt+1} on Δc_{it+1} decays according to $\sqrt{N}$ if $\zeta = 2$.

- if $\zeta = 1$, then:

$$\frac{\partial\left(\Delta c_{it+1}\right)}{\partial\varepsilon_{jt+1}} = \sigma_j\left(\frac{j^{-\frac{1}{\zeta}}}{N^{1-\frac{1}{\zeta}}}\right) = \frac{\sigma_j}{j} > 0 \tag{7.11}$$

  Note that the above inequality holds asymptotically as $N \to \infty$.

The implications of the results obtained above are stated more formally in the following proposition:

Proposition 8. *Assuming CARA preferences, the partial effect of an idiosyncratic shock to the j-th largest endowment on the optimal consumption of the i-th individual:*

(i) remains positive when the size of endowments follows a power-law distribution with $\zeta = 1$ *(i.e., a Zipf distribution) and*

(ii) decays according to $\frac{1}{j^{\frac{1}{\zeta}} N^{\frac{\zeta-1}{\zeta}}}$ *when the size of endowments follows a power-law distribution with* $\zeta > 1$.

7.2 Decreasing Absolute Risk Aversion (DARA)

The DARA or generalized Stone-Geary utility function that we consider in this section has the following form:

$$u\left(c_{it},\, b_{it}\right) = b_{it}\frac{\left(c_{it} - \delta\right)^{1+\gamma} - 1}{1+\gamma} \tag{7.12}$$

The expression above represents the case of Constant Relative Risk Aversion (CRRA) preferences, which have been examined by Mace (1991) and Cochrane (1991), when $\delta = 0$. Okagi and Zhang (2001) and Mazzocco and Saini (2012) have argued in favour of facilitating in the analysis of risk-sharing the possibility that δ can take values different than zero. More specifically, for a positive δ the utility function above exhibits Decreasing Relative Risk Aversion. In this case, the positive parameter δ is often referred to as the subsistence level of consumption. Note that the marginal utility of consumption tends to infinity when c_{it} approaches δ. In the case of a negative-valued δ, preferences exhibit Increasing Relative Risk Aversion and the individual is assumed to have a finite marginal utility when $c_{it} = 0$.

Substituting the partial derivative of the DARA utility function with respect to consumption into the first order condition (2.8) and solving for the difference of c_t^i from the 'subsitstence' parameter δ, we get:

$$c_{it} - \delta = \left(\frac{\kappa_t}{b_{it}\lambda_i\left(\beta_i\right)^t}\right)^{\frac{1}{\gamma}}. \tag{7.13}$$

Aggregating over the N agents of the economy, we have:

$$\begin{aligned} \sum_{i=1}^{N} c_{it} - \delta N &= \sum_{i=1}^{N}\left(\frac{\kappa_t}{b_{it}\lambda_i\left(\beta_i\right)^t}\right)^{\frac{1}{\gamma}} \\ \Rightarrow Y_t - \delta N &= \left(\kappa_t\right)^{\frac{1}{\gamma}}\sum_{i=1}^{N}\left(b_{it}\lambda_i\left(\beta_i\right)^t\right)^{-\frac{1}{\gamma}} \end{aligned} \tag{7.14}$$

The last equality can be used to obtain an expression for $\ln\left(\kappa_t\right)$:

$$\ln\left(\kappa_t\right) = \gamma\ln\left(Y_t - \delta N\right) - \gamma\ln\left[\sum_{i=1}^{N}\left(b_{it}\lambda_i\left(\beta_i\right)^t\right)^{-\frac{1}{\gamma}}\right]. \tag{7.15}$$

Denote $\Theta_{it} \equiv b_{it}\lambda_i\left(\beta_i\right)^t$ and use equations (7.26), (7.15) and (2.14) to obtain the following expression for the logarithmic growth of optimal individual consumption:

$$\begin{aligned}
\Delta \ln\left(c_{it+1}-\delta\right) &= \frac{1}{\gamma}\Delta\left(\ln \kappa_{t+1}\right)-\Delta \ln\left(\Theta_{it+1}\right) \\
&= \Delta \ln\left(Y_{t+1}-\delta N\right)-\Delta \ln\left[\sum_{i=1}^{N}\left(\Theta_{it+1}\right)^{-\frac{1}{\gamma}}\right]-\frac{1}{\gamma}\Delta \ln\left(\Theta_{it+1}\right) \qquad (7.16)
\end{aligned}$$

To be precise, the expression above represents the logarithmic growth of optimal individual consumption when $\delta = 0$. This is the case of Constant Relative Risk Aversion (CRRA) preferences mentioned earlier, which has been extensively explored in the risk-sharing literature. For a positive δ, the expression provides the logarithmic growth of the optimal consumption in excess of the subsistence parameter δ, which is often the case explored in more recent studies.

The logarithmic growth of total endowments in the expression above can be calculated based on equation (2.4) as follows:

$$\Delta \ln\left(Y_{t+1}-\delta N\right) \simeq \frac{\Delta Y_{t+1}}{Y_t-\delta N}=\frac{\sum_{i=1}^{N}\left(\Delta y_{it+1}\right)}{Y_t-\delta N}=\frac{\sum_{i=1}^{N}\left(y_{it}\sigma_i\varepsilon_{it+1}\right)}{Y_t-\delta N}. \qquad (7.17)$$

Substituting (7.16) into (7.17), we get:

$$\Delta \ln\left(c_{it+1}-\delta\right)=\frac{\sum_{i=1}^{N}\left(y_{it}\sigma_i\varepsilon_{it+1}\right)}{Y_t-\delta N}-\Delta \ln\left[\sum_{i=1}^{N}\left(\Theta_{it+1}\right)^{-\frac{1}{\gamma}}\right]-\frac{1}{\gamma}\Delta \ln\left(\Theta_{it+1}\right) \qquad (7.18)$$

Similarly to Section 7.1, we demonstrate in what follows that when the size of endowments is distributed uniformly across agents or with a finite variance, then the idiosyncratic shocks cancel out in the aggregate. The idiosyncratic shocks are shown to matter for optimal consumption allocations when the size of endowments follows the Zipf distribution.

7.2.1 Identical-sized endowments

Assuming that $\overline{y} = y_{it} = (1/N)Y_t$ for every agent i, then the logarithmic growth of total endowments can be written as:

$$\Delta \ln\left(c_{it+1}-\delta\right)=\frac{\overline{y}}{N\left(\overline{y}-\delta\right)}\sum_{i=1}^{N}\left(\sigma_i\varepsilon_{it+1}\right)-\ln\left[\sum_{i=1}^{N}\left(\Theta_{it+1}\right)^{-\frac{1}{\gamma}}\right]-\frac{1}{\gamma}\Delta \ln\left(\Theta_{it+1}\right). \qquad (7.19)$$

The partial derivative of the changes in the optimal consumption of individual i with respect to the idiosyncratic endowment shock ε_{jt+1} is equal to:

$$\frac{\partial\left(\Delta \ln\left(c_{it+1}-\delta\right)\right)}{\partial \varepsilon_{jt+1}}=\frac{\overline{y}\sigma_j}{N\left(\overline{y}-\delta\right)}.$$

The partial effect of ε^{j}_{t+1} becomes 0 as $N \to \infty$.

7.2.2 Endowments distributed with finite variance

If endowments are distributed with a finite variance, then $N^{-1}\sum_{i=1}^{N}(y_{it}) \xrightarrow{a.s.} \mathbb{E}(y_t)$. Thus, we can express total endowments Y_t as follows:

$$Y_t = \sum_{i=1}^{N}(y_{it}) = N\mathbb{E}(y_t)$$

The partial derivative of equation (7.6) with respect to the idiosyncratic endowment shock ε_{jt+1} can be written as:

$$\frac{\partial(\Delta \ln(c_{it+1}-\delta))}{\partial \varepsilon_{jt+1}} = \frac{\sigma_j y_{jt}}{Y_t - \delta N} = \frac{\sigma_j y_{jt}}{N(\mathbb{E}(y_t)-\delta)}.$$

Given the finite variance assumption, then the term $\frac{\sigma_j y_{jt}}{\mathbb{E}(y_t)-\delta}$ remains a positive real number and the partial effect of ε_{jt+1} becomes 0 as $N \to \infty$.

Proposition 9. *Assuming DARA preferences, the partial effect of an idiosyncratic shock to the j-th agent's endowment on the optimal consumption of the i-th individual becomes 0 as $N \to \infty$ when:*

(i) endowments are distributed uniformly across agents or

(ii) endowments are distributed with a finite variance across agents.

7.2.3 Endowments following a power law distribution

Under the assumption that endowments follow a power law distribution, the j-th largest endowment in the population of N agents is approximately equal to:

$$y_{jt} = \left(\frac{j}{N}\right)^{-\frac{1}{\zeta}}, \tag{7.20}$$

where $\zeta \geq 1$ denotes the exponent of the power law distribution of endowments. The size of aggregate endowments as the size of the economy grows large (i.e., $N \to \infty$) will depend on the value of parameter ζ.

- if $\zeta > 1$, then the mean value of the endowment size takes a finite value, as the size of the economy grows large (i.e., $N^{-1}\sum_{i=1}^{N}(y_{it}) \xrightarrow{a.s.} \mathbb{E}(y_t)$). Thus, for large N, total endowments Y_t can be expressed as $Y_t = \sum_{i=1}^{N}(y_{it}) = N\mathbb{E}(y_t)$ and the logarithmic growth of total endowments can be written as:

$$\Delta \ln(c_{it+1}-\delta) = \frac{1}{N\mathbb{E}(y_t) - N\delta}\sum_{i=1}^{N}\left(\left(\frac{i}{N}\right)^{-\frac{1}{\zeta}}\sigma_i\varepsilon_{it+1}\right) - \ln\left[\sum_{i=1}^{N}(\Theta_{it+1})^{-\frac{1}{\gamma}}\right] - \frac{1}{\gamma}\Delta\ln(\Theta_{it+1}). \tag{7.21}$$

The partial derivative of equation (7.21) with respect to the idiosyncratic endowment shock

ε_{jt+1} becomes equal to:

$$\frac{\partial\left(\Delta\ln\left(c_{it+1}-\delta\right)\right)}{\partial\varepsilon_{jt+1}}=\frac{1}{N^{\frac{\zeta-1}{\zeta}}}\frac{\sigma_j}{j^{\frac{1}{\zeta}}\left(\mathbb{E}\left(y_t\right)-\delta\right)}.$$

Given that $\zeta>1$ and that $\frac{\sigma_j}{j^{\frac{1}{\zeta}}\left(\mathbb{E}(y_t)-\delta\right)}$ is a positive finite value, then the partial effect becomes 0 as $N\to\infty$.

- if $\zeta=1$, then $y_{jt}=\left(\frac{N}{j}\right)$ and the expected value of endowments (for a large N) is equal to $\mathbb{E}\left(y_t\right)=\int_1^N yf\left(y\right)dy=\int_1^N yy^{-2}dy=\ln N$, where $f\left(y\right)=y^{-2}$ is the probability density function of the distribution of endowment sizes. Therefore, the aggregate endowment is equal to $Y_t=N\mathbb{E}\left(y_t\right)=N\ln N$ and the logarithmic growth of total endowments can be expressed as:

$$\Delta\ln\left(c_{it+1}-\delta\right)=\frac{\sum_{i=1}^N\left(\frac{N}{i}\sigma_i\varepsilon_{it+1}\right)}{N\ln N-\delta N}-\ln\left[\sum_{i=1}^N\left(\Theta_{it+1}\right)^{-\frac{1}{\gamma}}\right]-\frac{1}{\gamma}\Delta\ln\left(\Theta_{it+1}\right). \tag{7.22}$$

The partial derivative of equation (7.22) with respect to the idiosyncratic endowment shock ε_{jt+1} is equal to:

$$\frac{\partial\left(\Delta\ln\left(c_{it+1}-\delta\right)\right)}{\partial\varepsilon_{jt+1}}=\frac{\sigma_j}{j\left(\ln N-\delta\right)}. \tag{7.23}$$

Equation (7.23) leads to the following proposition:

Proposition 10. *Assuming CRRA preferences, the partial effect of an idiosyncratic shock to the j-th largest endowment on the optimal consumption of the i-th individual decays according to:*

(i) $\frac{1}{j(\ln N-\delta)}$ *when the size of endowments follows a power-law distribution with $\zeta=1$ (i.e., a Zipf distribution) and*

(ii) $\frac{1}{j^{\frac{1}{\zeta}}N^{\frac{\zeta-1}{\zeta}}}$ *when the size of endowments follows a power-law distribution with $\zeta>1$.*

7.3 Heterogeneous risk preferences

In this section we consider the Constant Relative Risk Aversion (CRRA) with heterogeneous risk preference parameters γ_i:

$$u\left(c_{it},\,b_{it}\right)=b_{it}\frac{\left(c_{it}\right)^{1-\gamma_i}-1}{1-\gamma_i}. \tag{7.24}$$

Substitute the partial derivative of the CRRA utility function with respect to consumption into the first order condition (2.8) and solve for c_{it} to obtain:

$$c_{it}=\left(\frac{b_{it}\lambda_i\left(\rho_i\right)^t}{\kappa_t}\right)^{\frac{1}{\gamma_i}}. \tag{7.25}$$

Taking the logarithm of both sides yields:

$$\ln c_{it} = -\frac{1}{\gamma_i} \ln \kappa_t + \frac{1}{\gamma_i} \ln \left(b_{it} \lambda_i \left(\rho_i \right)^t \right). \tag{7.26}$$

The growth rate of individual consumption can be obtained by taking the first difference of the equation above:

$$\Delta \ln \left(c_{it+1} \right) \quad = \quad -\frac{1}{\gamma_i} \Delta \ln \left(\kappa_{t+1} \right) + \frac{1}{\gamma_i} \Delta \ln \left(b_{it} \lambda_i \left(\rho_i \right)^t \right). \tag{7.27}$$

The second term can be ignored under the assumption that b_{it} is constant across time and $\rho_i = 1$.

The change in the level of individual consumption, taking consumption from last period c_{it} as given, can be expressed as:

$$\Delta c_{it+1} \simeq c_{it} \Delta \ln \left(c_{it+1} \right) \quad = \quad -\frac{c_{it}}{\gamma_i} \Delta \ln \left(\kappa_{t+1} \right).$$

Aggregating over the N agents of the economy, we obtain the following expression for the change in aggregate consumption:

$$\Delta C_t = \sum_{i=1}^{N} \left(\Delta c_{it} \right) \quad = \quad -\sum_{i=1}^{N} \left(\frac{c_{it}}{\gamma_i} \Delta \ln \left(\kappa_{t+1} \right) \right).$$

Using the binding aggregate resource constraint, we can express the change in aggregate income as the change in aggregate consumption:

$$\Delta Y_t = \Delta C_t = -\sum_{i=1}^{N} \left(\frac{c_{it}}{\gamma_i} \Delta \ln \left(\kappa_{t+1} \right) \right).$$

Rearranging terms, we obtain:

$$\Delta \ln \left(\kappa_{t+1} \right) = -\frac{\Delta Y_t}{\sum_{i=1}^{N} \left(\frac{c_{it}}{\gamma_i} \right)}. \tag{7.28}$$

Substituting $\Delta \ln \kappa_{t+1}$ into equation (7.27), we get:

$$\begin{aligned}\Delta \ln (c_{it+1}) &= -\frac{1}{\gamma_i}\Delta \ln (\kappa_{t+1}) \\ &= \frac{\Delta Y_t}{\gamma_i \sum_{i=1}^{N}\left(\frac{c_{it}}{\gamma_i}\right)}. \end{aligned} \tag{7.29}$$

If we assume that the risk aversion parameter, γ_i (strictly greater than zero), is i.i.d. with finite variance and, further, that it is independent of consumption, c_i and endowments, y_i (distributed with $\zeta \geq 1$), then for the variable $\left(\frac{c_{it}}{\gamma_i}\right)$,

$$\mathbb{P}(|\frac{c_{it}}{\gamma_i}| > x) = \mathbb{P}\left(c_{it} > \left(\frac{x}{|\frac{1}{\gamma_i}|}\right)\right) \tag{7.30}$$

$$\sim \mathbb{E}\left[\left(\frac{1}{|\gamma_i|}\right)^{\zeta}\right] x^{-\zeta} \tag{7.31}$$

Therefore, for $\zeta > 1$,

$$\sum_{i=1}^{N}\left(\frac{c_{it}}{\gamma_i}\right) = N\mathbb{E}[c_{it}]\mathbb{E}\left[\left(\frac{1}{\gamma_i}\right)\right] \tag{7.32}$$

$$> \frac{N\mathbb{E}[c_{it}]}{\mathbb{E}[\gamma_i]} \tag{7.33}$$

The last inequality follows from Jensen's Inequality.

Then, we can rewrite (7.29) as:

$$\Delta \ln (c_{it+1}) > \frac{\Delta (Y_{it})}{\frac{\gamma_i}{\mathbb{E}[\gamma_i]} N\mathbb{E}[c_{it}]} \tag{7.34}$$

7.3.1 Endowments following a power law distribution

Under the assumption that endowments follow a power law distribution, the j-th largest endowment in the population of N agents is approximately equal to:

$$y_{jt} = \left(\frac{j}{N}\right)^{-\frac{1}{\zeta}}, \tag{7.35}$$

where $\zeta \geq 1$ denotes the exponent of the power law distribution of endowments. The size of aggregate endowments as the size of the economy grows large (i.e., $N \to \infty$) will depend on the value of parameter ζ.

- if $\zeta > 1$, then the mean value of the endowment size takes a finite value, as the size of the economy grows large (i.e., $N^{-1}\sum_{i=1}^{N}(y_{it}) \xrightarrow{a.s} \mathbb{E}(y_t)$). Thus, for large N, total endowments Y_t can be expressed as $Y_t = \sum_{i=1}^{N}(y_{it}) = N\mathbb{E}(y_t)$ and he partial derivative of equation (7.34) with respect to the idiosyncratic endowment shock ε_{jt+1} becomes equal to:

$$\frac{\partial\left(\Delta\ln\left(c_{it+1}\right)\right)}{\partial\varepsilon_{jt+1}} > \frac{\sigma_j}{N^{\frac{\zeta-1}{\zeta}}\cdot j^{\frac{1}{\zeta}}\left(\frac{\gamma_i}{\mathbb{E}[\gamma_i]}\mathbb{E}[c_{it}]\right)}.$$

  Given that $\zeta > 1$ and that $\frac{\sigma_j}{j^{\frac{1}{\zeta}}(\mathbb{E}(y_t)-\delta)}$ is a positive finite value, then the partial effect becomes 0 as $N \to \infty$.

- if $\zeta = 1$, then $y_{jt} = \left(\frac{N}{j}\right)$ and the expected value of endowments (for a large N) is equal to $\mathbb{E}(y_t) = \int_1^N yf(y)\,dy = \int_1^N yy^{-2}dy = \ln N$, where $f(y) = y^{-2}$ is the probability density function of the distribution of endowment sizes. Therefore, the aggregate endowment is equal to $Y_t = N\mathbb{E}(y_t) = N\ln N$ and the partial derivative of equation (7.34) with respect to the idiosyncratic endowment shock ε_{jt+1} is equal to:

$$\frac{\partial\left(\Delta\ln\left(c_{it+1}\right)\right)}{\partial\varepsilon_{jt+1}} > \frac{\sigma_j}{\ln N\cdot j\left(\frac{\gamma_i}{\mathbb{E}[\gamma_i]}\mathbb{E}[c_{it}]\right)}. \tag{7.36}$$

  Equation (7.36) leads to the following proposition:

  Proposition 11. *Assuming CRRA preferences, the partial effect of an idiosyncratic shock to the j-th largest endowment on the optimal consumption of the i-th individual decays according to:*

  (i) $\frac{1}{j\cdot\ln N}$ *when the size of endowments follows a power-law distribution with $\zeta = 1$ (i.e., a Zipf distribution) and*

  (ii) $\frac{1}{j^{\frac{1}{\zeta}}N^{\frac{\zeta-1}{\zeta}}}$ *when the size of endowments follows a power-law distribution with $\zeta > 1$.*

Apart from asymptotics, the difference in the effect of an i.i.d shock on consumption growth, when allowing for risk heterogeneity (compare Eq. (7.34) to Eq. (2.20)), is that the denominator now depends on the agent's risk aversion coefficient γ_j relative to the mean $\mathbb{E}[\gamma_j]$. Therefore, the effect of ε_{jt+1} on $\Delta\ln(c_{it+1})$ is mitigated to an extent by the relative value of agent j's risk aversion. Extremely risk-averse agents, for example with $\gamma_j > \mathbb{E}[\gamma_j]$, are affected less by their own shock relative to risk-loving agents by being allocated a larger share of aggregate resources. Consistent with Mazzocco (2012), the social planner pools resources with the view to optimally insure and then allocates them according to agents' preferences for risk. Note that in this case, the proportional allocation is able to provide some insurance against aggregate shocks. When aggregate shocks are themselves affected by idiosyncratic shocks, the implication is that that consumption growth is affected less by the idiosyncratic shocks of risk averse agents (those whose risk aversion is above the mean) relative to the case when all agents have constant risk aversion. The counterpart, however, is that consumption growth is affected more by the idiosyncratic shocks of risk loving agents (those whose risk aversion is below the mean). Future research could extend the analysis with a flexible model, that ties risk aversion of each agent to the amount of endowments they hold, to generate predictions for the effect of redistributive policies.