American Economic Development Since The Civil War

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Abstract

This paper assesses the joint dynamics of population, education and economic growth in the United States since 1860. A structural model describes how are interacting the conditional distributions of fertility, mortality and children education, conditionnally on parental education. Theoretically, a trade-off between quantity and quality of children takes place, and possibly generates multiple equilibria in the economy depending on the cost of education. The model is then calibrated and trajectories across time of GDP per capita, schooling attainment for various grades, fertility rate, survival probabilities at different ages, are well captured. This allows to investigate the long run influence of income inequality on growth, as well as fertility or mortality differentials. In the context of the United States, the latter are found to have a modest negative impact on long-term income growth, contrary to income inequality which has a strong negative effect. Moreover, some counterfactual simulations assess the central role of the educational cost, which turn out to be a very effective lever for long-term economic development.

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1 Introduction

The original goal of the so-called unified growth theory has been to give some microfoundations to a recurrent fact in economic history of nations, namely the combination of an economic take-off and a demographic transition. Its causes have been much debated by demographers, historians or economists, and it is only recently that some theoretical models have emerged to explain both population and income dynamics on the long run. Nevertheless, those models have only received a partial assessment until now.

The goal of this paper is to fill this gap by providing a model of the joint dynamics of economic growth, technological progress, the distributions of schooling, fertility, and mortality, as well as to calibrate this model on data from the United States over the period 1860-2000. What is the influence of income inequality, fertility or mortality differentials, on future performances of that country? In what extent can a given public policy alleviate the weight of the past? By producing counterfactuals based on the US calibration, on a first step the paper aims at assessing the relationship between income inequality, or fertility and mortality differentials, and the growth path of the economy. On a second step, it states upon the efficiency of a given public policy, namely the subsidization of education.

The framework encompasses major traits of former theoretical literature. Oded Galor and Omer Weil (2000) provide a theoretical microfunded explanation for both the economic take-off and the demographic transition: the relationships between income and population is driven by a children quantity-quality trade-off, triggered by technological change that raises the demand for human capital, thus enhancing its supply and leading on the same time to a reduction of fertility. Nevertheless, an important aspect of the demographic transition, mortality reduction, is absent from this framework, which neither allows to study the role of inequality since income distribution is ignored.

In a following paper, Oded Galor and Omer Moav (2003) study the relationship between inequality and growth through the relative importance of returns to physical and human capital in the long-term process of development. In early stages, the economy is capital-intensive and inequality has a positive impact on growth since the rich save more. In latter stages, the economy is human-capital intensive and inequality has a negative impact on growth since human capital has important diminishing returns and should be widely spread among individuals: its accumulation is fostered by the reduction of credit constraints that limit human capital investments. For the sake of simplicity, population dynamics is ignored in their analysis, as well as its potential consequences on growth.

On the contrary, David de la Croix and Matthias Doepke (2003) envisage the interaction between fertility, income distribution and growth. They show a causal impact of the income distribution on growth via differential fertility between rich and poor people: this differential is likely to increase as a consequence of higher inequality, creating more low educated people on the long-term, finally reducing aggregate human capital and growth. Importantly, reduction in mortality is not taken into account. As it explains half of the demographic transition, it cannot be neglected from an empirical perspective. If they derive some qualitatively plausible results, their framework also appears to fit the data only approximatively.

Lastly, Jesus Fernandez-Villaverde (2001) is the only one paper that studies both endogenous fertility and mortality from an empirical perspective, but his analysis abstracts from the link between income distribution and growth. To sum up, there does not exist any model assessing the joint dynamics of economic growth, population dynamics, and inequality.

In this paper, households choose between quantity and quality of children and save for future consumption, subject to mortality risks. Those choices and the latter risks differ across households since they are conditional on parental education. Dynamically, the heterogeneity of constraints and behaviours has important repercussions since the model can exhibit multiple equilibria depending on the cost of education. If it is low, every household will invest in children education, if it is high, poor households will more likely invest in quantity of children, which is relatively cheaper than education. In that case, a poverty trap is likely to be created for poor dynasties and a polarized society to emerge on the long run. This original yet simple mechanism does not rely on any imperfection of the credit-market, but simply puts forward the idea that fertility choices can generate multiple equilibria in growth dynamics, because they are not independant from educational investment choices.

As a result, the model fits in a satisfactory way the distributions of fertility, mortality at different ages, education, and average income. In the context of the United States, it turns out that fertility and mortality differentials have a moderate negative impact on growth, while income inequality has a strong negative effect. This is because it prevents education accumulation among poor households, and slowdowns the convergence towards mass higher education. Considering this, any policy that diminishes the cost of education will have a strong and positive impact on long-term growth and human capital accumulation, as some simulations demonstrate.

Section 2 describes the historical facts relevant to this model, which is presented in Section 3 together with its major dynamical properties. Section 4 depicts its parametrization, while Section 5 proposes some counterfactual analyses in order to gauge the impact of inequality on growth. Last section concludes

2 Trends in population dynamics and educational attainment since 1860

2.1 Education

In the middle of the nineteenth century, the United-States were the most educated country in the world. In 1870, illiteracy was about 20 % and mass primary schooling was already completed. According to the US Department of Education (1993), enrolment rate of young people aged between 5 and 17 was about 75% in 1900, a figure that reflects full enrolment in elementary schools for the 5-13 years old, and a 10% enrolment rate in secondary school for the remaining group. Then in the first half of the twentieth century, the United States experienced a dramatic development of secondary schooling. The percentage of high school graduates relatively to the 17 years old population was about 25% in 1925, 50% at the beginning of Second World War, and reached its current level of 80% in the early 1970s. Following this wave, the college graduation rate has gone from about 10% in 1960 to 30% in 2000, the bulk of this increase taking place in the first half of this period (about two thirds).

Factors that have contributed to the fast development of education are numerous. In spite of the long-lasting racial segregation of Black, it is believed that a widely shared and deeply enrooted egalitarian vision of education contributed a lot to its spread. In its "Bill for the more general diffusion of knowledge" (1779), Thomas Jefferson advocated the benefits of education for democracy in such terms: "the most effectual means of preventing [tyranny] would be, to illuminate, as far as practicable, the minds of people at large, and more especially to give them knowledge of those facts...they may be enabled to know". In an enlightning paper, Claudia Goldin and Lawrence F. Katz (2003) explain the causes of the american educational system's success story: apart from the latter cultural trait that promoted egalitarianism and, in practice, gender equality, decentralization and local financing of schools, which became free very early in time, haved played a key role in US educational achievement. Interestingly, it appears that the development of elementary and secondary education were not the outcome of mandatory schooling laws, which often lagged behind phases of enrolment increase, and were not binding anyway. Similarly, the contribution of the state to funding was modest at the beginning. Indeed, the share of the state in the funding of elementary and secondary schools has raised from 20% in 1890 up to 50% in 2000, and has stayed almost constant and equal to 30% for higher education. To sum up, the most plausible way of viewing education in the US over that period would be to consider it as a private good, more than as a public good.

Importantly, it is likely that the price of this private good has varied across time, but it is hard to decide in which way. On one hand, there has been a clear increase in expenditures per pupil, following the real increases of teachers' wage and administrative costs, as well as the reduction of class size. For instance the pupil/teacher ratio in elementary and secondary schooling has gone from 35 in 1870 to 15 in 1995. This entails a clear rise of the direct cost of education. On the other hand, it is very likely that the opportunity cost of education has decreased along the period, as mentionned hereafter.

2.2 Fertility and Mortality

In most countries, the demographic transition is composed of a first phase where mortality is decreasing, followed by fertility in a second phase. A traditional explanation advocated by demographers is that the decline in infant mortality triggers that in fertility, building on the hypothesis that each household has a desired number of surviving offspring. In the United-States fertility rates have declined from 7 children in 1800 to 3.5 in 1900 with an acceleration around 1840s, to reach its current value of about two children per woman in 2000. In contrast, data on infant mortality start only in 1850, which is too late in time to state upon their long-run relationships.

However, this explanation is challenged by the lack of regularity in demographic patterns of Western European countries. In some countries, fertility started rising before declining, or stayed constant in some others, a picture hard to reconcile with a steadily decreasing infant mortality. In that respect, Jesus Fernandez-Vilaverde (2001) shows that in the English case fertility granger-causes infant mortality rather than the other way around, and Matthias Doepke (2004) assesses via some structural analysis that infant or child mortality are unable to explain fertility's decline.

The main trigger of fertility's decline is often advocated to be increased investments into children's human capital. As argued by Oded Galor and Omer Weil (1999, 2000), an rising demand for education in the course of technological change has triggered parental investments in education as well as a substitution effect from children quantity to children quality. It is unfortunate that there exists no educational data for the United-States in the first half of the nineteenth century, where the development of mass primary schooling has taken place. In parallel of the rising demand for human capital, it is often mentionned in the literature that a falling opportunity cost of education is likely to have fostered fertility decline. Hazan and Berdugo (2002) and Matthias Doepke (2005) provides some quantitative evidence that the rising wage's gap between adult and children along the development process, as well as educational and child labor policies, had a strong impact on fertility rates. Similarly, Oded Galor and Omer Weil (1996) argue that the increase of women labor participation along with the rise of their real wage, has increased the cost of bearing children, making the choice of high fertility relatively less attractive than that of high educational investments.

The fall of mortality rates is mainly due to the reduction of normal causes of death rather than to the elimination of famines, as explained in Wrigley and Schofield (1981). There has been a controversy on whether this should be attributed to an increase of calories consumption or to scientific progress of medecine. Since consumption includes both of them in the model, this undeterminacy does not appear very problematic in the following model.

3 The Model

3.1 Consumer's program

There are 4 periods in life of 20 years each. Work takes place between 20 and 60 (period 1 and 2), consumption and fertility choices at the beginning of period 1. Lifetime is stochastic, and s^t_{t+j} stands for the probability that an individual born in period t will be alive at the end of period $t+j$, with $j \in \{0, 1, 2\}$. For empirical motives, these survival probabilities are derived from the Cox model, and death rate are constant within each cohort of age. If λ^t_{t+j} is the death rate j periods after birth at period t and z_{t+j} are some observed characteristics, then $\lambda^t_{t+j} = \lambda_j \exp(-z'_{t+j}\beta)$ where λ_j is a given constant, and one has

$$\begin{aligned} s^t_t &= \exp(-\lambda_0 \exp(-z'_t \pi_0)) \\ s^t_{t+j} &= s^t_{t+j-1} \exp(-\lambda_j \exp(-z'_{t+j}\pi_j)), \quad j = 1, 2 \end{aligned}$$

In practice one retains a single individual component z_{t+j}, individual consumption, but other factors such as education or average education in the society could be considered.

At the beginning of period 2 a representative agent born in year t with human capital h^t maximizes utility over lifetime, while caring about both quantity n_{t+1} and human capital h^{t+1} of her children. Each agent represents a high number of comparable households such that n_{t+1} can be a continous variable (the mean number of born children among that kind of household). Raising $n_{t+1}s_{t+1}^{t+1}$ surviving children has a time cost $\phi w_{t+1}^{t}h^{t}n_{t+1}s_{t+1}^{t+1}$, where ϕ stands for the proportion of lost wage for each child. Providing e_{t+1} years of schooling to these children has a monetary cost $\tau_{t+1}w_{t+1}e_{t+1}n_{t+1}s_{t+1}^{t+1}$, which is fixed, ie does not depend on parental human capital. The parameter τ_{t+1} depends on the pupils/teachers ratio, on teachers' human capital, as well as on other institutional features that affect the opportunity cost of education such as child labor policies. It may vary across time, it is parametrized by its initial and final values in respectively 1860 and 2000, and it decreases linearly between those dates[1]. Raising children is subject to the time constraint $\phi n_{t+1}s_{t+1}^{t+1} < 1$ (each household has one unit of time). Production of human capital is a concave fonction of years of schooling. Two others externalities are likely to happen: one is the social return to education, ie spillovers of the community average human capital, and the other is a direct transmission of human capital from parents to children. Following David de La Croix and Matthias Doepke (2003) we retain the following production function for human capital

$$h^{t+1} = (\theta + e_{t+1})^{\eta} (h^t)^{\rho} (\bar{h}^t)^{\kappa} \varepsilon$$

where ε is an log-normal ability shock of mean 1 and variance σ^2. Abstracting from the latter and from education spillovers, an uneducated household has therefore θ^{η} units of human capital.

[1] Other functional forms has been tested such as heterogeneity in the parameter τ with respect to grades, ie primary, secondary and higher education. In terms of fitting the data, no significant benefits have been noticed and the simplest form has been retained.

An household also saves assets a^t_{t+i} during period i of her working life in order to consume during her retirement. Due to accidental death before 60, an household can receive unvolontary transfers. All these transfers tr_{t+i} are pooled and mutualized so that there is no intergenerational transfers from parents to children[2]. The utility function u is assumed to be Constant Relative Risk Aversion with parameter C. The program thus writes:

$$\underset{c^t_{t+1},c^t_{t+2},c^t_{t+3},e_{t+1},n_{t+1}}{Max} \mathrm{E}\ \left[u(c^t_{t+1}) + \beta u(c^t_{t+2}) + \beta^2 u(c^t_{t+3}) + \gamma u(n_{t+1}h^{t+1})\right]$$

$$\begin{aligned}
s.t.\ c^t_{t+1} + a^t_{t+1} + \tau_{t+1}w_{t+1}e_{t+1}n_{t+1}s^{t+1}_{t+1} &= w_{t+1}h^t(1-\phi n_{t+1}s^{t+1}_{t+1}) + tr_{t+1} \\
c^t_{t+2} + a^t_{t+2} &= w^e_{t+2}h^t + (1+r_{t+1})a^t_{t+1} + tr_{t+2} \\
c^t_{t+3} &= (1+r^e_{t+2})a^t_{t+2} \\
h^{t+1} &= (1+e_{t+1})^\eta\,(h^t)^\tau(\bar{h}^t)^\kappa \\
0 \le e_{t+1} &\le 16 \\
e_{t+1}n_{t+1} &< 1/s^{t+1}_{t+1}
\end{aligned}$$

Notice that the solutions to this program are conditionnal on parental human capital: the joint distribution of $(c^t_{t+i}, h^{t+1}, n_{t+1})$ is conditionnal on h^t. From the same perspective recall that life expectancy also depends on h^t.

The structure of information is the following: at the beginning of each period the agent observes the amount of aggregate inputs, namely labor and capital, as well as their rental prices. To solve her program she has to form expectations on capital returns and wages at the beginning of the following periods. Rational expectations would greatly complicate the empirical tractability of the program

[2] for simplicity we should assume that these transfers are mutualized within each cohort of age, otherwise this would mean that each cohort of age would rationally forecast what will be the savings of the following cohorts of age, conditionally on their own human capital investments.

since rental prices on subsequent periods would logically depend on consumers' decisions; therefore each agent should solve not only her program but also that of every agent in order to derive the recursive equilibria of the economy. A certain degree of myopia is believed to be a more realistic assumption. Therefore one simply assumes that agents assume stationarity of wages and capital returns, which gives

$$\begin{aligned} w_{t+2}^e &= w_{t+1} \\ r_{t+2}^e &= r_{t+1} \end{aligned}$$

In further extensions both adaptative and extrapolative expections are examined, and do not change the spirit of the model[3].

3.2 Aggregates

At each date the working population is composed of two vintages of human capital distributions. Let us call the cumulative distribution function of human capital for the new cohort born in $t+1$. Then

$$F_{new}^{t+1}(h) = \int_0^{+\infty} 1_{h^{t+1}(h^t) \leq h} dF_{new}^t(h^t)$$

Total human capital among the surviving labor force is simply the mixture of two distributions

$$dF_{t+1}(h) = p_{t+1} \frac{s_{t+1}^t(h) dF_{new}^t(h)}{\int_0^{+\infty} s_{t+1}^t(h) dF_{new}^t(h)} + (1 - p_{t+1}) \frac{s_{t+1}^{t-1}(h) dF_{new}^{t-1}(h)}{\int_0^{+\infty} s_{t+1}^{t-1}(h) dF_{new}^{t-1}(h)}$$

where p_{t+1} is the relative weight of the cohort aged between 20 and 40 with respect to population aged between 20 and 60 at date $t+1$. The number of

[3] see appendix

births is the sum of children conditionnally on the survival of parents

$$N_{t+1} = N_t \int_0^{+\infty} s_{t+1}^t(h) n_{t+1}(h) dF_{new}^t(h)$$

which provides the total population at date $t+1$

$$P_{t+1} = \sum_{i=0}^{3} N_{t+1-i} \int_0^{+\infty} s_{t+1}^{t+1-i}(h) dF_{new}^{t+1-i}(h) = \sum_{i=0}^{3} P_{t+1}^i$$

where P_{t+1}^i the total population born in $t+1-i$ that has survived until the end of period $t+1$. Consequently, the relative weight of cohort aged 20-40 in the active population is simply

$$p_{t+1} = \frac{P_{t+1}^1}{P_{t+1}^1 + P_{t+1}^2}$$

Production uses a Cobb-Douglas production function

$$Y_{t+1} = K_{t+1}^{\alpha} (A_{t+1} L_{t+1})^{1-\alpha}$$

Firms maximize profits, wages and interest rates are equal to their marginal product

$$\begin{aligned} w_{t+1} &= (1-\alpha) k_{t+1}^{\alpha} \\ r_{t+1} &= \alpha k_{t+1}^{\alpha-1} \end{aligned}$$

where $k_{t+1} = K_{t+1}/A_{t+1}L_{t+1}$is physical capital per unit of effective labor. The market-clearing conditions for labor and capital apply. Labor is equal to the total of work hours times human capital

$$L_{t+1} = N_t \int_0^{+\infty} h s_{t+1}^t(h)(1 - \phi n_{t+1}(h) s_{t+1}^{t+1}(h)) dF_{new}^t(h) + N_{t-1} \int_0^{+\infty} h s_{t+1}^{t-1}(h) dF_{new}^{t-1}(h)$$

while physical capital depreciates at rate δ and is augmented by households' savings during working life

$$K_{t+1} = (1-\delta)K_t + N_t \int_0^{+\infty} s_{t+1}^t(h) a_{t+1}^t(h) dF_{new}^t(h) + N_{t-1} \int_0^{+\infty} s_{t+1}^{t-1}(h) a_{t+1}^{t-1}(h) dF_{new}^{t-1}(h)$$

The rate of growth in technology A_t may depend on mean aggregate human capital in an endogenous growth context, as well as on other determinants such as lagged technology or population externality. Following Jerome Vandenbussche et al. (2004) and Philippe Aghion et al. (2005), the role of higher education on technological growth is disentangled from that of primary and secondary education. A simple specification is

$$A_{t+1} = A_t e^{\nu (H_t^{P-S})^{\mu} (H_t^{H})^{\mu_S}}$$

where H_t^{P-S} and H_t^H are human capital derived from agents with respectively primary/secondary schooling or higher education. Therefore this model encompasses both exogenous and endogenous growth models, if μ and μ_S are respectively equal to or different from 0.

3.3 The Poverty Trap and the Economic Take-off

Consumer's program is solved with the help of a stochastic algorithm which has two advantages: it gets rid of the usual initial conditions problem that undermine gradient-based optimizers, and is quick to converge given a proper calibration of its underlying acceptance parameters. This calibration is nevertheless tricky and a proper description of the procedure is given into annex. In a first step a simpler model is studied in order to understand the relative role of basic components of the model. Therefore we assume

(A1) No mortality risks: all agents live three periods

(A2) No human capital externalities: $\rho = \kappa = 0$

(A3) Exogenous growth: $\mu = \mu_S = 0$

(A4) The opportunity cost of education does not vary across time : $\tau_{t+1} = \tau$

The introduction of a fixed monetary cost for education and a time cost for raising children naturally generates a trade-off between quantity and quality of children, which is conditional on parental wage: the fixed cost of education is relatively higher for poor parents, while the cost of raising a large number of children is higher for rich households because each child consumes a fixed proportion of parental wage. In annex are derived explicit solutions of the individual program with a logarithmic utility function and under assumption (A1-A4). In particular one has

$$
\begin{aligned}
e_{t+1} &= e^*_{t+1} \text{ if } 0 \leq e^*_{t+1} \leq 16 \\
&= 0 \text{ if } e^*_{t+1} < 0 \\
&= 16 \text{ if } e^*_{t+1} \geq 16 \\
e^*_{t+1} &= \left(\frac{-1}{1-\eta} + \frac{\eta\phi}{\tau(1-\eta)} h^t \right)
\end{aligned}
$$

Education thus depends linearly and positively on parental human capital[4], and on three other parameters, the return to education η, the time cost of children ϕ, the opportunity cost of education τ.

What are the dynamical properties of the economy? There are three possible cases depicted on figure 1.

[4] decreasing returns to schooling are a fundamental requirement for that.

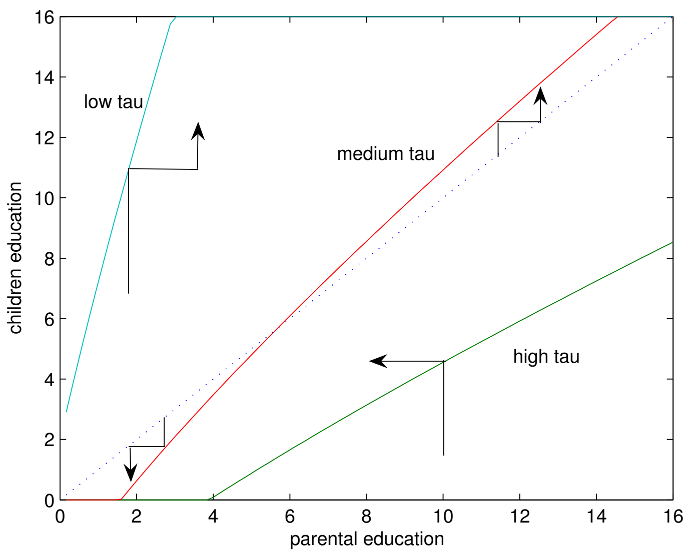

Figure 1: The Three Dynamics

When the cost of education is very high, the only equilibrium is illiteracy for everyone, when it is medium, a fraction gets some education on the long term and the others become illiterate, when it is low, everybody reaches higher education on the long term. The first two regimes are characterized by the existence of a poverty trap for at least a fraction of the population, the third one is called the economic take-off. Formally one has the following proposition

Proposition 1 *Given some initial distribution of education* $(e_{i,0})_i$*, there exists two tresholds* τ^m *and* τ^M *such that* $\forall i$ $e_{i,\infty} = 16$ *if* $\tau < \tau^m$ *and* $e_{i,\infty} = 0$ *if* $\tau > \tau^M$*. Otherwise, there are two regimes characterized by a given initial threshold* $\bar{e}$ *such that* $e_{i,\infty} = 0$ *if* $e_{i,0} < \bar{e}$ *and* $e_{i,\infty} = e^{\infty} > 0$ *if* $e_{i,0} \geq \bar{e}$

Proof. There exists τ^M such that for any $\tau > \tau^M$ $e^*_{t+1} < e_t$ whatever e_0 comprised between 0 and 16, τ^M is simply the solution of $\frac{-1}{1-\eta} + \frac{\eta\phi}{\tau(1-\eta)}(1+16)^\eta =$ 16. In that case the sequence $(e^*_t)_t$ converges towards $-\infty$, hence $(e_t)_t$ towards 0 whatever the initial condition e_0. Similarly there exists a threshold τ^m such that for any $\tau < \tau^m$, $e^{*,m}_{t+1} > e_t$ whatever e_0 comprised between 0 and 16, τ^m is the solution of $\frac{-1}{1-\eta} + \frac{\eta\phi}{\tau^m(1-\eta)} = 0$. Then $(e^*_t)_t$ converges towards $+\infty$, hence $(e^m_t)_t$ towards 16 whatever the initial condition e_0. In the intermediate case where $\tau^m < \tau < \tau^M$, $e^*_{t+1} = e_t$ has at least a solution for some e_t. It is well-known that the first solution, if not the only one, to this equation is a non-stable equilibrium since the function $e_t \longmapsto (1 + e_t)^\eta$ is concave. So there exists a value $\bar{e}$ such that $(e^*_t)_t$ tends to $-\infty$ for any initial condition below $\bar{e}$, and $(e^*_t)_t$ tends to a limit $l \in]0, +\infty]$ for any initial condition above $\bar{e}$. Hence $(e_t)_t$ converge towards 0 if $e_0 < \bar{e}$ and $(e_t)_t$ converge towards a strictly positive number if $e_0 > \bar{e}$. ■

Next figure illustrates the poverty trap regime: it displays the consumer's optimal choices in quantity and quality of children for ten periods that represent two centuries, with respect to the initial uniform distribution of education. Thus, the arrows represent the trajectory of dynasties' choices across time.

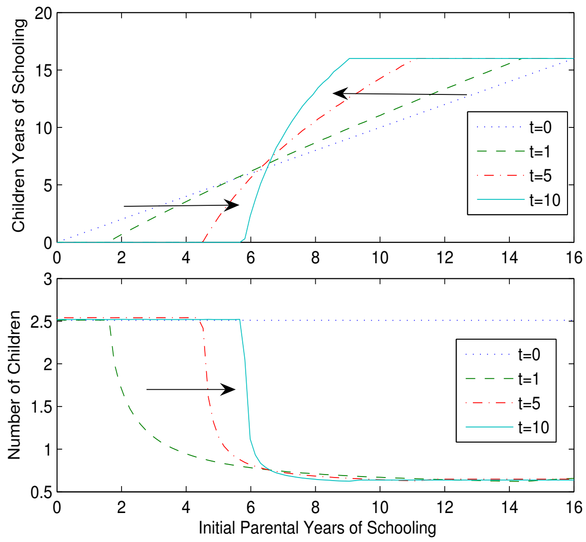

Figure 2: The Poverty Trap

One clearly notices that society becomes more and more polarized over the generations, since parents with an education over a threshold of approximatively 8 years of schooling invest a lot in children education, while low educated parents invest little or nothing at all. On the long term, the society will be clearly split into two groups, those who have higher education (16 years of schooling) and those who are illiterate (0 years of schooling).

This overall behaviour of the economy differs from that described by David de la Croix and Matthias Doepke (2002) who predict a convergence towards perfect equality given some limitations of the parameters' domain, then analyse theoretically the dynamics of the economy along a balanced growth path. First, restricting the domain of parameters evacuate poverty traps which are a relevant economic fact. Second, as the goal of the paper is not to provide a theoretical analysis but rather a realistic calibration, focusing on a balanced growth path appears to be restrictive. Indeed, from an historical perspective many countries have experienced dramatic variations in their lon-term growth path, even in the case of the United States where growh varies significatively when computed over 20 years-long periods since 1840.

This poverty trap mechanism channeled by educational investment echoes other stories, where the same result is driven by a non-convexity in the saving rate as described by Francois Bourguignon (1981), or by imperfections on the credit market as depicted by Oded Galor and Joseph Zeira (1993). However, the mechanism presented here is original in the litterature and shows that poverty trap are likely to be created simply because of agents' preferences and fertility choices. Robert J. Barro and Gary Becker (1989) mention multiple equilibria but in a global context where agents are similar within each time period.

Modifying the environment characteristics such as the cost of education is likely to trigger economic take-off. In that case, every household invests in education rather than in children quantity, a process that historically fits the socio-economic dynamics of the United States.

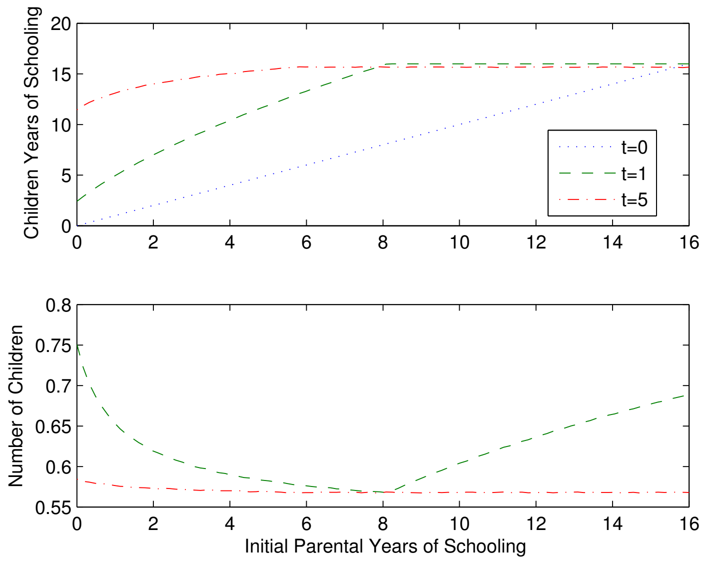

Figure 3: The Economic Take-off

As shown by Figure 3, generations invest in education so that from one generation to another children are at least as educated as their parents. Here dynamics of fertility turn out to be non monotonic with respect to education, since education is capped to 16 grades and households invest in more children quantity when they are richer and on the same time constrained in their educational investments.

4 Assessing the Economic Development of the United States 1860-2000

4.1 The data

Data are derived from a collection of statistical sources. The educational distribution is given by Christian Morrisson and Fabrice Murtin (2006) who use total enrolments in primary, secondary and higher education as well as age pyramids to derive enrolment rates at the world level since 1870. Interestingly their statistics match those of Claudia Goldin in the Historical Statistics of the United-States (2006) at the starting point of the latter source, which is based on a survey run by the US Census. Indeed, they find that in 1940 people displaying respectively elementary schooling (grades 1 to 8), secondary and higher education represent 61.2, 27.5 and 11.3 percents of the labour force, while official figures are respectively 62.9, 26.7 and 10.4 percents. Therefore one reasonnably can think that the original series on education are accurate, even since their beginning in 1870.

Age pyramids are taken from Brian R. Mitchell (2003) and aggregated into 4 groups as in the model. Fertility rates and mortality, more precisely survival probabilities at the age of 20, 40 and 60, are taken from the Historical Statistics of the United-States (2006). Last, the net migration flows by age are derived from figures deduced from Jean-Claude Chesnais (1986), Imre Ferenczi and Walter F. Willcox (1931) for the period 1840-1930, and from the Historical Statistics of the United-States (2006) afterwards. They are needed in order to adjust the simulated population at different ages with migrations flows taken as exogenous variables. All figures, GDP per capita, population, enrolment rates and demographic rates, are then averaged over the last 20 years; thus figures in 2000 represent in fact an average value over the period 1980-2000[5].

4.2 The Calibration

Given the important number of parameters, the calibration proved to be a difficult exercise, because of the extremely heavy numerical burden of the program: if the distribution of human capital is discretized into hundred elements, there

[5] educational figures for the period 1840-1860 are partly extrapolations based on a constant level of enrolment among White 5-17 years old pupils on the period 1850-1900.

are hundred dynamical programming problems to solve for each period and each combination of the structural parameters. Priors calibrations on subsets of parameters were executed while setting some variables of interest equal to the observed one. For instance, in order to calibrate the survival probabilities parameters, the simulated education distribution as well as GDP were exchanged with their observed counterparts in the simulations.

Another arising difficulty was the initial values of state variables such as factor prices or the standard errors of initial differences in fertility Δn or survival probabilities Δs. Unknown initial factor prices were derived numerically and taken equal to their stationary values in an economy where the other observed state variables are set constant at each period; and initial demographic differences were obtained by identifying Black people with the first decile of the human capital distribution. Because of ability shocks, this assumption is clearly unaccurate, but it provides the best benchmark given data constraints.

Risk aversion is taken equal to 1, time preference parameter β is 0.4, which corresponds to a 4.5% annual discount rate; a slightly higher value of 0.5 turned out to be a good fit for the altruistic parameter γ, Last, capital elasticity α is set equal to 0.3, elasticity of total factor productivity relatively to stocks of human capital to respectively 0.35 and 0.5, and the capital depreciation rate is $\delta = 0.85$, or about 10% annually which is in tune with Greenwood et al. (1997).

Other parameters were calibrated in the following way. Following Haveman and Wolfe (1995) I set $\phi = 0.14$ which means that each child costs 14% of parents' time endowment; the costs of education turned out to be a very sensible parameter and the calibration suggests that it has been reduced by 50% along the period, while its value in 2000 corresponds to a pupil/teacher ratio of 18.2, were teachers paid exclusively by pupils' parents. Human capital of uneducated people is set equal to 0.6; taking a value smaller than 1 clearly augments the return of the first years of schooling, a feature in tune with high returns to primary education observed worlwide and depicted by George Psacharopoulos and Harry A. Patrinos (2002). The elasticity of the human production function is taken equal to 0.57, which is coherent with the estimates given by Martin Browning et al. (1999). As the externatlity of human capital on society is found to be low by Daron Acemoglu and Joshua D. Angrist (2000), I set it at 0.04; $\rho = 0.05$ turned out to be a satisfactory value in practice, which suggests that intergenerational transmission of human capital are mostly driven by incentives and/or preferences rather than by a direct transmission of education. The variance of the ability shocks is calibrated to match a Gini of the income distribution close to 0.35. Table 1 sums up parameters' value

Table 1 - Initial Calibration Parameters

Fertility	ϕ	Δn					
	0.14	-0.5					
Mortality	λ_0	λ_1	λ_2	π_0	π_1	π_2	Δs
	1.25	0.3	0.6	13.0	1.5	0.7	0.15
Education	τ_{1860}	τ_{2000}	θ	σ	η	κ	ρ
	0.083	0.055	0.40	0.5	0.57	0.04	0.05
Technology	α	δ	μ	μ^S	ν		
	0.3	0.85	0.35	0.5	0.1		
Preferences	C	γ	β				
	1	0.5	0.4				

Given this calibration and initial conditions, a GMM procedure will be achieved in a latter version of the paper in order to estimate the stuctural parameters. The difficulty relies on the fact that the matched moments are not easily computable and have to be simulated. Those moments are the mean of: income (GDP divided by labour force), fertility, conditional survival probabilities between 0-20, 20-40 and 40-60, attainment in primary, secondary and higher education. The procedure is the following: a sample of parameters is drawn randomly, moments are simulated and average deviation from observed moments is computed, and so on until sufficient convergence. Then a Newey-West correction is applied after numerical estimation of the optimal weighting matrix, and the procedure is launched again.

As a result, I obtain the following dynamics for log-income, fertility, mortality, population and education. All predicted figures are plain lines and observed figures are in dots. The model turns out to fit the data in a satisfying way.

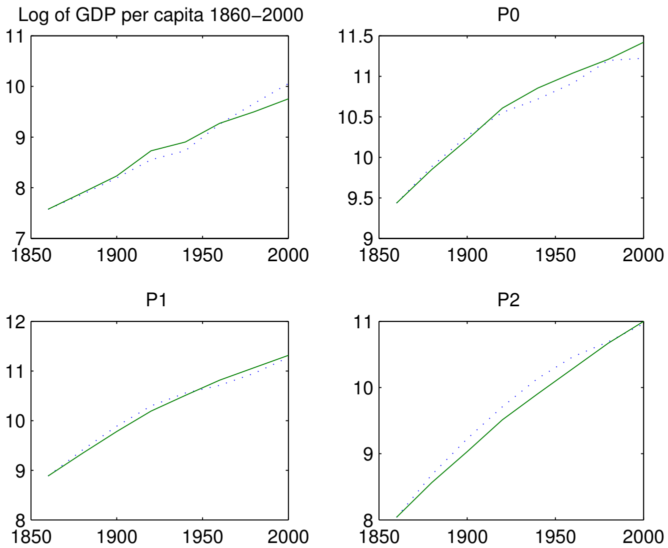

Figure 4: Observed and Simulated Income and Population by Age

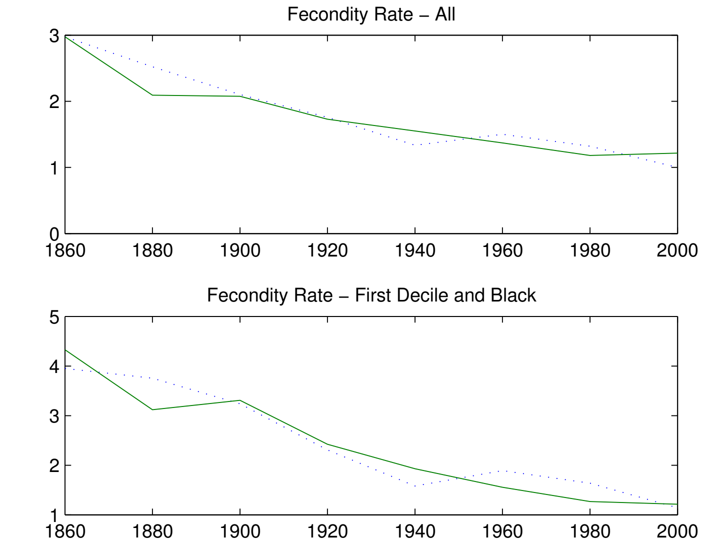

Figure 5 : Observed and Simulated Fertility Rates

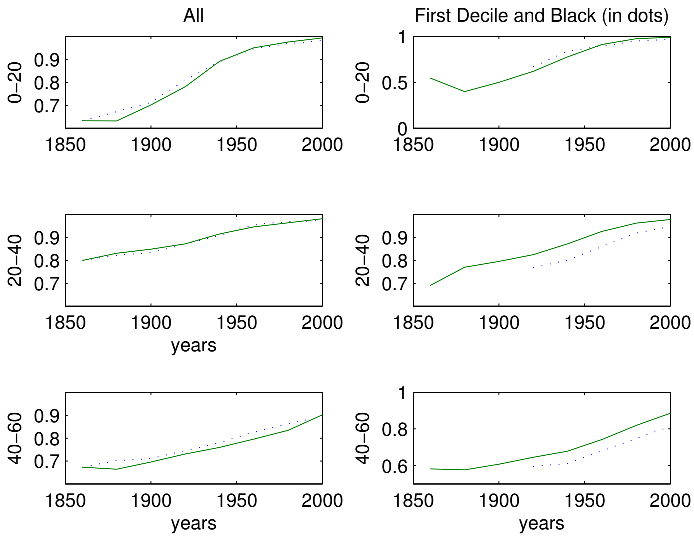

Figure 6: Observed and Simulated Conditional Survival Probabilities

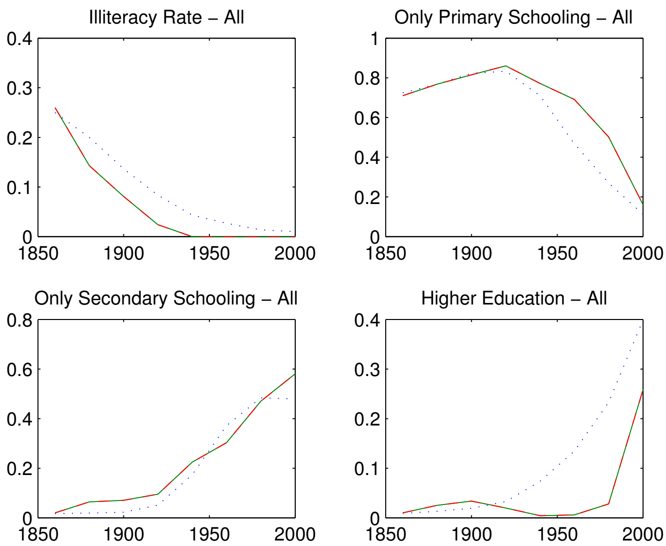

Figure 7: Observed and Simulated Distributions of Education

5 Counterfactuals

In this section I test the sensitivity of the model with respect to key elements: inequality in ability, differentials in fertility, mortality, and the cost of education.

5.1 Population dynamics and growth

Differences in fertility and in mortality are key variables to economic development, but how much is their influence of long-term growth? I run two Monte-Carlo experiments where differences in fertility and in mortality are allowed to vary arbitrarily. A mean-preserving spread is applied to the observed averages of fertility and conditional survival probabilities and the latter variables are considered to be exogenous. In particular, this means that in the first simulation consumers maximize a program where the number of children is constrained and predetermined at each period, and that the gap of fertility remains the same across time.

I found low or moderate influence of demographic factors on growth. Consider for instance the two extreme values -0.3 and - 0.7 for the spread of the fertility rate. Those figures mean that the 10th percentile of the income distribution will have respectively 30% and 190% more children than the 90th percentile at any date. Then the simulated annual growth rate for 1860-2000 is respectively equal to 1.64% and 1.45%, which entails a GDP per capita of respectively 26 600$ and 19 900$ in 2000. This is a significant difference, albeit not a huge one. This negative effect is easy to interpret: inequality of fertility entails more people with low education, hence slows down growth.

Concerning mortality, differences are even smaller. Equalizing the conditional survival probabilities or inputting a 60% gap that cumulates at each period of life provide comparable growth rates of respectively 1.29% and 1.23%. Mortality inequality may nevertheless have a bigger impact in a poverty trap regime, which remains to be analyzed.

5.2 Inequality and growth

What is the impact of inequality on long-term growth? In order to answer to this question, some Monte Carlo experiments are run where the variance of ability shocks vary from 0 to 1. Table 2 reports the annual long-term growth rate for the period 1860-2000, as well as average educational attainment in 2000. All initial conditions in 1860 are the same across simulations. To give an idea, the R^2 of Mincerian Regression in 1860 varies from 0.82 when $\sigma = 0.2$ to 0.35 when $\sigma = 1$, whereas it is close to 0.40 for all simulations in 2000, of course excepted for $\sigma = 0$.

Table 2 - Income Inequality and Growth

	Standard Error of Ability Shocks σ					
	0.0	0.2	0.4	0.6	0.8	1.0
Gini in 1860	0.328	0.351	0.405	0.475	0.538	0.577
Gini in 2000	0.019	0.145	0.268	0.405	0.498	0.644
Annual Growth Rate 1860-2000	1.42	1.36	1.33	1.09	0.72	0.83
Average Years of Schooling in 2000	15.9	13.8	11.9	11.1	8.8	12.4

It turns out that inequality in ability, in other words income inequality, has a very strong and negative impact on growth and educational attainment, both of them beeing divided by about two when σ goes from 0 to 0.8. The case $\sigma = 1.0$ is an extreme one, where inequality in ability is so high that a large fraction of the population (60% in 2000) can reach higher education if their parents draw a positive and sufficiently high ability shock at the period before. Thus it is a case of very high inequality mixed with very high intergenerational mobility. A more realistic domain sets ability's standard error between 0.4 and 0.6. Over this domain, the correlation between inequality and growth is very similar to that found by Robert J. Barro (2000) in his reduced-form regressions.

Dynamically, it turns out that inequality declines steadily and monotonically, thus does not exhibit any Kuznetz curve. It is maybe because the parameter ρ driving intergenerational persistence of human capital is low, as suggested by the simulations of David de la Croix and Matthias Doepke (2003). Table 1 shows that the smaller initial inequality, the more important the reduction of inequality in absolute levels over the period. That is, inequality introduces a noise in the economy that slowdowns convergence towards mass higher education and fast long-term growth. It is fruitful to compare the long-run outcomes of two simulations for respectively $\sigma = 0.2$ and $\sigma = 0.8$, which are two acceptable extreme cases. The percentages of people attaining higher education in 2000 are respectively 55% and 20%, the average fertility rates are respectively 2.1 and 5.0, the survival probability at 20 respectively 0.99 and 0.7. These results mean that the important levels of inequality at each period prevent investments into education that cumulate over generations, thus maintain a sizeable proportion of agents into poverty. For instance in the most equal economy illiteracy disappears by 1900 while in the most unequal illiteracy rate is still equal to 10% in 2000. Thus inequality has a deterrent effect on growth via the reduction of education accumulation on the long-term. Next section illustrates some policy implication of this.

5.3 Reforming education

As it has been largely emphasized, the key parameter of this simulated economy is the cost of education. I provide hereafter some sensitivity analysis with respect to initial and long-term cost of education, and Table 3 reports the simulated average years of schooling in 2000. Not surprisingly, it turns out that long-term attainment is non-increasing in both initial and final cost of education. Also, long-term outcomes seem to be quite sensitive to those parameters around the region of interest, for $\tau_0 = 0.83$ and $\tau_T = 0.55$. Had the education cost been smaller by about 10% in 2000 then the average years of schooling would have reached a level about 15.5, which means nearly achieving mass higher education. Thus, the cost of education seems to be the key lever of public policy in this economy in order to promote long-term growth and socio-economic development.

Table 3 - Simulated Average Years of Schooling in 2000 with Respect to the Educational Cost

	Final Value τ_T					
	0.03	0.04	0.05	0.06	0.07	0.08
Initial Value τ_0						
0.05	15.9	15.9	15.9	15.9	15.9	15.9
0.06	15.9	15.9	15.9	15.9	15.8	15.7
0.07	15.8	15.8	15.7	15.4	14.7	14.0
0.08	14.8	13.8	13.6	12.7	11.9	10.4
0.09	12.5	12.1	11.0	9.9	9.2	8.4
0.10	10.4	9.3	8.7	8.2	7.9	7.7

6 Conclusion

The paper provides a consistent theoretical framework of the dynamic interactions between economic growth, fertility, mortality and the education distributions, which embeds all major traits of unified growth theory. Theoretically, this model shows that the crucial lever of long-term development is the cost of education, that can lead to an homogenous distribution of education in the long run - made of either illiterate people or Doctors -, or on the contrary generate a polarized society. This model has been calibrated on data from the United States since 1860, and all state variables have been predicted satisfactorily, with plausible values of the structural parameters. I show that demographic differences in fertility or mortality do not make a very big difference in terms of long-term growth, given low levels of risk aversion. On the contrary, income inequality have a strong and negative impact on growth and average educational attainment, because it slowdowns accumulation of human capital within dynasties.

This framework can provide an interesting evaluation tool of mid and long-term consequences of public policies, especially for developping countries. However, there are many ways through which the United States differ from "average" countries. One important singularity of this country has been the role of democratic ideals shared and promoted by the founding fathers, for whom education had to play a key role in the establishment of the New Republic. At the same period, only Prussia, in a lesser extent France, had promoted education for moral purposes, or military ones...The development of mass education in England for instance had much to do with a capitalist lobbying on the government, which aimed at raising workers' productivity through enhanced education. Over the twenthieth century, the extension of mandatory schooling reflected most often competition or contagion effects between states, or a Schumpeterian catch-up effect as shown by Fabrice Murtin and Martina Viarengo (2006). Therefore, in a more general framework the role of the State should probably be refined. A setup for endogenous taxation and public investments into education should be considered, as well as the formation of governments themselves as it is theoretically achieved by Francois Bourguignon and Thierry Verdier (2000). This is left aside for future research.

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A Some Basic Simulations of the Model

The growth path

The growth path of the economy can be described through the reaction of the economy to shocks. One considers respectively an educational shock where an illiterate population gets suddenly a uniform distribution of years of schooling, as well as a productivity shock, namely a doubling of the exogenous technological rate.

In the first case we observe on Figure 3 a come back to the same long-term growth path after a century, despite average fertility and education have been modified. The schooling reform had just a transitory effect on growth, a conclusion that clearly would not hold if technological progress was endogenous, namely if its growth rate was depending on the stock of schooling. In the second case the transition to the new long-term growth path is achieved within less than a century, but fertility reaches its initial level on the long run.

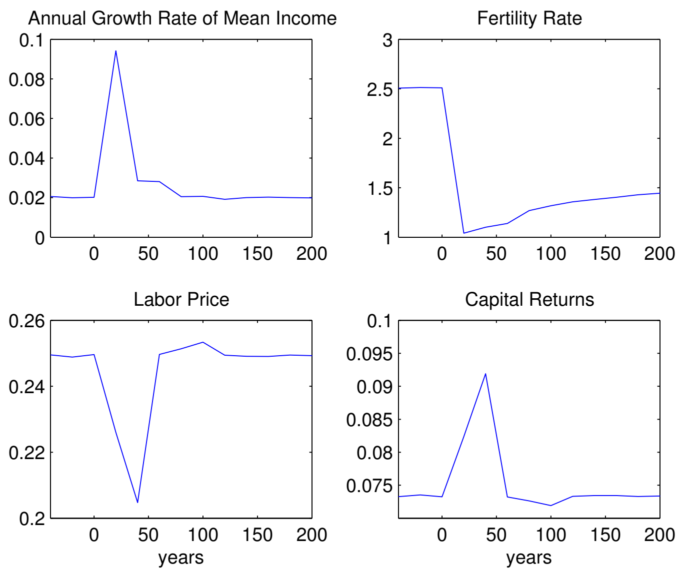

Figure 8: An Educational Shock in the Economy

As mentionned above, an important motivation of the paper is the dynamical aspect of endogenous growth, above all with respect to schooling reforms. When does the take-off start? Next figure plots the illiteracy rate after 20 iterations with respect to τ[6] for the same level of ϕ. After so many iterates, simulations show that the economy has attained its long-term growth path. For lower levels of the cost of schooling, the economy moves to a single equilibria which is full investment in education regardless of parental educational level.

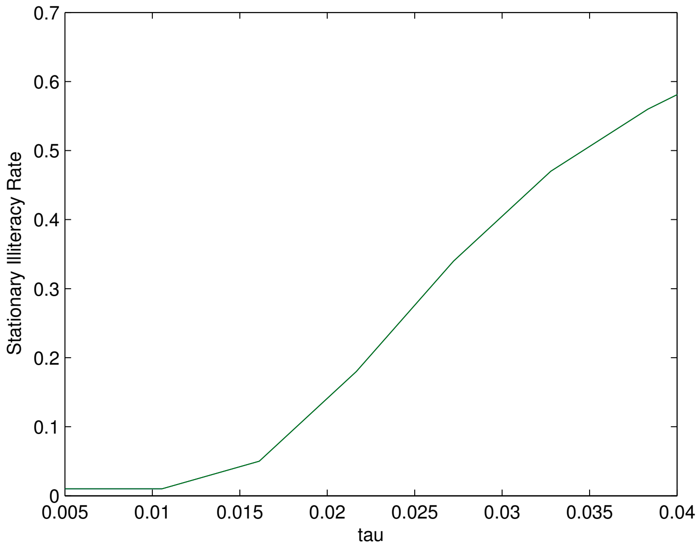

Figure 9: The Long-Term Illiteracy Rate

Endogenous growth

With endogenous growth the economy no longer returns to its steady-state exogenous growth path after some schooling reforms that has modified the long-term level of human capital. Let us reconsider the introduction of a uniform schooling distribution ex-nihilo. Here the long-term growth rate no longer re-

[6] In the appendix, values of τ have been normalized by average human capital and are not comparable with those obtained in the paper.

turns to its 2% original value but equals 4.5% now.

Figure 10: Endogeneous Growth

Human Capital Externalities

We begin by considering an exogenous technological progress and the introduction of a positive externality of average human capital on each individual's human capital. As this externality is probably low, we set $\kappa = 0.3$. Figure 6 reports few differences with the benchmark case (figure 3), except that the come-back to the 2% steady-state level is slower due to autocorrelation introduced by the human capital externality.

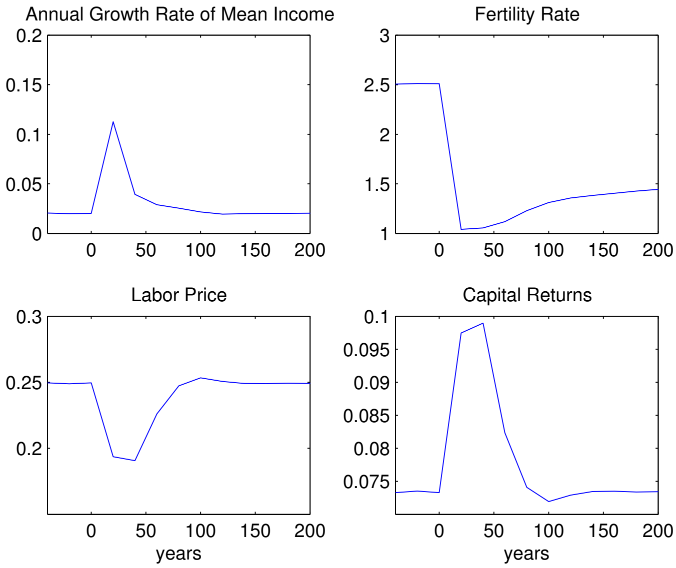

Figure 11: Direct Transmission of Human Capital

Let us turn to the other kind of human capital externality, namely that children receive human capital directly from parents inside the household. This externality drives in part social mobility. Its implications for long-run growth are quite similar to what precedes and we do not report them. On the other hand, it has interesting consequences on the equilibrium regimes of the economy. Next figure reports the stationnary illiteracy rate with respect to τ for three different levels of parental education externality: none ($\rho = 0$), medium ($\rho = 0.3$) and high ($\rho = 0.6$)

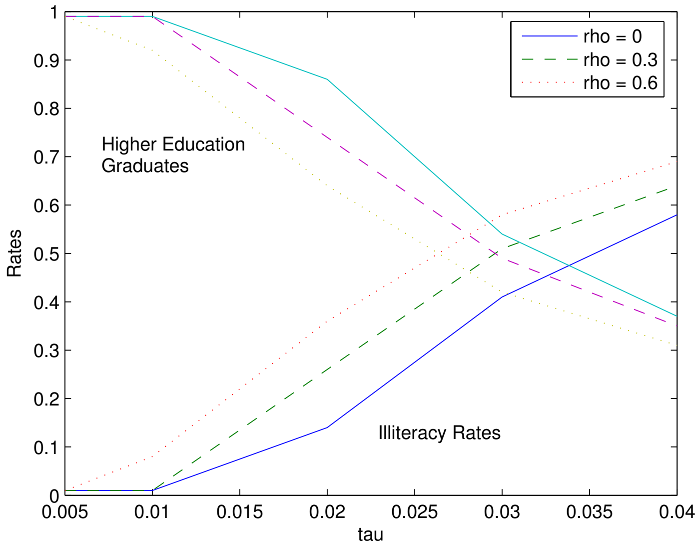

Figure 12: The Influence of Human Capital Direct Transmission on the Economic Regime

Does intergenerational transmission of human capital entail a slowdown of convergence towards the same level as without this externality? or are the long-term levels different when one introduces some autocorrelation in human capital between generations? Simulations with $\rho = 0.6$ show that 20 generations after the educational shock the distribution of schooling has indeed converged towards its limit distribution, as shown by figure 7. Therefore some extent of human capital externality modifies the long-term educational distribution. It can be understooden by the fact that at final date, illiterates are issued from dynasties that have invested less and less across time in children education. If one allows for direct transmission of human capital, at the margin some households will invest less than what they would have done in absence of this externality.But this gap cumulates along generations, and at one point in time, it might turn that an offspring receives less education than her parents have (see how the dynamics of the dynasty with 8 years of schooling at $t = 0$ has been modified).

Diminishing the incentive to accumulate human capital in each period therefore turns out to higher stationary illiteracy rate.

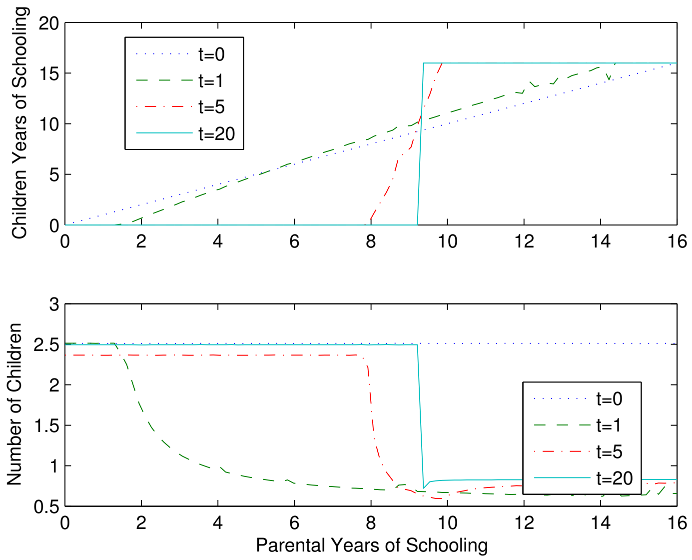

Figure 13: The Poverty Trap with Direct Transmission of Human Capital

Expectations

Here agents forecast wages and capital returns variations either on extrapolating their variations over the last two periods. Formally this leads to

$$\begin{aligned} w^e_{t+2} &= w_{t+1}\frac{w_{t+1}}{w_t} \\ r^e_{t+2} &= r_{t+1}\frac{r_{t+1}}{r_t} \end{aligned}$$

Introducing extrapolative expectations entail some cyclical fluctuations in the economy. This is due to the fact that there is no learning across the period: adaptative rather than extrapolative expectations would certainly make those cycles vanish in the long term, exactely as there is a cobweb convergence of prices based on adaptative expectations towards rational expectations prices in the classical Muth (1961) model. Consider for instance a productivity shock in period 0 with exogenous technological progress (= 10% in period 0) and no social externality. Figure 9 illustrates the convergence process towards the long-term growth path (2%). After the first burst of productivity, we converge slowly towards stationary cycles.

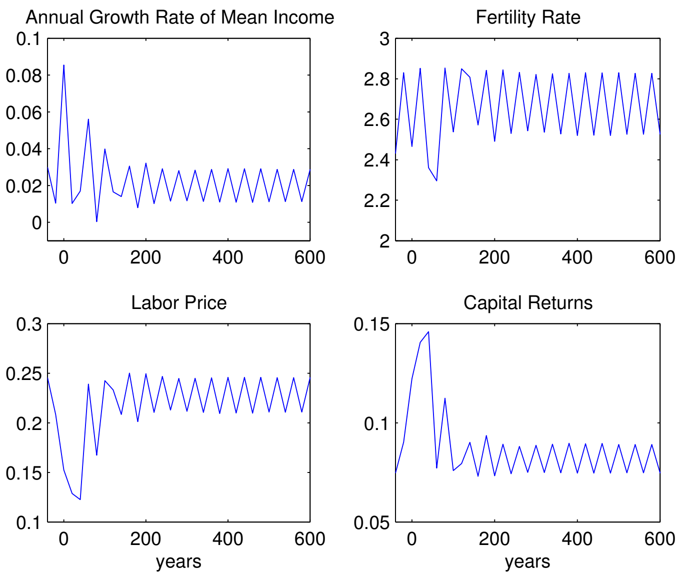

Figure 14: Extrapolative Expectations and Economic Cycles

If the agent has some adaptatives expectations, cycles may disappear and the economy converge towards a unique equilibrium, as in the cobweb model.

B A Simple Case without Mortality Risks and Logarithmic Utility Function

This simple case is useful in order to understand the functionning of the model, as well as to provide some relevant initial conditions for the stochastic maximization algorithm. First-order conditions provide

$$e^*_{t+1} = \frac{-1}{1-\eta} + \frac{\eta\phi}{\tau(1-\eta)} h^t$$

When the constraints $0 < e^*_{t+1} < 16$ are not binding, the solutions to the consumer's problem are given by

$$\begin{aligned}
c_{t+1} &= \frac{[(1+r_{t+1})w_{t+1} + w^e_{t+2}]h^t}{(1+r_{t+1})(1+\beta+\beta^2+\gamma)} \\
c_{t+2} &= \beta \frac{[(1+r_{t+1})w_{t+1} + w^e_{t+2}]h^t}{(1+\beta+\beta^2+\gamma)} \\
a_{t+1} &= \frac{[(\beta+\beta^2)(1+r_{t+1})w_{t+1} - (1+\gamma)w^e_{t+2}]h^t}{(1+r_{t+1})(1+\beta+\beta^2+\gamma)} \\
a_{t+2} &= \beta c_{t+2} \\
n_{t+1} &= \frac{(1-\eta)\gamma}{\tau w_{t+1}(-1+\frac{\phi}{\tau}h^t)(1+\beta+\beta^2+\gamma)}[w_{t+1}h^t + \frac{w^e_{t+2}}{(1+r_{t+1})}h^t]
\end{aligned}$$

In that case the education given to children and their number turn out to be respectively an increasing and decreasing functions of parental human capital. It is interesting to describe what happens when $e^*_{t+1} \leq 0$, ie when children receive no education. That occurs for all households such that

$$h^t \leq \frac{\tau}{\phi\eta}$$

In that case, the number of children is constant and equal to

$$n_{t+1} = \frac{\gamma[(1+r_{t+1})w_{t+1} + w^e_{t+2}]}{\phi w_{t+1}(1+r_{t+1})(1+\beta+\beta^2+\gamma)}$$

C A Stochastic Maximization Algorithm

A maximization of the consumer's utility function can hardly be achieved through a discretization of the state space and the explicit computation of expected utility. The size of the state space is indeed 5 (two periods consumptions, number of children, education provided, plus the conditionning by parental education), which would cause important numerical problems since a precise discretization would request a huge amount of memory and computing time.

Any solver based on gradient algorithm would be very dependant of initial conditions, since it is not guaranted that the value function is convex. Instead, I use a stochastic algorithm based on the ratio of likelihood method, where the search locus shrinks over time as in simulated annealing. The method performs extremely well in practice and is fast to converge. More precisely here is the algorithm

1. Consider initial values of education provided, number of children and savings given by the explicit formulas derived in the case where we exclude mortality risks and retain a logarithmic utility function. At iteration 0 one has 4 components times the number of households H taken equal to 100 in practice, ie one has a matrix $(x^0_{i,h})_{i\leq 4, h\leq H}$.

2. At iteration m : for each component $(x^m_{i,h})_{i\leq 4, h\leq H}$ draw a candidate matrix $(x^{m,c}_{i,h})_{i\leq 4, h\leq H}$ given by

$$\forall i, h \; x^{m,c}_{i,h} = (1+\theta_i(m).U_{[-1,1]})x^m_{i,h}$$

where $U_{[-1,1]}$ is a random number taken from a uniform distribution and $\theta_i(m)$ is a cooling parameter that narrows the domain of search across iterations. In practice the cooling is calibrated to bridge initial and final values at a sufficiently slow speed in order to insure proper exploration of the domain.

3. Compute the likelihood ratio $r = L(x_{i,h}^{m,c})/L(x_{i,h}^{m})$ and set

$$\begin{aligned} x_{i,h}^{m+1} &= x_{i,h}^{m} \text{ if } r > 1 \\ x_{i,h}^{m+1} &= x_{i,h}^{m,c} \text{ otherwise} \end{aligned}$$

4. Repeat 2-3 until $m = 1000$.

5. Return $x_{i,h}^{*} = x_{i,h}^{1000}$

In all practical cases envisaged the shape of the resulting 4 curves $(x_{i,h}^{m})_{h \leq H}$ is very regular with respect to parental education, which indicates a good convergence in practice.